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# Convergent Filter Bases

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**Summary.** We are inspired by the work of Henri Cartan [16], Bourbaki [10] (TG. I Filtres) and Claude Wagschal [34]. We define the base of filter, image filter, convergent filter bases, limit filter and the filter base of tails (fr: *filtre des sections*).

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The notation and terminology used in this paper have been introduced in the following articles: [24], [1], [2], [33], [20], [18], [28], [11], [12], [13], [29], [3], [37], [25], [26], [4], [17], [30], [5], [14], [23], [35], [36], [22], [31], [6], [7], [9], [19], [27], and [15].

## 1. Filters – Set-Theoretical Approach

From now on $X$ denotes a non empty set, $\mathcal{F}$ denotes a filter of $X$, and $S$ denotes a family of subsets of $X$.

Let $X$ be a set and $S$ be a family of subsets of $X$. We say that $S$ is upper if and only if

(Def. 1) for every subsets $Y_1$, $Y_2$ of $X$ such that $Y_1 \in S$ and $Y_1 \subseteq Y_2$ holds $Y_2 \in S$.

Let us note that there exists a $\cap$-closed family of subsets of $X$ which is non empty and there exists a non empty, $\cap$-closed family of subsets of $X$ which is upper.

Let $X$ be a non empty set. Let us note that there exists a non empty, upper, $\cap$-closed family of subsets of $X$ which has non empty elements.

Now we state the propositions:




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(1) $S$ is a non empty, upper, $\cap$-closed family of subsets of $X$ with non empty elements if and only if $S$ is a filter of $X$.

(2) Let us consider non empty sets $X_1$, $X_2$, a filter $\mathcal{F}_1$ of $X_1$, and a filter $\mathcal{F}_2$ of $X_2$. Then the set of all $f_1 \times f_2$ where $f_1$ is an element of $\mathcal{F}_1$, $f_2$ is an element of $\mathcal{F}_2$ is a non empty family of subsets of $X_1 \times X_2$.

Let $X$ be a non empty set. We say that $X$ is $\cap$-finite closed if and only if

(Def. 2) for every finite, non empty subset $S_1$ of $X$, $\bigcap S_1 \in X$.

One can check that there exists a non empty set which is $\cap$-finite closed.

Now we state the proposition:

(3) Let us consider a non empty set $X$. If $X$ is $\cap$-finite closed, then $X$ is $\cap$-closed.

Note that every non empty set which is $\cap$-finite closed is also $\cap$-closed.

(4) Let us consider a set $X$, and a family $S$ of subsets of $X$. Then $S$ is $\cap$-closed and $X \in S$ if and only if $\mathrm{FinMeetCl}(S) \subseteq S$.

(5) Let us consider a non empty set $X$, and a non empty subset $A$ of $X$. Then $\{B$, where $B$ is a subset of $X : A \subseteq B\}$ is a filter of $X$.

Let $X$ be a non empty set. Note that every filter of $X$ is $\cap$-closed.

(6) Let us consider a set $X$, and a family $B$ of subsets of $X$. If $B = \{X\}$, then $B$ is upper.

(7) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then $\mathcal{F}' \neq 2^X$.

Let $X$ be a non empty set. The functor $\mathrm{Filt}(X)$ yielding a non empty set is defined by the term

(Def. 3) the set of all $\mathcal{F}'$ where $\mathcal{F}'$ is a filter of $X$.

Let $I$ be a non empty set and $M$ be a $(\mathrm{Filt}(X))$-valued many sorted set indexed by $I$. The intersection of the family of filters $M$ yielding a filter of $X$ is defined by the term

(Def. 4) $\bigcap \mathrm{rng}\, M$.

Let $\mathcal{F}_1$, $\mathcal{F}_2$ be filters of $X$. We say that $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$ if and only if

(Def. 5) $\mathcal{F}_1 \subseteq \mathcal{F}_2$.

One can verify that the predicate is reflexive. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 6) $\mathcal{F}_2 \subseteq \mathcal{F}_1$.

Observe that the predicate is reflexive.

Now we state the propositions:

(8) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a filter $\mathcal{F}$ of $X$. Suppose $\mathcal{F} = \{X\}$. Then $\mathcal{F}$ is coarser than $\mathcal{F}'$.

(9) Let us consider a non empty set $X$, a non empty set $I$, a $(\mathrm{Filt}(X))$-valued many sorted set $M$ indexed by $I$, an element $i$ of $I$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F}' = M(i)$. Then the intersection of the family of filters $M$ is coarser than $\mathcal{F}'$.

