

Convergent Filter Bases

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. We are inspired by the work of Henri Cartan [16], Bourbaki [10] (TG. I Filtres) and Claude Wagschal [34]. We define the base of filter, image filter, convergent filter bases, limit filter and the filter base of tails (fr: *filtre des sections*).

MSC: 54A20 03B35

Keywords: convergence; filter; filter base; Frechet filter; limit; net; sequence

MML identifier: CARDFIL2, version: 8.1.04 5.32.1246

The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [2], [33], [20], [18], [28], [11], [12], [13], [29], [3], [37], [25], [26], [4], [17], [30], [5], [14], [23], [35], [36], [22], [31], [6], [7], [9], [19], [27], and [15].

1. FILTERS – SET-THEORETICAL APPROACH

From now on X denotes a non empty set, \mathcal{F} denotes a filter of X , and S denotes a family of subsets of X .

Let X be a set and S be a family of subsets of X . We say that S is upper if and only if

(Def. 1) for every subsets Y_1, Y_2 of X such that $Y_1 \in S$ and $Y_1 \subseteq Y_2$ holds $Y_2 \in S$.

Let us note that there exists a \cap -closed family of subsets of X which is non empty and there exists a non empty, \cap -closed family of subsets of X which is upper.

Let X be a non empty set. Let us note that there exists a non empty, upper, \cap -closed family of subsets of X which has non empty elements.

Now we state the propositions:

- (1) S is a non empty, upper, \cap -closed family of subsets of X with non empty elements if and only if S is a filter of X .
- (2) Let us consider non empty sets X_1, X_2 , a filter \mathcal{F}_1 of X_1 , and a filter \mathcal{F}_2 of X_2 . Then the set of all $f_1 \times f_2$ where f_1 is an element of \mathcal{F}_1 , f_2 is an element of \mathcal{F}_2 is a non empty family of subsets of $X_1 \times X_2$.

Let X be a non empty set. We say that X is \cap -finite closed if and only if

- (Def. 2) for every finite, non empty subset S_1 of X , $\cap S_1 \in X$.

One can check that there exists a non empty set which is \cap -finite closed.

Now we state the proposition:

- (3) Let us consider a non empty set X . If X is \cap -finite closed, then X is \cap -closed.

Note that every non empty set which is \cap -finite closed is also \cap -closed.

- (4) Let us consider a set X , and a family S of subsets of X . Then S is \cap -closed and $X \in S$ if and only if $\text{FinMeetCl}(S) \subseteq S$.
- (5) Let us consider a non empty set X , and a non empty subset A of X . Then $\{B, \text{ where } B \text{ is a subset of } X : A \subseteq B\}$ is a filter of X .

Let X be a non empty set. Note that every filter of X is \cap -closed.

- (6) Let us consider a set X , and a family B of subsets of X . If $B = \{X\}$, then B is upper.
- (7) Let us consider a non empty set X , and a filter \mathcal{F}' of X . Then $\mathcal{F}' \neq 2^X$.

Let X be a non empty set. The functor $\text{Filt}(X)$ yielding a non empty set is defined by the term

- (Def. 3) the set of all \mathcal{F}' where \mathcal{F}' is a filter of X .

Let I be a non empty set and M be a $(\text{Filt}(X))$ -valued many sorted set indexed by I . The intersection of the family of filters M yielding a filter of X is defined by the term

- (Def. 4) $\cap \text{rng } M$.

Let $\mathcal{F}_1, \mathcal{F}_2$ be filters of X . We say that \mathcal{F}_1 is coarser than \mathcal{F}_2 if and only if

- (Def. 5) $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

One can verify that the predicate is reflexive. We say that \mathcal{F}_1 is finer than \mathcal{F}_2 if and only if

- (Def. 6) $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

Observe that the predicate is reflexive.

Now we state the propositions:

- (8) Let us consider a non empty set X , a filter \mathcal{F}' of X , and a filter \mathcal{F} of X . Suppose $\mathcal{F} = \{X\}$. Then \mathcal{F} is coarser than \mathcal{F}' .

- (9) Let us consider a non empty set X , a non empty set I , a $(\text{Filt}(X))$ -valued many sorted set M indexed by I , an element i of I , and a filter \mathcal{F}' of X . Suppose $\mathcal{F}' = M(i)$. Then the intersection of the family of filters M is coarser than \mathcal{F}' .
- (10) Let us consider a set X , and a family S of subsets of X . Suppose $\text{FinMeetCl}(S)$ has non empty elements. Then S has non empty elements.
- (11) Let us consider a non empty set X , a family G of subsets of X , and a filter \mathcal{F}' of X . Suppose $G \subseteq \mathcal{F}'$. Then
 - (i) $\text{FinMeetCl}(G) \subseteq \mathcal{F}'$, and
 - (ii) $\text{FinMeetCl}(G)$ has non empty elements.

The theorem is a consequence of (4).

Let X be a non empty set, \mathcal{F}' be a filter of X , and B be a non empty subset of \mathcal{F}' . We say that B is filter basis if and only if

- (Def. 7) for every element f of \mathcal{F}' , there exists an element b of B such that $b \subseteq f$.

