

Polynomially Bounded Sequences and Polynomial Sequences

Hiroyuki Okazaki
Shinshu University
Nagano, Japan

Yuichi Futa
Japan Advanced Institute
of Science and Technology
Ishikawa, Japan

Summary. In this article, we formalize polynomially bounded sequences that plays an important role in computational complexity theory. Class P is a fundamental computational complexity class that contains all polynomial-time decision problems [11], [12]. It takes polynomially bounded amount of computation time to solve polynomial-time decision problems by the deterministic Turing machine. Moreover we formalize polynomial sequences [5].

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The notation and terminology used in this paper have been introduced in the following articles: [26], [18], [16], [17], [6], [22], [10], [7], [8], [24], [14], [1], [2], [3], [13], [20], [27], [28], [21], [25], and [9].

1. PRELIMINARIES

Now we state the proposition:

- (1) Let us consider natural numbers m, k . If $1 \leq m$, then $1 \leq m^k$.

Let us consider natural numbers m, n . Now we state the propositions:

- (2) $m \leq m^{n+1}$.
(3) If $2 \leq m$, then $n + 1 \leq m^n$.

(4) Let us consider a natural number k . Then $2 \cdot k \leq 2^k$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv 2 \cdot \mathbb{N}_1 \leq 2^{\mathbb{N}_1}$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (25)], [24, (5)], [1, (14)], (2). For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(5) Let us consider natural numbers k, n . If $k \leq n$, then $n+k \leq 2^n$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathbb{N}_1 + k + k \leq 2^{\mathbb{N}_1+k}$. $2 \cdot k \leq 2^k$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n+1]$ by [20, (27), (25), (24)]. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(6) Let us consider natural numbers k, m . If $2 \cdot k + 1 \leq m$, then $2^k \leq 2^m/m$. The theorem is a consequence of (5).

(7) Let us consider real numbers a, b, c . If $1 < a$ and $0 < b \leq c$, then $\log_a b \leq \log_a c$.

Let us consider a natural number n and a real number a . Now we state the propositions:

(8) If $1 < a$, then $a^n < a^{n+1}$.

(9) If $1 \leq a$, then $a^n \leq a^{n+1}$.

(10) There exists a partial function g from \mathbb{R} to \mathbb{R} such that

(i) $\text{dom } g =]0, +\infty[$, and

(ii) for every real number x such that $x \in]0, +\infty[$ holds $g(x) = \log_2 x$, and

(iii) g is differentiable on $]0, +\infty[$, and

(iv) for every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and $0 < g'(x)$.

PROOF: Set $g = \log_2 e \cdot (\text{the function } \ln)$. For every real number d such that $d \in]0, +\infty[$ holds $g(d) = \log_2 d$ by [20, (56)]. For every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and $0 < g'(x)$ by [23, (18)], [22, (15)], [20, (57)], [23, (11)]. \square

(11) There exists a partial function f from \mathbb{R} to \mathbb{R} such that

(i) $]e, +\infty[= \text{dom } f$, and

(ii) for every real number x such that $x \in \text{dom } f$ holds $f(x) = x/\log_2 x$, and

(iii) f is differentiable on $]e, +\infty[$, and

(iv) for every real number x_0 such that $x_0 \in]e, +\infty[$ holds $0 \leq f'(x_0)$, and

(v) f is non-decreasing.