(10) Let us consider a set $X$, and a family $S$ of subsets of $X$. Suppose $\mathrm{FinMeetCl}(S)$ has non empty elements. Then $S$ has non empty elements.

(11) Let us consider a non empty set $X$, a family $G$ of subsets of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $G \subseteq \mathcal{F}'$. Then

  (i) $\mathrm{FinMeetCl}(G) \subseteq \mathcal{F}'$, and

  (ii) $\mathrm{FinMeetCl}(G)$ has non empty elements.

The theorem is a consequence of (4).

Let $X$ be a non empty set, $\mathcal{F}'$ be a filter of $X$, and $B$ be a non empty subset of $\mathcal{F}'$. We say that $B$ is filter basis if and only if

(Def. 7) for every element $f$ of $\mathcal{F}'$, there exists an element $b$ of $B$ such that $b \subseteq f$.

Now we state the proposition:

(12) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a non empty subset $B$ of $\mathcal{F}'$. Then $\mathcal{F}'$ is coarser than $B$ if and only if $B$ is filter basis.

Let $X$ be a non empty set and $\mathcal{F}'$ be a filter of $X$. Observe that there exists a non empty subset of $\mathcal{F}'$ which is filter basis.

A generalized basis of $\mathcal{F}'$ is a filter basis, non empty subset of $\mathcal{F}'$. Now we state the proposition:

(13) Let us consider a non empty set $X$. Then every filter of $X$ is a generalized basis of $\mathcal{F}'$.

Let $X$ be a set and $B$ be a family of subsets of $X$. The functor $[B]$ yielding a family of subsets of $X$ is defined by

(Def. 8) for every subset $x$ of $X$, $x \in it$ iff there exists an element $b$ of $B$ such that $b \subseteq x$.

Now we state the propositions:

(14) Let us consider a set $X$, and a family $S$ of subsets of $X$. Then $[S] = \{x$, where $x$ is a subset of $X$ : there exists an element $b$ of $S$ such that $b \subseteq x\}$.

(15) Let us consider a set $X$, and an empty family $B$ of subsets of $X$. Then $[B] = 2^X$.

(16) Let us consider a set $X$, and a family $B$ of subsets of $X$. If $\emptyset \in B$, then $[B] = 2^X$.

## 2. Filters – Lattice-Theoretical Approach

Now we state the propositions:

(17) Let us consider a set $X$, a non empty family $B$ of subsets of $X$, and a subset $L$ of $2^X_{\subseteq}$. If $B = L$, then $[B] = \uparrow L$.

(18) Let us consider a set $X$, and a family $B$ of subsets of $X$. Then $B \subseteq [B]$.

Let $X$ be a set and $B_1$, $B_2$ be families of subsets of $X$. We say that $B_1$ and $B_2$ are equivalent generators if and only if

(Def. 9) for every element $b_1$ of $B_1$, there exists an element $b_2$ of $B_2$ such that $b_2 \subseteq b_1$ and for every element $b_2$ of $B_2$, there exists an element $b_1$ of $B_1$ such that $b_1 \subseteq b_2$.

Let us note that the predicate is reflexive and symmetric.

Let us consider a set $X$ and families $B_1$, $B_2$ of subsets of $X$.

Let us assume that $B_1$ and $B_2$ are equivalent generators. Now we state the propositions:

(19) $[B_1] \subseteq [B_2]$.

(20) $[B_1] = [B_2]$.

Let $X$ be a non empty set, $\mathcal{F}'$ be a filter of $X$, and $B$ be a non empty subset of $\mathcal{F}'$. The functor $\# B$ yielding a non empty family of subsets of $X$ is defined by the term

(Def. 10) $B$.

Now we state the propositions:

(21) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Then $\mathcal{F}' = [\# B]$.

(22) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and a family $B$ of subsets of $X$. If $\mathcal{F}' = [B]$, then $B$ is a generalized basis of $\mathcal{F}'$.

(23) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, a generalized basis $B$ of $\mathcal{F}'$, a family $S$ of subsets of $X$, and a subset $S_1$ of $\mathcal{F}'$. Suppose $S = S_1$ and $\# B$ and $S$ are equivalent generators. Then $S_1$ is a generalized basis of $\mathcal{F}'$. The theorem is a consequence of (19), (21), and (22).

(24) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, and generalized bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $\mathcal{F}'$. Then $\# \mathcal{B}_1$ and $\# \mathcal{B}_2$ are equivalent generators. The theorem is a consequence of (21).

Let $X$ be a set and $B$ be a family of subsets of $X$. We say that $B$ is quasi basis if and only if

(Def. 11) for every elements $b_1$, $b_2$ of $B$, there exists an element $b$ of $B$ such that $b \subseteq b_1 \cap b_2$.

