

Polynomially Bounded Sequences and Polynomial Sequences

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Summary. In this article, we formalize polynomially bounded sequences that plays an important role in computational complexity theory. Class P is a fundamental computational complexity class that contains all polynomial-time decision problems [11], [12]. It takes polynomially bounded amount of computation time to solve polynomial-time decision problems by the deterministic Turing machine. Moreover we formalize polynomial sequences [5].

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The notation and terminology used in this paper have been introduced in the following articles: [26], [18], [16], [17], [6], [22], [10], [7], [8], [24], [14], [1], [2], [3], [13], [20], [27], [28], [21], [25], and [9].

1. PRELIMINARIES

Now we state the proposition:

- (1) Let us consider natural numbers m, k . If $1 \leq m$, then $1 \leq m^k$.

Let us consider natural numbers m, n . Now we state the propositions:

- (2) $m \leq m^{n+1}$.
(3) If $2 \leq m$, then $n + 1 \leq m^n$.

(4) Let us consider a natural number k . Then $2 \cdot k \leq 2^k$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2 \cdot k \leq 2^k$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (25)], [24, (5)], [1, (14)], (2). For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(5) Let us consider natural numbers k, n . If $k \leq n$, then $n+k \leq 2^n$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv k + k \leq 2^{k+k}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (27), (25), (24)]. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(6) Let us consider natural numbers k, m . If $2 \cdot k + 1 \leq m$, then $2^k \leq 2^m/m$. The theorem is a consequence of (5).

(7) Let us consider real numbers a, b, c . If $1 < a$ and $0 < b \leq c$, then $\log_a b \leq \log_a c$.

Let us consider a natural number n and a real number a . Now we state the propositions:

(8) If $1 < a$, then $a^n < a^{n+1}$.

(9) If $1 \leq a$, then $a^n \leq a^{n+1}$.

(10) There exists a partial function g from \mathbb{R} to \mathbb{R} such that

(i) $\text{dom } g =]0, +\infty[$, and

(ii) for every real number x such that $x \in]0, +\infty[$ holds $g(x) = \log_2 x$, and

(iii) g is differentiable on $]0, +\infty[$, and

(iv) for every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and $0 < g'(x)$.

PROOF: Set $g = \log_2 e \cdot (\text{the function } \ln)$. For every real number d such that $d \in]0, +\infty[$ holds $g(d) = \log_2 d$ by [20, (56)]. For every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and $0 < g'(x)$ by [23, (18)], [22, (15)], [20, (57)], [23, (11)]. \square

(11) There exists a partial function f from \mathbb{R} to \mathbb{R} such that

(i) $]e, +\infty[= \text{dom } f$, and

(ii) for every real number x such that $x \in \text{dom } f$ holds $f(x) = x/\log_2 x$, and

(iii) f is differentiable on $]e, +\infty[$, and

(iv) for every real number x_0 such that $x_0 \in]e, +\infty[$ holds $0 \leq f'(x_0)$, and

(v) f is non-decreasing.

PROOF: Consider g being a partial function from \mathbb{R} to \mathbb{R} such that $\text{dom } g =]0, +\infty[$ and for every real number x such that $x \in]0, +\infty[$ holds $g(x) = \log_2 x$ and g is differentiable on $]0, +\infty[$ and for every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and $0 < g'(x)$. Set $g_0 = g]e, +\infty[$. For every object x such that $x \in]e, +\infty[$ holds $x \in]0, +\infty[$ by [23, (11)]. Set $f = \text{id}_{\Omega_{\mathbb{R}}/g_0} \cdot g_0^{-1}(\{0\}) = \emptyset$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in \text{dom } f$ holds $f(x) = x/\log_2 x$ by [7, (49)]. For every real number x such that $x \in]e, +\infty[$ holds f is differentiable in x and $f'(x) = \log_2 x - \log_2 e/(\log_2 x)^2$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in]e, +\infty[$ holds $0 \leq f'(x)$ by [20, (57)], [23, (11)]. \square

- (12) Let us consider real numbers x, y . If $e < x \leq y$, then $x/\log_2 x \leq y/\log_2 y$. The theorem is a consequence of (11).
- (13) Let us consider a natural number k . Suppose $e < k$. Then there exists a natural number N such that for every natural number n such that $N \leq n$ holds $2^k \leq n/\log_2 n$. The theorem is a consequence of (12) and (6).