Now we state the proposition:

- (12) Let us consider a non empty set X , a filter \mathcal{F}' of X , and a non empty subset B of \mathcal{F}' . Then \mathcal{F}' is coarser than B if and only if B is filter basis.

Let X be a non empty set and \mathcal{F}' be a filter of X . Observe that there exists a non empty subset of \mathcal{F}' which is filter basis.

A generalized basis of \mathcal{F}' is a filter basis, non empty subset of \mathcal{F}' . Now we state the proposition:

- (13) Let us consider a non empty set X . Then every filter of X is a generalized basis of \mathcal{F}' .

Let X be a set and B be a family of subsets of X . The functor $[B]$ yielding a family of subsets of X is defined by

- (Def. 8) for every subset x of X , $x \in [B]$ iff there exists an element b of B such that $b \subseteq x$.

Now we state the propositions:

- (14) Let us consider a set X , and a family S of subsets of X . Then $[S] = \{x, \text{ where } x \text{ is a subset of } X : \text{ there exists an element } b \text{ of } S \text{ such that } b \subseteq x\}$.
- (15) Let us consider a set X , and an empty family B of subsets of X . Then $[B] = 2^X$.
- (16) Let us consider a set X , and a family B of subsets of X . If $\emptyset \in B$, then $[B] = 2^X$.

2. FILTERS – LATTICE-THEORETICAL APPROACH

Now we state the propositions:

- (17) Let us consider a set X , a non empty family B of subsets of X , and a subset L of 2^X . If $B = L$, then $[B] = \uparrow L$.
- (18) Let us consider a set X , and a family B of subsets of X . Then $B \subseteq [B]$.

Let X be a set and B_1, B_2 be families of subsets of X . We say that B_1 and B_2 are equivalent generators if and only if

- (Def. 9) for every element b_1 of B_1 , there exists an element b_2 of B_2 such that $b_2 \subseteq b_1$ and for every element b_2 of B_2 , there exists an element b_1 of B_1 such that $b_1 \subseteq b_2$.

Let us note that the predicate is reflexive and symmetric.

Let us consider a set X and families B_1, B_2 of subsets of X .

Let us assume that B_1 and B_2 are equivalent generators. Now we state the propositions:

- (19) $[B_1] \subseteq [B_2]$.
- (20) $[B_1] = [B_2]$.

Let X be a non empty set, \mathcal{F}' be a filter of X , and B be a non empty subset of \mathcal{F}' . The functor $\# B$ yielding a non empty family of subsets of X is defined by the term

- (Def. 10) B .

Now we state the propositions:

- (21) Let us consider a non empty set X , a filter \mathcal{F}' of X , and a generalized basis B of \mathcal{F}' . Then $\mathcal{F}' = [\# B]$.
- (22) Let us consider a non empty set X , a filter \mathcal{F}' of X , and a family B of subsets of X . If $\mathcal{F}' = [B]$, then B is a generalized basis of \mathcal{F}' .
- (23) Let us consider a non empty set X , a filter \mathcal{F}' of X , a generalized basis B of \mathcal{F}' , a family S of subsets of X , and a subset S_1 of \mathcal{F}' . Suppose $S = S_1$ and $\# B$ and S are equivalent generators. Then S_1 is a generalized basis of \mathcal{F}' . The theorem is a consequence of (19), (21), and (22).
- (24) Let us consider a non empty set X , a filter \mathcal{F}' of X , and generalized bases $\mathcal{B}_1, \mathcal{B}_2$ of \mathcal{F}' . Then $\# \mathcal{B}_1$ and $\# \mathcal{B}_2$ are equivalent generators. The theorem is a consequence of (21).

Let X be a set and B be a family of subsets of X . We say that B is quasi basis if and only if

- (Def. 11) for every elements b_1, b_2 of B , there exists an element b of B such that $b \subseteq b_1 \cap b_2$.

Let X be a non empty set. Let us note that there exists a non empty family of subsets of X which is quasi basis and there exists a non empty, quasi basis family of subsets of X which has non empty elements.

A filter base of X is a non empty, quasi basis family of subsets of X with non empty elements. Now we state the proposition:

- (25) Let us consider a non empty set X , and a filter base B of X . Then $[B]$ is a filter of X .

Let X be a non empty set and B be a filter base of X . The functor $[B]$ yielding a filter of X is defined by the term

(Def. 12) $[B]$.

Now we state the propositions:

- (26) Let us consider a non empty set X , and filter bases $\mathcal{B}_1, \mathcal{B}_2$ of X . Suppose $[\mathcal{B}_1] = [\mathcal{B}_2]$. Then \mathcal{B}_1 and \mathcal{B}_2 are equivalent generators.
- (27) Let us consider a non empty set X , a filter base \mathcal{F} of X , and a filter \mathcal{F}' of X . Suppose $\mathcal{F} \subseteq \mathcal{F}'$. Then $[\mathcal{F}]$ is coarser than \mathcal{F}' .
- (28) Let us consider a non empty set X , and a family G of subsets of X . Suppose $\text{FinMeetCl}(G)$ has non empty elements. Then
 - (i) $\text{FinMeetCl}(G)$ is a filter base of X , and
 - (ii) there exists a filter \mathcal{F}' of X such that $\text{FinMeetCl}(G) \subseteq \mathcal{F}'$.