Let $X$ be a non empty set. Let us note that there exists a non empty family of subsets of $X$ which is quasi basis and there exists a non empty, quasi basis family of subsets of $X$ which has non empty elements.

A filter base of $X$ is a non empty, quasi basis family of subsets of $X$ with non empty elements. Now we state the proposition:

(25) Let us consider a non empty set $X$, and a filter base $B$ of $X$. Then $[B]$ is a filter of $X$.

Let $X$ be a non empty set and $B$ be a filter base of $X$. The functor $[B)$ yielding a filter of $X$ is defined by the term

(Def. 12) $[B]$.

Now we state the propositions:

(26) Let us consider a non empty set $X$, and filter bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $X$. Suppose $[\mathcal{B}_1) = [\mathcal{B}_2)$. Then $\mathcal{B}_1$ and $\mathcal{B}_2$ are equivalent generators.

(27) Let us consider a non empty set $X$, a filter base $\mathcal{F}$ of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F} \subseteq \mathcal{F}'$. Then $[\mathcal{F})$ is coarser than $\mathcal{F}'$.

(28) Let us consider a non empty set $X$, and a family $G$ of subsets of $X$. Suppose $\mathrm{FinMeetCl}(G)$ has non empty elements. Then

(i) $\mathrm{FinMeetCl}(G)$ is a filter base of $X$, and

(ii) there exists a filter $\mathcal{F}'$ of $X$ such that $\mathrm{FinMeetCl}(G) \subseteq \mathcal{F}'$.

The theorem is a consequence of (4).

(29) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then every generalized basis of $\mathcal{F}'$ is a filter base of $X$.

(30) Let us consider a non empty set $X$. Then every filter base of $X$ is a generalized basis of $[B)$.

(31) Let us consider a non empty set $X$, a filter $\mathcal{F}'$ of $X$, a generalized basis $B$ of $\mathcal{F}'$, and a subset $L$ of $2^X_{\subseteq}$. If $L = \# B$, then $\mathcal{F}' = \uparrow L$. The theorem is a consequence of (21) and (17).

(32) Let us consider a non empty set $X$, a filter base $B$ of $X$, and a subset $L$ of $2^X_{\subseteq}$. If $L = B$, then $[B) = \uparrow L$.

(33) Let us consider a non empty set $X$, filters $\mathcal{F}_1$, $\mathcal{F}_2$ of $X$, a generalized basis $\mathcal{B}_1$ of $\mathcal{F}_1$, and a generalized basis $\mathcal{B}_2$ of $\mathcal{F}_2$. Then $\mathcal{F}_1$ is coarser than $\mathcal{F}_2$ if and only if $\mathcal{B}_1$ is coarser than $\mathcal{B}_2$. The theorem is a consequence of (21).

(34) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Then

(i) $f^\circ(\# B)$ is a filter base of $Y$, and

(ii) $[f^\circ(\# B)]$ is a filter of $Y$, and

(iii) $[f^\circ(\# B)] = \{M$, where $M$ is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$.

PROOF: Set $\mathcal{F} = f^\circ(\# B)$. $\mathcal{F}$ is a quasi basis, non empty family of subsets of $Y$ by (29), [35, (123), (121)]. $\mathcal{F}$ has non empty elements by [35, (118)]. $[\mathcal{F}] = \{M$, where $M$ is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$ by [35, (143)], [12, (42)], (21), [35, (123)]. □

Let $X$, $Y$ be non empty sets, $f$ be a function from $X$ into $Y$, and $\mathcal{F}'$ be a filter of $X$. The image of filter $\mathcal{F}'$ under $f$ yielding a filter of $Y$ is defined by the term

(Def. 13) $\{M$, where $M$ is a subset of $Y : f^{-1}(M) \in \mathcal{F}'\}$.

Now we state the propositions:

(35) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, and a filter $\mathcal{F}'$ of $X$. Then

(i) $f^\circ\mathcal{F}'$ is a filter base of $Y$, and

(ii) $[f^\circ\mathcal{F}']$ = the image of filter $\mathcal{F}'$ under $f$.

The theorem is a consequence of (13) and (34).

(36) Let us consider a non empty set $X$, and a filter base $B$ of $X$. If $B = [B)$, then $B$ is a filter of $X$.

(37) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Then

(i) $f^\circ(\# B)$ is a generalized basis of the image of filter $\mathcal{F}'$ under $f$, and

(ii) $[f^\circ(\# B)]$ = the image of filter $\mathcal{F}'$ under $f$.

The theorem is a consequence of (34) and (30).