Let us consider a natural number x . Let us assume that $1 < x$.

- (14) There exists a natural number N such that for every natural number n such that $N \leq n$ holds $4 < n/\log_x n$.
- (15) There exist natural numbers N, c such that for every natural number n such that $N \leq n$ holds $n^x \leq c \cdot x^n$.
- (16) Let us consider a natural number x . Suppose $1 < x$. Then there exist no natural numbers N, c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$.

PROOF: Consider N being a natural number such that there exists a natural number c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$. $N \neq 0$ by [20, (42), (24)]. Consider c being a natural number such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$. There exists an element n of \mathbb{N} such that $N \leq n$ and $0 < n - (x/4)$ by [24, (6), (3)]. Consider n being an element of \mathbb{N} such that $N \leq n$ and $0 < n - (x/4)$. $0 < c$ by [20, (34)]. For every natural number k such that $1 \leq k$ holds $2^{k \cdot n} \leq c \cdot (k \cdot n)^x$. For every natural number k such that $1 \leq k$ holds $k \cdot n \leq \log_2 c + x \cdot \log_2 k + x \cdot \log_2 n$ by [20, (34)], (7), [20, (55), (52), (53)]. Consider Z being an element of \mathbb{N} such that for every natural number k such that $Z \leq k$ holds $4 < k/\log_2 k$. There exists a natural number k such that $Z \leq k$ and $\log_2 c + x \cdot \log_2 n/n - (x/4) \leq k$ by [24, (6), (3)]. There exists a natural number k such that $Z \leq k$ and $\log_2 c + x \cdot \log_2 n/n - (x/4) \leq k$ and $1 < k$ by [1, (11)]. Consider k being a natural number such that $Z \leq k$ and $1 < k$ and $\log_2 c + x \cdot \log_2 n/n - (x/4) \leq k$. \square

(17) Let us consider natural numbers a, b . If $a \leq b$, then $\{n^a\}_{n \in \mathbb{N}} \in O(\{n^b\}_{n \in \mathbb{N}})$.

(18) Let us consider a natural number x . Suppose $1 < x$. Then there exist no natural numbers N, c such that for every natural number n such that $N \leq n$ holds $x^n \leq c \cdot n^x$.

PROOF: There exist natural numbers N, c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$ by [24, (7)]. \square

(19) Let us consider a non negative real number a , and a natural number n . If $1 \leq n$, then $0 < \{n^a\}_{n \in \mathbb{N}}(n)$.

2. POLYNOMIALLY BOUNDED SEQUENCES

Let p be a sequence of real numbers. We say that p is polynomially bounded if and only if

(Def. 1) there exists a natural number k such that $p \in O(\{n^k\}_{n \in \mathbb{N}})$.

Now we state the propositions:

(20) Let us consider a sequence f of real numbers. Suppose f is not polynomially bounded. Let us consider a natural number k . Then $f \notin O(\{n^k\}_{n \in \mathbb{N}})$.

(21) Let us consider a sequence f of real numbers. Suppose for every natural number k , $f \notin O(\{n^k\}_{n \in \mathbb{N}})$. Then f is not polynomially bounded.

(22) Let us consider a positive real number a . Then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is positive.

Let us consider a real number a . Now we state the propositions:

(23) If $1 \leq a$, then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is non-decreasing. The theorem is a consequence of (9).

(24) If $1 < a$, then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is increasing. The theorem is a consequence of (8).

(25) Let us consider a natural number a . If $1 < a$, then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is not polynomially bounded.