The theorem is a consequence of (4).

- (29) Let us consider a non empty set X , and a filter \mathcal{F}' of X . Then every generalized basis of \mathcal{F}' is a filter base of X .
- (30) Let us consider a non empty set X . Then every filter base of X is a generalized basis of $[B]$.
- (31) Let us consider a non empty set X , a filter \mathcal{F}' of X , a generalized basis B of \mathcal{F}' , and a subset L of 2_{\subseteq}^X . If $L = \# B$, then $\mathcal{F}' = \uparrow L$. The theorem is a consequence of (21) and (17).
- (32) Let us consider a non empty set X , a filter base B of X , and a subset L of 2_{\subseteq}^X . If $L = B$, then $[B] = \uparrow L$.
- (33) Let us consider a non empty set X , filters $\mathcal{F}_1, \mathcal{F}_2$ of X , a generalized basis \mathcal{B}_1 of \mathcal{F}_1 , and a generalized basis \mathcal{B}_2 of \mathcal{F}_2 . Then \mathcal{F}_1 is coarser than \mathcal{F}_2 if and only if \mathcal{B}_1 is coarser than \mathcal{B}_2 . The theorem is a consequence of (21).
- (34) Let us consider non empty sets X, Y , a function f from X into Y , a filter \mathcal{F}' of X , and a generalized basis B of \mathcal{F}' . Then
 - (i) $f^\circ(\# B)$ is a filter base of Y , and
 - (ii) $[f^\circ(\# B)]$ is a filter of Y , and

(iii) $[f^\circ(\# B)] = \{M, \text{ where } M \text{ is a subset of } Y : f^{-1}(M) \in \mathcal{F}'\}$.

PROOF: Set $\mathcal{F} = f^\circ(\# B)$. \mathcal{F} is a quasi basis, non empty family of subsets of Y by (29), [35, (123), (121)]. \mathcal{F} has non empty elements by [35, (118)]. $[\mathcal{F}] = \{M, \text{ where } M \text{ is a subset of } Y : f^{-1}(M) \in \mathcal{F}'\}$ by [35, (143)], [12, (42)], (21), [35, (123)]. \square

Let X, Y be non empty sets, f be a function from X into Y , and \mathcal{F}' be a filter of X . The image of filter \mathcal{F}' under f yielding a filter of Y is defined by the term

(Def. 13) $\{M, \text{ where } M \text{ is a subset of } Y : f^{-1}(M) \in \mathcal{F}'\}$.

Now we state the propositions:

(35) Let us consider non empty sets X, Y , a function f from X into Y , and a filter \mathcal{F}' of X . Then

- (i) $f^\circ \mathcal{F}'$ is a filter base of Y , and
- (ii) $[f^\circ \mathcal{F}'] = \text{the image of filter } \mathcal{F}' \text{ under } f$.

The theorem is a consequence of (13) and (34).

(36) Let us consider a non empty set X , and a filter base B of X . If $B = [B]$, then B is a filter of X .

(37) Let us consider non empty sets X, Y , a function f from X into Y , a filter \mathcal{F}' of X , and a generalized basis B of \mathcal{F}' . Then

- (i) $f^\circ(\# B)$ is a generalized basis of the image of filter \mathcal{F}' under f , and
- (ii) $[f^\circ(\# B)] = \text{the image of filter } \mathcal{F}' \text{ under } f$.

The theorem is a consequence of (34) and (30).

(38) Let us consider non empty sets X, Y , a function f from X into Y , and filter bases $\mathcal{B}_1, \mathcal{B}_2$ of X . Suppose \mathcal{B}_1 is coarser than \mathcal{B}_2 . Then $[\mathcal{B}_1]$ is coarser than $[\mathcal{B}_2]$. The theorem is a consequence of (30) and (33).

(39) Let us consider non empty sets X, Y , a function f from X into Y , and a filter \mathcal{F}' of X . Then $f^\circ \mathcal{F}'$ is a filter of Y if and only if $Y = \text{rng } f$.

PROOF: Reconsider $f_3 = f^\circ \mathcal{F}'$ as a filter base of Y . $[f_3] \subseteq f_3$ by [35, (143)], [11, (76), (77)]. \square

(40) Let us consider a non empty set X , a non empty subset A of X , a filter \mathcal{F}' of A , and a generalized basis B of \mathcal{F}' . Then

- (i) $(\overset{A}{\hookrightarrow})^\circ(\# B)$ is a filter base of X , and
- (ii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)]$ is a filter of X , and
- (iii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)] = \{M, \text{ where } M \text{ is a subset of } X : (\overset{A}{\hookrightarrow})^{-1}(M) \in \mathcal{F}'\}$.