(38) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, and filter bases $\mathcal{B}_1$, $\mathcal{B}_2$ of $X$. Suppose $\mathcal{B}_1$ is coarser than $\mathcal{B}_2$. Then $[\mathcal{B}_1)$ is coarser than $[\mathcal{B}_2)$. The theorem is a consequence of (30) and (33).

(39) Let us consider non empty sets $X$, $Y$, a function $f$ from $X$ into $Y$, and a filter $\mathcal{F}'$ of $X$. Then $f^\circ\mathcal{F}'$ is a filter of $Y$ if and only if $Y = \operatorname{rng} f$.
PROOF: Reconsider $f_3 = f^\circ\mathcal{F}'$ as a filter base of $Y$. $[f_3) \subseteq f_3$ by [35, (143)], [11, (76), (77)]. □

(40) Let us consider a non empty set $X$, a non empty subset $A$ of $X$, a filter $\mathcal{F}'$ of $A$, and a generalized basis $B$ of $\mathcal{F}'$. Then

(i) $(\overset{A}{\hookrightarrow})^\circ(\# B)$ is a filter base of $X$, and

(ii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)]$ is a filter of $X$, and

(iii) $[(\overset{A}{\hookrightarrow})^\circ(\# B)] = \{M$, where $M$ is a subset of $X : (\overset{A}{\hookrightarrow})^{-1}(M) \in \mathcal{F}'\}$.

Let $L$ be a non empty relational structure. The functor $\operatorname{Tails}(L)$ yielding a non empty family of subsets of $L$ is defined by the term

(Def. 14) the set of all $\uparrow i$ where $i$ is an element of $L$.

Now we state the proposition:

(41) Let us consider a non empty, transitive, reflexive relational structure $L$. Suppose $\Omega_L$ is directed. Then $[\mathrm{Tails}(L)]$ is a filter of $\Omega_L$.
PROOF: $\mathrm{Tails}(L)$ is non empty family of subsets of $L$ and quasi basis and has non empty elements by [6, (22)]. $\square$

Let $L$ be a non empty, transitive, reflexive relational structure. Assume $\Omega_L$ is directed. The functor TailsFilter $L$ yielding a filter of $\Omega_L$ is defined by the term

(Def. 15) $[\mathrm{Tails}(L)]$.

Now we state the proposition:

(42) Let us consider a non empty, transitive, reflexive relational structure $L$. Suppose $\Omega_L$ is directed. Then $\mathrm{Tails}(L)$ is a generalized basis of TailsFilter $L$. The theorem is a consequence of (22).

Let $L$ be a relational structure and $x$ be a family of subsets of $L$. The functor $\# x$ yielding a family of subsets of $\Omega_L$ is defined by the term

(Def. 16) $x$.

Now we state the proposition:

(43) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, and a function $f$ from $\Omega_L$ into $X$. Suppose $\Omega_L$ is directed. Then $f^{\circ}(\# \mathrm{Tails}(L))$ is a generalized basis of the image of filter TailsFilter $L$ under $f$. The theorem is a consequence of (42) and (37).

Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, and a subset $x$ of $X$. Now we state the propositions:

(44) Suppose $\Omega_L$ is directed and $x \in f^{\circ}(\# \mathrm{Tails}(L))$. Then there exists an element $j$ of $L$ such that for every element $i$ of $L$ such that $i \geqslant j$ holds $f(i) \in x$.

(45) Suppose $\Omega_L$ is directed and there exists an element $j$ of $L$ such that for every element $i$ of $L$ such that $i \geqslant j$ holds $f(i) \in x$. Then there exists an element $b$ of $\mathrm{Tails}(L)$ such that $f^{\circ}b \subseteq x$.

(46) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, a filter $\mathcal{F}'$ of $X$, and a generalized basis $B$ of $\mathcal{F}'$. Suppose $\Omega_L$ is directed. Then $\mathcal{F}'$ is coarser than the image of filter TailsFilter $L$ under $f$ if and only if $B$ is coarser than $f^{\circ}(\# \mathrm{Tails}(L))$. The theorem is a consequence of (43) and (33).

(47) Let us consider a non empty set $X$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into $X$, and a filter base $B$ of

$X$. Suppose $\Omega_L$ is directed. Then $B$ is coarser than $f^{\circ}(\#\,\mathrm{Tails}(L))$ if and only if for every element $b$ of $B$, there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (44) and (45).

Let $X$ be a non empty set and $s$ be a sequence of $X$. The elementary filter of $s$ yielding a filter of $X$ is defined by the term

(Def. 17) the image of filter FrechetFilter($\mathbb{N}$) under $s$.