PROOF: Consider k being a natural number such that $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}} \in O(\{n^k\}_{n \in \mathbb{N}})$. Reconsider $f = \{n^k\}_{n \in \mathbb{N}}$ as an eventually positive sequence of real numbers. Reconsider $t = \{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ as an eventually non-negative sequence of real numbers. $t \in O(f)$ and for every element n of \mathbb{N} such that $1 \leq n$ holds $0 < f(n)$. Consider c being a real number such that $c > 0$ and for every element n of \mathbb{N} such that $n \geq 1$ holds $(\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}})(n) \leq c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$. For every natural number n such that $n \geq 1$ holds $2^n \leq c \cdot n^k$ by [24, (7)]. There exist natural numbers N, b such that for every natural number n such that $N \leq n$ holds $2^n \leq b \cdot n^k$ by [24, (3)]. \square

3. POLYNOMIAL SEQUENCES

Now we state the proposition:

(26) Let us consider a finite 0-sequence x of \mathbb{R} , and a sequence y of real numbers. Then

(i) $x \cdot y$ is a finite transfinite sequence of elements of \mathbb{R} , and

(ii) $\text{dom}(x \cdot y) = \text{dom } x$, and

(iii) for every object i such that $i \in \text{dom } x$ holds $(x \cdot y)(i) = x(i) \cdot y(i)$.

Let x be a finite 0-sequence of \mathbb{R} and y be a sequence of real numbers. Observe that the functor $x \cdot y$ yields a finite 0-sequence of \mathbb{R} . Now we state the proposition:

(27) Let us consider a finite 0-sequence d of \mathbb{R} , and natural numbers x, i . Suppose $i \in \text{dom } d$. Then $(d \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})(i) = d(i) \cdot x^i$. The theorem is a consequence of (26).

Let c be a finite 0-sequence of \mathbb{R} . The functor $\text{Seq}_{\text{poly}}(c)$ yielding a sequence of real numbers is defined by

(Def. 2) for every natural number x , $it(x) = \sum(c \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})$.

Let us consider a finite 0-sequence d of \mathbb{R} and a natural number k . Now we state the propositions:

(28) Suppose $\text{len } d = k + 1$. Then there exists a real number a and there exists a finite 0-sequence d_1 of \mathbb{R} and there exists a sequence y of real numbers such that $\text{len } d_1 = k$ and $d_1 = d \upharpoonright k$ and $a = d(k)$ and $d = d_1 \hat{\ } \langle a \rangle$ and $\text{Seq}_{\text{poly}}(d) = \text{Seq}_{\text{poly}}(d_1) + y$ and for every natural number i , $y(i) = a \cdot i^k$. PROOF: Consider a being a real number, d_1 being a finite 0-sequence of \mathbb{R} such that $\text{len } d_1 = k$ and $d_1 = d \upharpoonright k$ and $a = d(k)$ and $d = d_1 \hat{\ } \langle a \rangle$. Define $\mathcal{F}(\text{natural number}) = a \cdot i^k$. Consider y being a sequence of real numbers such that for every natural number x , $y(x) = \mathcal{F}(x)$ from [15, Sch. 1]. For every element x of \mathbb{N} , $(\text{Seq}_{\text{poly}}(d))(x) = (\text{Seq}_{\text{poly}}(d_1) + y)(x)$ by (26), [1, (13), (44)], (27). \square

(29) If $\text{len } d = 1$, then there exists a real number a such that $a = d(0)$ and for every natural number x , $(\text{Seq}_{\text{poly}}(d))(x) = a$. The theorem is a consequence of (26).

(30) If $\text{len } d = 1$ and d is non-negative yielding, then $\text{Seq}_{\text{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}})$. The theorem is a consequence of (29).

(31) Let us consider a natural number k , a real number a , and a sequence y of real numbers. Suppose $0 \leq a$ and for every natural number i , $y(i) = a \cdot i^k$. Then $y \in O(\{n^k\}_{n \in \mathbb{N}})$.

(32) Let us consider natural numbers k, n . If $k \leq n$, then $O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^n\}_{n \in \mathbb{N}})$.