Let L be a non empty relational structure. The functor $\text{Tails}(L)$ yielding a non empty family of subsets of L is defined by the term

(Def. 14) the set of all $\uparrow i$ where i is an element of L .

Now we state the proposition:

(41) Let us consider a non empty, transitive, reflexive relational structure L . Suppose Ω_L is directed. Then $[\text{Tails}(L)]$ is a filter of Ω_L .

PROOF: $\text{Tails}(L)$ is non empty family of subsets of L and quasi basis and has non empty elements by [6, (22)]. \square

Let L be a non empty, transitive, reflexive relational structure. Assume Ω_L is directed. The functor $\text{TailsFilter } L$ yielding a filter of Ω_L is defined by the term

(Def. 15) $[\text{Tails}(L)]$.

Now we state the proposition:

(42) Let us consider a non empty, transitive, reflexive relational structure L . Suppose Ω_L is directed. Then $\text{Tails}(L)$ is a generalized basis of $\text{TailsFilter } L$. The theorem is a consequence of (22).

Let L be a relational structure and x be a family of subsets of L . The functor $\# x$ yielding a family of subsets of Ω_L is defined by the term

(Def. 16) x .

Now we state the proposition:

(43) Let us consider a non empty set X , a non empty, transitive, reflexive relational structure L , and a function f from Ω_L into X . Suppose Ω_L is directed. Then $f^\circ(\# \text{Tails}(L))$ is a generalized basis of the image of filter $\text{TailsFilter } L$ under f . The theorem is a consequence of (42) and (37).

Let us consider a non empty set X , a non empty, transitive, reflexive relational structure L , a function f from Ω_L into X , and a subset x of X . Now we state the propositions:

(44) Suppose Ω_L is directed and $x \in f^\circ(\# \text{Tails}(L))$. Then there exists an element j of L such that for every element i of L such that $i \geq j$ holds $f(i) \in x$.

(45) Suppose Ω_L is directed and there exists an element j of L such that for every element i of L such that $i \geq j$ holds $f(i) \in x$. Then there exists an element b of $\text{Tails}(L)$ such that $f^\circ b \subseteq x$.

(46) Let us consider a non empty set X , a non empty, transitive, reflexive relational structure L , a function f from Ω_L into X , a filter \mathcal{F}' of X , and a generalized basis B of \mathcal{F}' . Suppose Ω_L is directed. Then \mathcal{F}' is coarser than the image of filter $\text{TailsFilter } L$ under f if and only if B is coarser than $f^\circ(\# \text{Tails}(L))$. The theorem is a consequence of (43) and (33).

(47) Let us consider a non empty set X , a non empty, transitive, reflexive relational structure L , a function f from Ω_L into X , and a filter base B of

X . Suppose Ω_L is directed. Then B is coarser than $f^\circ(\# \text{Tails}(L))$ if and only if for every element b of B , there exists an element i of L such that for every element j of L such that $i \leq j$ holds $f(j) \in b$. The theorem is a consequence of (44) and (45).

Let X be a non empty set and s be a sequence of X . The elementary filter of s yielding a filter of X is defined by the term

(Def. 17) the image of filter $\text{FrechetFilter}(\mathbb{N})$ under s .

Now we state the propositions:

(48) There exists a sequence \mathcal{F}' of $2^{\mathbb{N}}$ such that for every element x of \mathbb{N} , $\mathcal{F}'(x) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : x \leq y\}$.

PROOF: Define $\mathcal{F}(\text{object}) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : \text{there exists an element } x_0 \text{ of } \mathbb{N} \text{ such that } x_0 = \$_1 \text{ and } x_0 \leq y\}$. There exists a function f from \mathbb{N} into $2^{\mathbb{N}}$ such that for every object x such that $x \in \mathbb{N}$ holds $f(x) = \mathcal{F}(x)$ from [12, Sch. 2]. Consider \mathcal{F}' being a function from \mathbb{N} into $2^{\mathbb{N}}$ such that for every object x such that $x \in \mathbb{N}$ holds $\mathcal{F}'(x) = \mathcal{F}(x)$. For every element x of \mathbb{N} , $\mathcal{F}'(x) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : x \leq y\}$. \square

(49) Let us consider a natural number n . Then $\mathbb{N} \setminus \{t, \text{ where } t \text{ is an element of } \mathbb{N} : n \leq t\}$ is finite.

PROOF: $\mathbb{N} \setminus \{t, \text{ where } t \text{ is an element of } \mathbb{N} : n \leq t\} \subseteq n + 1$ by [8, (3), (5)], [32, (4)]. \square

(50) Let us consider an element p of the ordered \mathbb{N} . Then $\{x, \text{ where } x \text{ is an element of } \mathbb{N} : \text{there exists an element } p_0 \text{ of } \mathbb{N} \text{ such that } p = p_0 \text{ and } p_0 \leq x\} = \uparrow p$.