Now we state the propositions:

(48) There exists a sequence $\mathcal{F}'$ of $2^{\mathbb{N}}$ such that for every element $x$ of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where $y$ is an element of $\mathbb{N} : x \leqslant y\}$.

PROOF: Define $\mathcal{F}(\text{object}) = \{y$, where $y$ is an element of $\mathbb{N}$ : there exists an element $x_0$ of $\mathbb{N}$ such that $x_0 = \$_1$ and $x_0 \leqslant y\}$. There exists a function $f$ from $\mathbb{N}$ into $2^{\mathbb{N}}$ such that for every object $x$ such that $x \in \mathbb{N}$ holds $f(x) = \mathcal{F}(x)$ from [12, Sch. 2]. Consider $\mathcal{F}'$ being a function from $\mathbb{N}$ into $2^{\mathbb{N}}$ such that for every object $x$ such that $x \in \mathbb{N}$ holds $\mathcal{F}'(x) = \mathcal{F}(x)$. For every element $x$ of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where $y$ is an element of $\mathbb{N} : x \leqslant y\}$. □

(49) Let us consider a natural number $n$. Then $\mathbb{N} \setminus \{t$, where $t$ is an element of $\mathbb{N} : n \leqslant t\}$ is finite.

PROOF: $\mathbb{N} \setminus \{t$, where $t$ is an element of $\mathbb{N} : n \leqslant t\} \subseteq n+1$ by [8, (3), (5)], [32, (4)]. □

(50) Let us consider an element $p$ of the ordered $\mathbb{N}$. Then $\{x$, where $x$ is an element of $\mathbb{N}$ : there exists an element $p_0$ of $\mathbb{N}$ such that $p = p_0$ and $p_0 \leqslant x\} = {\uparrow}p$.

PROOF: For every element $p$ of the carrier of the ordered $\mathbb{N}$, $\{x$, where $x$ is an element of the carrier of the ordered $\mathbb{N} : p \leqslant x\} = {\uparrow}p$ by [6, (18)]. □

Observe that $\Omega_{\text{the ordered } \mathbb{N}}$ is directed and the ordered $\mathbb{N}$ is reflexive.

Now we state the proposition:

(51) Let us consider a denumerable set $X$. Then FrechetFilter($X$) = the set of all $X \setminus A$ where $A$ is a finite subset of $X$.

Let us consider a sequence $\mathcal{F}'$ of $2^{\mathbb{N}}$.

Let us assume that for every element $x$ of $\mathbb{N}$, $\mathcal{F}'(x) = \{y$, where $y$ is an element of $\mathbb{N} : x \leqslant y\}$. Now we state the propositions:

(52) $\operatorname{rng} \mathcal{F}'$ is a generalized basis of FrechetFilter($\mathbb{N}$).

PROOF: FrechetFilter($\mathbb{N}$) = the set of all $\mathbb{N} \setminus A$ where $A$ is a finite subset of $\mathbb{N}$. For every object $t$ such that $t \in \operatorname{rng} \mathcal{F}'$ holds $t \in$ FrechetFilter($\mathbb{N}$). Reconsider $\mathcal{F}_1 = \operatorname{rng} \mathcal{F}'$ as a non empty subset of FrechetFilter($\mathbb{N}$). $\mathcal{F}_1$ is filter basis by [21, (2)], [4, (44)], [11, (3)]. □

(53) $\#$ Tails(the ordered $\mathbb{N}$) $=$ rng $\mathcal{F}'$. The theorem is a consequence of (50).

Now we state the proposition:

(54) (i) $\#$ Tails(the ordered $\mathbb{N}$) is a generalized basis of FrechetFilter($\mathbb{N}$), and

(ii) TailsFilter the ordered $\mathbb{N} =$ FrechetFilter($\mathbb{N}$).

The theorem is a consequence of (48), (53), (52), and (21).

The base of Frechet filter yielding a filter base of $\mathbb{N}$ is defined by the term

(Def. 18) $\#$ Tails(the ordered $\mathbb{N}$).

Now we state the propositions:

(55) $\mathbb{N} \in$ the base of Frechet filter.

(56) The base of Frechet filter is a generalized basis of FrechetFilter($\mathbb{N}$).

(57) Let us consider a non empty set $X$, filters $\mathcal{F}_1$, $\mathcal{F}_2$ of $X$, and a filter $\mathcal{F}'$ of $X$. Suppose $\mathcal{F}'$ is finer than $\mathcal{F}_1$ and $\mathcal{F}'$ is finer than $\mathcal{F}_2$. Let us consider an element $M_1$ of $\mathcal{F}_1$, and an element $M_2$ of $\mathcal{F}_2$. Then $M_1 \cap M_2$ is not empty.