PROOF: Consider i being a natural number such that $n = k + i$. Define $\mathcal{P}[\text{natural number}] \equiv O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^{(k+\$1)}\}_{n \in \mathbb{N}})$. For every natural number x such that $\mathcal{P}[x]$ holds $\mathcal{P}[x + 1]$. For every natural number x , $\mathcal{P}[x]$ from [1, Sch. 2]. \square

(33) Let us consider a natural number k , and a non-negative yielding finite 0-sequence c of \mathbb{R} . Suppose $\text{len } c = k + 1$. Then $\text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non-negative yielding finite 0-sequence c of \mathbb{R} such that $\text{len } c = \$1 + 1$ holds $\text{Seq}_{\text{poly}}(c) \in O(\{n^{\$1}\}_{n \in \mathbb{N}})$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by (28), [7, (47)], [1, (13), (39)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

(34) Let us consider a natural number k , and a finite 0-sequence c of \mathbb{R} . Then there exists a finite 0-sequence d of \mathbb{R} such that

- (i) $\text{len } d = \text{len } c$, and
- (ii) for every natural number i such that $i \in \text{dom } d$ holds $d(i) = |c(i)|$.

PROOF: Define $\mathcal{F}(\text{natural number}) = |c(\$1)| (\in \mathbb{R})$. Consider d being a finite 0-sequence of \mathbb{R} such that $\text{len } d = \text{len } c$ and for every natural number j such that $j \in \text{len } c$ holds $d(j) = \mathcal{F}(j)$ from [18, Sch. 1]. \square

(35) Let us consider a finite 0-sequence c of \mathbb{R} , and a finite 0-sequence d of \mathbb{R} . Suppose $\text{len } d = \text{len } c$ and for every natural number i such that $i \in \text{dom } d$ holds $d(i) = |c(i)|$. Let us consider a natural number n . Then $(\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(d))(n)$.

PROOF: $\text{dom}(d \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}}) = \text{dom } d$. For every natural number i such that $i \in \text{dom}(c \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})$ holds $(c \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})(i) \leq (d \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})(i)$ by (26), (27), [19, (4)]. \square

(36) Let us consider a natural number k , and a finite 0-sequence c of \mathbb{R} . Suppose $\text{len } c = k + 1$ and $\text{Seq}_{\text{poly}}(c)$ is eventually nonnegative. Then $\text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.

PROOF: Consider d being a finite 0-sequence of \mathbb{R} such that $\text{len } d = \text{len } c$ and for every natural number i such that $i \in \text{dom } d$ holds $d(i) = |c(i)|$. For every natural number i such that $i \in \text{dom } d$ holds $0 \leq d(i)$ by [6, (46)]. For every real number r such that $r \in \text{rng } d$ holds $0 \leq r$. $\text{Seq}_{\text{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}})$. Consider t being an element of $\mathbb{R}^{\mathbb{N}}$ such that $\text{Seq}_{\text{poly}}(d) = t$ and there exists a real number c and there exists an element N of \mathbb{N} such that $c > 0$ and for every element n of \mathbb{N} such that $n \geq N$ holds $t(n) \leq c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geq 0$. Consider N_1 being a natural number such that for every natural number n such that $N_1 \leq n$ holds $0 \leq (\text{Seq}_{\text{poly}}(c))(n)$.

Consider a being a real number, N_2 being an element of \mathbb{N} such that $a > 0$ and for every element n of \mathbb{N} such that $n \geq N_2$ holds $t(n) \leq a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geq 0$. Set $N = N_1 + N_2$. For every element n of \mathbb{N} such that $n \geq N$ holds $(\text{Seq}_{\text{poly}}(c))(n) \leq a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $(\text{Seq}_{\text{poly}}(c))(n) \geq 0$ by [1, (11)], (35). \square

(37) Let us consider natural numbers k, n . If $0 < n$, then $n \cdot \{n^k\}_{n \in \mathbb{N}}(n) = \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$.

(38) Let us consider a finite 0-sequence c of \mathbb{R} . Suppose $\text{len } c = 0$. Let us consider a natural number x . Then $(\text{Seq}_{\text{poly}}(c))(x) = 0$.