PROOF: For every element p of the carrier of the ordered \mathbb{N} , $\{x, \text{ where } x \text{ is an element of the carrier of the ordered } \mathbb{N} : p \leq x\} = \uparrow p$ by [6, (18)]. \square

Observe that $\Omega_{\text{the ordered } \mathbb{N}}$ is directed and the ordered \mathbb{N} is reflexive.

Now we state the proposition:

(51) Let us consider a denumerable set X . Then $\text{FrechetFilter}(X) =$ the set of all $X \setminus A$ where A is a finite subset of X .

Let us consider a sequence \mathcal{F}' of $2^{\mathbb{N}}$.

Let us assume that for every element x of \mathbb{N} , $\mathcal{F}'(x) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : x \leq y\}$. Now we state the propositions:

(52) $\text{rng } \mathcal{F}'$ is a generalized basis of $\text{FrechetFilter}(\mathbb{N})$.

PROOF: $\text{FrechetFilter}(\mathbb{N}) =$ the set of all $\mathbb{N} \setminus A$ where A is a finite subset of \mathbb{N} . For every object t such that $t \in \text{rng } \mathcal{F}'$ holds $t \in \text{FrechetFilter}(\mathbb{N})$. Reconsider $\mathcal{F}_1 = \text{rng } \mathcal{F}'$ as a non empty subset of $\text{FrechetFilter}(\mathbb{N})$. \mathcal{F}_1 is filter basis by [21, (2)], [4, (44)], [11, (3)]. \square

(53) $\#Tails(\text{the ordered } \mathbb{N}) = \text{rng } \mathcal{F}'$. The theorem is a consequence of (50).

Now we state the proposition:

(54) (i) $\#Tails(\text{the ordered } \mathbb{N})$ is a generalized basis of $\text{FrechetFilter}(\mathbb{N})$,
and

(ii) $TailsFilter \text{ the ordered } \mathbb{N} = \text{FrechetFilter}(\mathbb{N})$.

The theorem is a consequence of (48), (53), (52), and (21).

The base of Frechet filter yielding a filter base of \mathbb{N} is defined by the term

(Def. 18) $\#Tails(\text{the ordered } \mathbb{N})$.

Now we state the propositions:

(55) $\mathbb{N} \in$ the base of Frechet filter.

(56) The base of Frechet filter is a generalized basis of $\text{FrechetFilter}(\mathbb{N})$.

(57) Let us consider a non empty set X , filters $\mathcal{F}_1, \mathcal{F}_2$ of X , and a filter \mathcal{F}' of X . Suppose \mathcal{F}' is finer than \mathcal{F}_1 and \mathcal{F}' is finer than \mathcal{F}_2 . Let us consider an element M_1 of \mathcal{F}_1 , and an element M_2 of \mathcal{F}_2 . Then $M_1 \cap M_2$ is not empty.

(58) Let us consider a non empty set X , and filters $\mathcal{F}_1, \mathcal{F}_2$ of X . Suppose for every element M_1 of \mathcal{F}_1 for every element M_2 of \mathcal{F}_2 , $M_1 \cap M_2$ is not empty. Then there exists a filter \mathcal{F}' of X such that

(i) \mathcal{F}' is finer than \mathcal{F}_1 , and

(ii) \mathcal{F}' is finer than \mathcal{F}_2 .

Let X be a set and x be a subset of X . The functor $\text{SubsetToBooleSubset } x$ yielding an element of 2_{\subseteq}^X is defined by the term

(Def. 19) x .

Now we state the propositions:

(59) Let us consider an infinite set X . Then $X \in$ the set of all $X \setminus A$ where A is a finite subset of X .

(60) Let us consider a set X , and a subset A of X . Then $\{B, \text{ where } B \text{ is an element of } 2_{\subseteq}^X : A \subseteq B\} = \{B, \text{ where } B \text{ is a subset of } X : A \subseteq B\}$.

(61) Let us consider a set X , and an element a of 2_{\subseteq}^X . Then $\uparrow a = \{Y, \text{ where } Y \text{ is a subset of } X : a \subseteq Y\}$.

(62) Let us consider a set X , and a subset A of X . Then $\{B, \text{ where } B \text{ is an element of } 2_{\subseteq}^X : A \subseteq B\} = \uparrow \text{SubsetToBooleSubset } A$. The theorem is a consequence of (60).

(63) Let us consider a non empty set X , and a filter \mathcal{F}' of X . Then $\bigcup \mathcal{F}' = X$.

(64) Let us consider an infinite set X . Then the set of all $X \setminus A$ where A is a finite subset of X is a filter of X . The theorem is a consequence of (59).

Let us consider a set X . Now we state the propositions:

(65) 2^X is a filter of 2_{\subseteq}^X .

(66) $\{X\}$ is a filter of 2_{\subseteq}^X .

(67) Let us consider a non empty set X . Then $\{X\}$ is a filter of X .

Let us consider an element A of 2_{\subseteq}^X . Now we state the propositions:

(68) $\{Y, \text{ where } Y \text{ is a subset of } X : A \subseteq Y\}$ is a filter of 2_{\subseteq}^X .