(58) Let us consider a non empty set $X$, and filters $\mathcal{F}_1$, $\mathcal{F}_2$ of $X$. Suppose for every element $M_1$ of $\mathcal{F}_1$ for every element $M_2$ of $\mathcal{F}_2$, $M_1 \cap M_2$ is not empty. Then there exists a filter $\mathcal{F}'$ of $X$ such that

(i) $\mathcal{F}'$ is finer than $\mathcal{F}_1$, and

(ii) $\mathcal{F}'$ is finer than $\mathcal{F}_2$.

Let $X$ be a set and $x$ be a subset of $X$. The functor SubsetToBooleSubset $x$ yielding an element of $2^X_\subseteq$ is defined by the term

(Def. 19) $x$.

Now we state the propositions:

(59) Let us consider an infinite set $X$. Then $X \in$ the set of all $X \setminus A$ where $A$ is a finite subset of $X$.

(60) Let us consider a set $X$, and a subset $A$ of $X$. Then $\{B$, where $B$ is an element of $2^X_\subseteq : A \subseteq B\} = \{B$, where $B$ is a subset of $X : A \subseteq B\}$.

(61) Let us consider a set $X$, and an element $a$ of $2^X_\subseteq$. Then $\uparrow a = \{Y$, where $Y$ is a subset of $X : a \subseteq Y\}$.

(62) Let us consider a set $X$, and a subset $A$ of $X$. Then $\{B$, where $B$ is an element of $2^X_\subseteq : A \subseteq B\} = \uparrow$SubsetToBooleSubset $A$. The theorem is a consequence of (60).

(63) Let us consider a non empty set $X$, and a filter $\mathcal{F}'$ of $X$. Then $\bigcup \mathcal{F}' = X$.

(64) Let us consider an infinite set $X$. Then the set of all $X \setminus A$ where $A$ is a finite subset of $X$ is a filter of $X$. The theorem is a consequence of (59).

Let us consider a set $X$. Now we state the propositions:

(65) $2^X$ is a filter of $2^X_\subseteq$.

(66) $\{X\}$ is a filter of $2^X_\subseteq$.

(67) Let us consider a non empty set $X$. Then $\{X\}$ is a filter of $X$.

Let us consider an element $A$ of $2^X_\subseteq$. Now we state the propositions:

(68) $\{Y$, where $Y$ is a subset of $X : A \subseteq Y\}$ is a filter of $2^X_\subseteq$.

(69) $\{B$, where $B$ is an element of $2^X_\subseteq : A \subseteq B\}$ is a filter of $2^X_\subseteq$. The theorem is a consequence of (60) and (68).

Now we state the proposition:

(70) Let us consider a non empty set $X$, and a non empty subset $B$ of $2^X_\subseteq$. Then for every elements $x$, $y$ of $B$, there exists an element $z$ of $B$ such that $z \subseteq x \cap y$ if and only if $B$ is filtered.

PROOF: For every elements $x$, $y$ of $B$, there exists an element $z$ of $B$ such that $z \subseteq x \cap y$ by [19, (2)]. $\square$

Let us consider a non empty set $X$ and a non empty subset $\mathcal{F}'$ of the lattice of subsets of $X$. Now we state the propositions:

(71) $\mathcal{F}'$ is a filter of the lattice of subsets of $X$ if and only if for every elements $p$, $q$ of $\mathcal{F}'$, $p \cap q \in \mathcal{F}'$ and for every element $p$ of $\mathcal{F}'$ and for every element $q$ of the lattice of subsets of $X$ such that $p \subseteq q$ holds $q \in \mathcal{F}'$.

(72) $\mathcal{F}'$ is a filter of the lattice of subsets of $X$ if and only if for every subsets $Y_1$, $Y_2$ of $X$, if $Y_1, Y_2 \in \mathcal{F}'$, then $Y_1 \cap Y_2 \in \mathcal{F}'$ and if $Y_1 \in \mathcal{F}'$ and $Y_1 \subseteq Y_2$, then $Y_2 \in \mathcal{F}'$. The theorem is a consequence of (71).

Now we state the propositions:

(73) Let us consider a non empty set $X$, and a non empty family $\mathcal{F}$ of subsets of $X$. Suppose $\mathcal{F}$ is a filter of the lattice of subsets of $X$. Then $\mathcal{F}$ is a filter of $2^X_\subseteq$. The theorem is a consequence of (71).