(39) Let us consider an eventually nonnegative sequence f of real numbers, and a natural number k . Suppose $f \in O(\{n^k\}_{n \in \mathbb{N}})$. Then there exists a natural number N such that for every natural number n such that $N \leq n$ holds $f(n) \leq \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$. The theorem is a consequence of (37).

(40) Let us consider a finite 0-sequence c of \mathbb{R} . Then there exists a finite 0-sequence a_1 of \mathbb{R} such that

(i) $a_1 = |c|$, and

(ii) for every natural number n , $(\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(a_1))(n)$.

PROOF: Reconsider $a_1 = |c|$ as a finite 0-sequence of \mathbb{R} . Set $m_1 = c \cdot \{n^{1-n+0}\}_{n \in \mathbb{N}}$. Set $m_2 = a_1 \cdot \{n^{1-n+0}\}_{n \in \mathbb{N}}$. For every natural number x such that $x \in \text{dom } m_1$ holds $m_1(x) \leq m_2(x)$ by [19, (4)]. \square

(41) Let us consider finite 0-sequences c, a_1 of \mathbb{R} . Suppose $a_1 = |c|$. Let us consider a natural number n . Then $|(\text{Seq}_{\text{poly}}(c))(n)| \leq (\text{Seq}_{\text{poly}}(a_1))(n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite 0-sequences c, a_1 of \mathbb{R} such that $\text{len } c = \$_1$ and $a_1 = |c|$ for every natural number x , $|(\text{Seq}_{\text{poly}}(c))(x)| \leq (\text{Seq}_{\text{poly}}(a_1))(x)$. $\mathcal{P}[0]$ by (26), [6, (44)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by (28), [7, (47)], [15, (7)], [6, (56), (65)]. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(42) Let us consider a real number a . Suppose $0 < a$. Let us consider a natural number k , and a non-negative yielding finite 0-sequence d of \mathbb{R} . Suppose $\text{len } d = k$. Then there exists a natural number N such that for every natural number x such that $N \leq x$ for every natural number i such that $i \in \text{dom } d$ holds $d(i) \cdot x^i \cdot k \leq a \cdot x^k$.

PROOF: For every natural number i such that $i \in \text{dom } d$ holds $0 \leq d(i)$ by [7, (3)]. \square

(43) Let us consider a natural number k , a finite 0-sequence d of \mathbb{R} , a real number a , and a sequence y of real numbers. Suppose $0 < a$ and $\text{len } d = k$ and for every natural number x , $y(x) = a \cdot x^k$. Then there exists a natural number N such that for every natural number x such that $N \leq x$ holds

$|(\text{Seq}_{\text{poly}}(d))(x)| \leq y(x)$. The theorem is a consequence of (38), (42), (26), (27), and (41).

- (44) Let us consider a natural number k , and a finite 0-sequence d of \mathbb{R} . Suppose $\text{len } d = k+1$ and $0 < d(k)$. Then $\text{Seq}_{\text{poly}}(d)$ is eventually nonnegative. PROOF: Consider a being a real number, d_1 being a finite 0-sequence of \mathbb{R} , y being a sequence of real numbers such that $\text{len } d_1 = k$ and $d_1 = d \upharpoonright k$ and $a = d(k)$ and $d = d_1 \hat{\ } \langle a \rangle$ and $\text{Seq}_{\text{poly}}(d) = \text{Seq}_{\text{poly}}(d_1) + y$ and for every natural number i , $y(i) = a \cdot i^k$. Consider N being a natural number such that for every natural number i such that $N \leq i$ holds $|(\text{Seq}_{\text{poly}}(d_1))(i)| \leq y(i)$. For every natural number i such that $N \leq i$ holds $0 \leq (\text{Seq}_{\text{poly}}(d))(i)$ by [19, (4)], [15, (7)]. \square

Let us consider a natural number k and a finite 0-sequence c of \mathbb{R} .

Let us assume that $\text{len } c = k+1$ and $0 < c(k)$. Now we state the propositions:

- (45) $\text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.
 (46) $\text{Seq}_{\text{poly}}(c)$ is polynomially bounded. The theorem is a consequence of (36) and (44).

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