(69) $\{B, \text{ where } B \text{ is an element of } 2_{\subseteq}^X : A \subseteq B\}$ is a filter of 2_{\subseteq}^X . The theorem is a consequence of (60) and (68).

Now we state the proposition:

(70) Let us consider a non empty set X , and a non empty subset B of 2_{\subseteq}^X . Then for every elements x, y of B , there exists an element z of B such that $z \subseteq x \cap y$ if and only if B is filtered.

PROOF: For every elements x, y of B , there exists an element z of B such that $z \subseteq x \cap y$ by [19, (2)]. \square

Let us consider a non empty set X and a non empty subset \mathcal{F}' of the lattice of subsets of X . Now we state the propositions:

(71) \mathcal{F}' is a filter of the lattice of subsets of X if and only if for every elements p, q of \mathcal{F}' , $p \cap q \in \mathcal{F}'$ and for every element p of \mathcal{F}' and for every element q of the lattice of subsets of X such that $p \subseteq q$ holds $q \in \mathcal{F}'$.

(72) \mathcal{F}' is a filter of the lattice of subsets of X if and only if for every subsets Y_1, Y_2 of X , if $Y_1, Y_2 \in \mathcal{F}'$, then $Y_1 \cap Y_2 \in \mathcal{F}'$ and if $Y_1 \in \mathcal{F}'$ and $Y_1 \subseteq Y_2$, then $Y_2 \in \mathcal{F}'$. The theorem is a consequence of (71).

Now we state the propositions:

(73) Let us consider a non empty set X , and a non empty family \mathcal{F} of subsets of X . Suppose \mathcal{F} is a filter of the lattice of subsets of X . Then \mathcal{F} is a filter of 2_{\subseteq}^X . The theorem is a consequence of (71).

(74) Let us consider a non empty set X . Then every filter of 2_{\subseteq}^X is a filter of the lattice of subsets of X . The theorem is a consequence of (72).

(75) Let us consider a non empty set X , and a non empty subset \mathcal{F}' of the lattice of subsets of X . Then \mathcal{F}' is filter of the lattice of subsets of X and has non empty elements if and only if \mathcal{F}' is a filter of X . The theorem is a consequence of (72).

(76) Let us consider a non empty set X . Then every proper filter of 2_{\subseteq}^X is a filter of X .

PROOF: \mathcal{F}' has non empty elements by [19, (18)], [7, (4)]. \square

(77) Let us consider a non empty topological space T , and a point x of T . Then the neighborhood system of x is a filter of the carrier of T .

Let T be a non empty topological space and \mathcal{F}' be a proper filter of $2_{\subseteq}^{\Omega T}$. The functor $\text{BooleanFilterToFilter}(\mathcal{F}')$ yielding a filter of the carrier of T is defined by the term

(Def. 20) \mathcal{F}' .

Let \mathcal{F}_1 be a filter of the carrier of T and \mathcal{F}_2 be a proper filter of $2_{\subseteq}^{\Omega T}$. We say that \mathcal{F}_1 is finer than \mathcal{F}_2 if and only if

(Def. 21) $\text{BooleanFilterToFilter}(\mathcal{F}_2) \subseteq \mathcal{F}_1$.

3. LIMIT OF A FILTER

Let T be a non empty topological space and \mathcal{F}' be a filter of the carrier of T . The functor $\text{LimFilter}(\mathcal{F}')$ yielding a subset of T is defined by the term

(Def. 22) $\{x, \text{ where } x \text{ is a point of } T : \mathcal{F}' \text{ is finer than the neighborhood system of } x\}$.

Let B be a filter base of the carrier of T . The functor $\text{Lim } B$ yielding a subset of T is defined by the term

(Def. 23) $\text{LimFilter}(B)$.

Now we state the proposition:

(78) Let us consider a non empty topological space T , and a filter \mathcal{F}' of the carrier of T . Then there exists a proper filter \mathcal{F}_1 of 2_{\subseteq}^{α} such that $\mathcal{F}' = \mathcal{F}_1$, where α is the carrier of T . The theorem is a consequence of (73) and (75).

Let T be a non empty topological space and \mathcal{F}' be a filter of the carrier of T . The functor $\text{FilterToBooleanFilter}(\mathcal{F}', T)$ yielding a proper filter of $2_{\subseteq}^{\Omega T}$ is defined by the term

(Def. 24) \mathcal{F}' .

Let us consider a non empty topological space T , a point x of T , and a filter \mathcal{F}' of the carrier of T . Now we state the propositions:

(79) x is a convergence point of \mathcal{F}' and T if and only if x is a convergence point of $\text{FilterToBooleanFilter}(\mathcal{F}', T)$ and T .

(80) x is a convergence point of \mathcal{F}' and T if and only if $x \in \text{LimFilter}(\mathcal{F}')$. The theorem is a consequence of (78).