(74) Let us consider a non empty set $X$. Then every filter of $2^X_\subseteq$ is a filter of the lattice of subsets of $X$. The theorem is a consequence of (72).

(75) Let us consider a non empty set $X$, and a non empty subset $\mathcal{F}'$ of the lattice of subsets of $X$. Then $\mathcal{F}'$ is filter of the lattice of subsets of $X$ and has non empty elements if and only if $\mathcal{F}'$ is a filter of $X$. The theorem is a consequence of (72).

(76) Let us consider a non empty set $X$. Then every proper filter of $2^X_\subseteq$ is a filter of $X$.

PROOF: $\mathcal{F}'$ has non empty elements by [19, (18)], [7, (4)]. $\square$

(77) Let us consider a non empty topological space $T$, and a point $x$ of $T$. Then the neighborhood system of $x$ is a filter of the carrier of $T$.

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a proper filter of $2_{\subseteq}^{\Omega_T}$. The functor BooleanFilterToFilter($\mathcal{F}'$) yielding a filter of the carrier of $T$ is defined by the term

(Def. 20) $\mathcal{F}'$.

Let $\mathcal{F}_1$ be a filter of the carrier of $T$ and $\mathcal{F}_2$ be a proper filter of $2_{\subseteq}^{\Omega_T}$. We say that $\mathcal{F}_1$ is finer than $\mathcal{F}_2$ if and only if

(Def. 21) BooleanFilterToFilter($\mathcal{F}_2$) $\subseteq \mathcal{F}_1$.

## 3. Limit of a Filter

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. The functor LimFilter($\mathcal{F}'$) yielding a subset of $T$ is defined by the term

(Def. 22) $\{x$, where $x$ is a point of $T$ : $\mathcal{F}'$ is finer than the neighborhood system of $x\}$.

Let $B$ be a filter base of the carrier of $T$. The functor Lim $B$ yielding a subset of $T$ is defined by the term

(Def. 23) LimFilter($[B)$).

Now we state the proposition:

(78) Let us consider a non empty topological space $T$, and a filter $\mathcal{F}'$ of the carrier of $T$. Then there exists a proper filter $\mathcal{F}_1$ of $2_{\subseteq}^{\alpha}$ such that $\mathcal{F}' = \mathcal{F}_1$, where $\alpha$ is the carrier of $T$. The theorem is a consequence of (73) and (75).

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. The functor FilterToBooleanFilter($\mathcal{F}', T$) yielding a proper filter of $2_{\subseteq}^{\Omega_T}$ is defined by the term

(Def. 24) $\mathcal{F}'$.

Let us consider a non empty topological space $T$, a point $x$ of $T$, and a filter $\mathcal{F}'$ of the carrier of $T$. Now we state the propositions:

(79) $x$ is a convergence point of $\mathcal{F}'$ and $T$ if and only if $x$ is a convergence point of FilterToBooleanFilter($\mathcal{F}', T$) and $T$.

(80) $x$ is a convergence point of $\mathcal{F}'$ and $T$ if and only if $x \in$ LimFilter($\mathcal{F}'$). The theorem is a consequence of (78).

Let $T$ be a non empty topological space and $\mathcal{F}'$ be a filter of $2_{\subseteq}^{\Omega_T}$. The functor LimFilterB($\mathcal{F}'$) yielding a subset of $T$ is defined by the term

(Def. 25) $\{x$, where $x$ is a point of $T$ : the neighborhood system of $x \subseteq \mathcal{F}'\}$.

Let us consider a non empty topological space $T$ and a filter $\mathcal{F}'$ of the carrier of $T$. Now we state the propositions:

(81) $\operatorname{LimFilter}(\mathcal{F}') = \operatorname{LimFilterB}(\operatorname{FilterToBooleanFilter}(\mathcal{F}', T))$.

(82) Lim(the net of FilterToBooleanFilter($\mathcal{F}'$, $T$)) = LimFilter($\mathcal{F}'$).

(83) Let us consider a Hausdorff, non empty topological space $T$, a filter $\mathcal{F}'$ of the carrier of $T$, and points $p$, $q$ of $T$. If $p, q \in \operatorname{LimFilter}(\mathcal{F}')$, then $p = q$.

Let $T$ be a Hausdorff, non empty topological space and $\mathcal{F}'$ be a filter of the carrier of $T$. Note that LimFilter($\mathcal{F}'$) is trivial.

Let $X$ be a non empty set, $T$ be a non empty topological space, $f$ be a function from $X$ into the carrier of $T$, and $\mathcal{F}'$ be a filter of $X$. The functor $\lim_{\mathcal{F}'} f$ yielding a subset of $\Omega_T$ is defined by the term

(Def. 26) LimFilter(the image of filter $\mathcal{F}'$ under $f$).