Let T be a non empty topological space and \mathcal{F}' be a filter of $2_{\subseteq}^{\Omega T}$. The functor $\text{LimFilterB}(\mathcal{F}')$ yielding a subset of T is defined by the term

(Def. 25) $\{x, \text{ where } x \text{ is a point of } T : \text{ the neighborhood system of } x \subseteq \mathcal{F}'\}$.

Let us consider a non empty topological space T and a filter \mathcal{F}' of the carrier of T . Now we state the propositions:

(81) $\text{LimFilter}(\mathcal{F}') = \text{LimFilterB}(\text{FilterToBooleanFilter}(\mathcal{F}', T)).$

(82) $\text{Lim}(\text{the net of } \text{FilterToBooleanFilter}(\mathcal{F}', T)) = \text{LimFilter}(\mathcal{F}').$

(83) Let us consider a Hausdorff, non empty topological space T , a filter \mathcal{F}' of the carrier of T , and points p, q of T . If $p, q \in \text{LimFilter}(\mathcal{F}')$, then $p = q$.

Let T be a Hausdorff, non empty topological space and \mathcal{F}' be a filter of the carrier of T . Note that $\text{LimFilter}(\mathcal{F}')$ is trivial.

Let X be a non empty set, T be a non empty topological space, f be a function from X into the carrier of T , and \mathcal{F}' be a filter of X . The functor $\text{lim}_{\mathcal{F}'} f$ yielding a subset of Ω_T is defined by the term

(Def. 26) $\text{LimFilter}(\text{the image of filter } \mathcal{F}' \text{ under } f).$

Let L be a non empty, transitive, reflexive relational structure and f be a function from Ω_L into the carrier of T . The functor $\text{LimF}(f)$ yielding a subset of Ω_T is defined by the term

(Def. 27) $\text{LimFilter}(\text{the image of filter } \text{TailsFilter } L \text{ under } f).$

Now we state the proposition:

(84) Let us consider a non empty topological space T , a non empty, transitive, reflexive relational structure L , a function f from Ω_L into the carrier of T , a point x of T , and a generalized basis B of $\text{BooleanFilterToFilter}(\text{the neighborhood system of } x)$. Suppose Ω_L is directed. Then $x \in \text{LimF}(f)$ if and only if for every element b of B , there exists an element i of L such that for every element j of L such that $i \leq j$ holds $f(j) \in b$. The theorem is a consequence of (46), (29), and (47).

Let T be a non empty topological space and s be a sequence of T . The functor $\text{LimF}(s)$ yielding a subset of T is defined by the term

(Def. 28) $\text{LimFilter}(\text{the elementary filter of } s).$

Now we state the proposition:

(85) Let us consider a non empty topological space T , and a sequence s of T . Then $\text{lim}_{\text{FrechetFilter}(\mathbb{N})} s = \text{LimF}(s)$.

Let us consider a non empty topological space T and a point x of T .

(86) The neighborhood system of x is a filter base of Ω_T . The theorem is a consequence of (76), (13), and (29).

(87) Every generalized basis of $\text{BooleanFilterToFilter}(\text{the neighborhood system of } x)$ is a filter base of Ω_T .

(88) Let us consider a non empty set X , a sequence s of X , and a filter base B of X . Then B is coarser than s° (the base of Frechet filter) if and only if for every element b of B , there exists an element i of the ordered \mathbb{N} such that for every element j of the ordered \mathbb{N} such that $i \leq j$ holds $s(j) \in b$.

- (89) Let us consider a non empty topological space T , a sequence s of T , a point x of T , and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then $x \in \lim_{\text{FrechetFilter}(\mathbb{N})} s$ if and only if B is coarser than s° (the base of Frechet filter). The theorem is a consequence of (46) and (54).
- (90) Let us consider a non empty topological space T , a sequence s of Ω_T , a point x of T , and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Then B is coarser than s° (the base of Frechet filter) if and only if for every element b of B , there exists an element i of the ordered \mathbb{N} such that for every element j of the ordered \mathbb{N} such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (29) and (47).

Let us consider a non empty topological space T , a sequence s of the carrier of T , a point x of T , and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x).

- (91) $x \in \lim_{\text{FrechetFilter}(\mathbb{N})} s$ if and only if for every element b of B , there exists an element i of the ordered \mathbb{N} such that for every element j of the ordered \mathbb{N} such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (89) and (90).
- (92) $x \in \text{LimF}(s)$ if and only if for every element b of B , there exists an element i of the ordered \mathbb{N} such that for every element j of the ordered \mathbb{N} such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (91).

4. NETS

Let L be a 1-sorted structure and s be a sequence of the carrier of L . The net of s yielding a non empty, strict net structure over L is defined by the term

(Def. 29) $\langle \mathbb{N}, \leq_{\mathbb{N}}, s \rangle$.

Let L be a non empty 1-sorted structure. Let us note that the net of s is non empty.

Now we state the proposition:

- (93) Let us consider a non empty 1-sorted structure L , a set B , and a sequence s of the carrier of L . Then the net of s is eventually in B if and only if there exists an element i of the net of s such that for every element j of the net of s such that $i \leq j$ holds (the net of s)(j) $\in B$.

Let us consider a non empty topological space T , a sequence s of the carrier of T , a point x of T , and a generalized basis B of BooleanFilterToFilter(the neighborhood system of x). Now we state the propositions:

- (94) for every element b of B , there exists an element i of the ordered \mathbb{N} such that for every element j of the ordered \mathbb{N} such that $i \leq j$ holds $s(j) \in b$ if

and only if for every element b of B , there exists an element i of the net of s such that for every element j of the net of s such that $i \leq j$ holds (the net of s)(j) $\in b$.

- (95) $x \in \text{LimF}(s)$ if and only if for every element b of B , the net of s is eventually in b . The theorem is a consequence of (92), (94), and (93).
- (96) $x \in \text{LimF}(s)$ if and only if for every element b of B , there exists an element i of \mathbb{N} such that for every element j of \mathbb{N} such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (91).
- (97) $x \in \text{LimF}(s)$ if and only if for every element b of B , there exists a natural number i such that for every natural number j such that $i \leq j$ holds $s(j) \in b$. The theorem is a consequence of (96).

REFERENCES

- [1] Grzegorz Bancerek. Cardinal numbers. *Formalized Mathematics*, 1(2):377–382, 1990.
- [2] Grzegorz Bancerek. König’s theorem. *Formalized Mathematics*, 1(3):589–593, 1990.
- [3] Grzegorz Bancerek. Complete lattices. *Formalized Mathematics*, 2(5):719–725, 1991.
- [4] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [5] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [6] Grzegorz Bancerek. Directed sets, nets, ideals, filters, and maps. *Formalized Mathematics*, 6(1):93–107, 1997.
- [7] Grzegorz Bancerek. Prime ideals and filters. *Formalized Mathematics*, 6(2):241–247, 1997.
- [8] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [9] Grzegorz Bancerek, Noboru Endou, and Yuji Sakai. On the characterizations of compactness. *Formalized Mathematics*, 9(4):733–738, 2001.
- [10] Nicolas Bourbaki. *General Topology: Chapters 1–4*. Springer Science and Business Media, 2013.
- [11] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [12] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [13] Czesław Byliński. Basic functions and operations on functions. *Formalized Mathematics*, 1(1):245–254, 1990.
- [14] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [15] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [16] Henri Cartan. Théorie des filtres. *C. R. Acad. Sci.*, CCV:595–598, 1937.
- [17] Marek Chmur. The lattice of natural numbers and the sublattice of it. The set of prime numbers. *Formalized Mathematics*, 2(4):453–459, 1991.
- [18] Agata Darmochwał. Finite sets. *Formalized Mathematics*, 1(1):165–167, 1990.
- [19] Adam Grabowski and Robert Milewski. Boolean posets, posets under inclusion and products of relational structures. *Formalized Mathematics*, 6(1):117–121, 1997.
- [20] Gilbert Lee and Piotr Rudnicki. Dickson’s lemma. *Formalized Mathematics*, 10(1):29–37, 2002.
- [21] Yatsuka Nakamura and Hisashi Ito. Basic properties and concept of selected subsequence of zero based finite sequences. *Formalized Mathematics*, 16(3):283–288, 2008. doi:10.2478/v10037-008-0034-y.

- [22] Beata Padlewska. Families of sets. *Formalized Mathematics*, 1(1):147–152, 1990.
- [23] Beata Padlewska and Agata Darmochwał. Topological spaces and continuous functions. *Formalized Mathematics*, 1(1):223–230, 1990.
- [24] Alexander Yu. Shibakov and Andrzej Trybulec. The Cantor set. *Formalized Mathematics*, 5(2):233–236, 1996.
- [25] Andrzej Trybulec. Tuples, projections and Cartesian products. *Formalized Mathematics*, 1(1):97–105, 1990.
- [26] Andrzej Trybulec. On the sets inhabited by numbers. *Formalized Mathematics*, 11(4):341–347, 2003.
- [27] Andrzej Trybulec. Moore-Smith convergence. *Formalized Mathematics*, 6(2):213–225, 1997.
- [28] Andrzej Trybulec and Agata Darmochwał. Boolean domains. *Formalized Mathematics*, 1(1):187–190, 1990.
- [29] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [30] Wojciech A. Trybulec and Grzegorz Bancerek. Kuratowski – Zorn lemma. *Formalized Mathematics*, 1(2):387–393, 1990.
- [31] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [32] Tetsuya Tsunetou, Grzegorz Bancerek, and Yatsuka Nakamura. Zero-based finite sequences. *Formalized Mathematics*, 9(4):825–829, 2001.
- [33] Josef Urban. Basic facts about inaccessible and measurable cardinals. *Formalized Mathematics*, 9(2):323–329, 2001.
- [34] Claude Wagschal. *Topologie et analyse fonctionnelle*. Hermann, 1995.
- [35] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [36] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [37] Stanisław Żukowski. Introduction to lattice theory. *Formalized Mathematics*, 1(1):215–222, 1990.

Received June 30, 2015