Let $L$ be a non empty, transitive, reflexive relational structure and $f$ be a function from $\Omega_L$ into the carrier of $T$. The functor LimF($f$) yielding a subset of $\Omega_T$ is defined by the term

(Def. 27) LimFilter(the image of filter TailsFilter $L$ under $f$).

Now we state the proposition:

(84) Let us consider a non empty topological space $T$, a non empty, transitive, reflexive relational structure $L$, a function $f$ from $\Omega_L$ into the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Suppose $\Omega_L$ is directed. Then $x \in \operatorname{LimF}(f)$ if and only if for every element $b$ of $B$, there exists an element $i$ of $L$ such that for every element $j$ of $L$ such that $i \leqslant j$ holds $f(j) \in b$. The theorem is a consequence of (46), (29), and (47).

Let $T$ be a non empty topological space and $s$ be a sequence of $T$. The functor LimF($s$) yielding a subset of $T$ is defined by the term

(Def. 28) LimFilter(the elementary filter of $s$).

Now we state the proposition:

(85) Let us consider a non empty topological space $T$, and a sequence $s$ of $T$. Then $\lim_{\mathrm{FrechetFilter}(\mathbb{N})} s = \operatorname{LimF}(s)$.

Let us consider a non empty topological space $T$ and a point $x$ of $T$.

(86) The neighborhood system of $x$ is a filter base of $\Omega_T$. The theorem is a consequence of (76), (13), and (29).

(87) Every generalized basis of BooleanFilterToFilter(the neighborhood system of $x$) is a filter base of $\Omega_T$.

(88) Let us consider a non empty set $X$, a sequence $s$ of $X$, and a filter base $B$ of $X$. Then $B$ is coarser than $s^\circ$(the base of Frechet filter) if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$.

(89) Let us consider a non empty topological space $T$, a sequence $s$ of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Then $x \in \lim_{\text{FrechetFilter}(\mathbb{N})} s$ if and only if $B$ is coarser than $s^\circ$(the base of Frechet filter). The theorem is a consequence of (46) and (54).

(90) Let us consider a non empty topological space $T$, a sequence $s$ of $\Omega_T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Then $B$ is coarser than $s^\circ$(the base of Frechet filter) if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (29) and (47).

Let us consider a non empty topological space $T$, a sequence $s$ of the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$).

(91) $x \in \lim_{\text{FrechetFilter}(\mathbb{N})} s$ if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (89) and (90).

(92) $x \in \text{LimF}(s)$ if and only if for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (91).

## 4. Nets

Let $L$ be a 1-sorted structure and $s$ be a sequence of the carrier of $L$. The net of $s$ yielding a non empty, strict net structure over $L$ is defined by the term

(Def. 29) $\langle \mathbb{N}, \leqslant_{\mathbb{N}}, s \rangle$.

Let $L$ be a non empty 1-sorted structure. Let us note that the net of $s$ is non empty.

Now we state the proposition:

(93) Let us consider a non empty 1-sorted structure $L$, a set $B$, and a sequence $s$ of the carrier of $L$. Then the net of $s$ is eventually in $B$ if and only if there exists an element $i$ of the net of $s$ such that for every element $j$ of the net of $s$ such that $i \leqslant j$ holds (the net of $s$)$(j) \in B$.

Let us consider a non empty topological space $T$, a sequence $s$ of the carrier of $T$, a point $x$ of $T$, and a generalized basis $B$ of BooleanFilterToFilter(the neighborhood system of $x$). Now we state the propositions:

(94) for every element $b$ of $B$, there exists an element $i$ of the ordered $\mathbb{N}$ such that for every element $j$ of the ordered $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$ if

and only if for every element $b$ of $B$, there exists an element $i$ of the net of $s$ such that for every element $j$ of the net of $s$ such that $i \leqslant j$ holds (the net of $s$)$(j) \in b$.

(95) $x \in \mathrm{LimF}(s)$ if and only if for every element $b$ of $B$, the net of $s$ is eventually in $b$. The theorem is a consequence of (92), (94), and (93).

(96) $x \in \mathrm{LimF}(s)$ if and only if for every element $b$ of $B$, there exists an element $i$ of $\mathbb{N}$ such that for every element $j$ of $\mathbb{N}$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (91).

(97) $x \in \mathrm{LimF}(s)$ if and only if for every element $b$ of $B$, there exists a natural number $i$ such that for every natural number $j$ such that $i \leqslant j$ holds $s(j) \in b$. The theorem is a consequence of (96).

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