PROOF: Consider g being a partial function from \mathbb{R} to \mathbb{R} such that $\text{dom } g =]0, +\infty[$ and for every real number x such that $x \in]0, +\infty[$ holds $g(x) = \log_2 x$ and g is differentiable on $]0, +\infty[$ and for every real number x such that $x \in]0, +\infty[$ holds g is differentiable in x and $g'(x) = \log_2 e/x$ and $0 < g'(x)$. Set $g_0 = g]e, +\infty[$. For every object x such that $x \in]e, +\infty[$ holds $x \in]0, +\infty[$ by [23, (11)]. Set $f = \text{id}_{\Omega_{\mathbb{R}}/g_0} \cdot g_0^{-1}(\{0\}) = \emptyset$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in \text{dom } f$ holds $f(x) = x/\log_2 x$ by [7, (49)]. For every real number x such that $x \in]e, +\infty[$ holds f is differentiable in x and $f'(x) = \log_2 x - \log_2 e/(\log_2 x)^2$ by [23, (11)], [7, (49)], [4, (10)], [20, (52)]. For every real number x such that $x \in]e, +\infty[$ holds $0 \leq f'(x)$ by [20, (57)], [23, (11)]. \square

- (12) Let us consider real numbers x, y . If $e < x \leq y$, then $x/\log_2 x \leq y/\log_2 y$. The theorem is a consequence of (11).
- (13) Let us consider a natural number k . Suppose $e < k$. Then there exists a natural number N such that for every natural number n such that $N \leq n$ holds $2^k \leq n/\log_2 n$. The theorem is a consequence of (12) and (6).

Let us consider a natural number x . Let us assume that $1 < x$.

- (14) There exists a natural number N such that for every natural number n such that $N \leq n$ holds $4 < n/\log_x n$.
- (15) There exist natural numbers N, c such that for every natural number n such that $N \leq n$ holds $n^x \leq c \cdot x^n$.
- (16) Let us consider a natural number x . Suppose $1 < x$. Then there exist no natural numbers N, c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$.

PROOF: Consider N being a natural number such that there exists a natural number c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$. $N \neq 0$ by [20, (42), (24)]. Consider c being a natural number such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$. There exists an element n of \mathbb{N} such that $N \leq n$ and $0 < n - (x/4)$ by [24, (6), (3)]. Consider n being an element of \mathbb{N} such that $N \leq n$ and $0 < n - (x/4)$. $0 < c$ by [20, (34)]. For every natural number k such that $1 \leq k$ holds $2^{k \cdot n} \leq c \cdot (k \cdot n)^x$. For every natural number k such that $1 \leq k$ holds $k \cdot n \leq \log_2 c + x \cdot \log_2 k + x \cdot \log_2 n$ by [20, (34)], (7), [20, (55), (52), (53)]. Consider Z being an element of \mathbb{N} such that for every natural number k such that $Z \leq k$ holds $4 < k/\log_2 k$. There exists a natural number k such that $Z \leq k$ and $\log_2 c + x \cdot \log_2 n/n - (x/4) \leq k$ by [24, (6), (3)]. There exists a natural number k such that $Z \leq k$ and $\log_2 c + x \cdot \log_2 n/n - (x/4) \leq k$ and $1 < k$ by [1, (11)]. Consider k being a natural number such that $Z \leq k$ and $1 < k$ and $\log_2 c + x \cdot \log_2 n/n - (x/4) \leq k$. \square

- (17) Let us consider natural numbers a, b . If $a \leq b$, then $\{n^a\}_{n \in \mathbb{N}} \in O(\{n^b\}_{n \in \mathbb{N}})$.
- (18) Let us consider a natural number x . Suppose $1 < x$. Then there exist no natural numbers N, c such that for every natural number n such that $N \leq n$ holds $x^n \leq c \cdot n^x$.
- PROOF: There exist natural numbers N, c such that for every natural number n such that $N \leq n$ holds $2^n \leq c \cdot n^x$ by [24, (7)]. \square
- (19) Let us consider a non negative real number a , and a natural number n . If $1 \leq n$, then $0 < \{n^a\}_{n \in \mathbb{N}}(n)$.

2. POLYNOMIALLY BOUNDED SEQUENCES

Let p be a sequence of real numbers. We say that p is polynomially bounded if and only if

(Def. 1) there exists a natural number k such that $p \in O(\{n^k\}_{n \in \mathbb{N}})$.

Now we state the propositions:

- (20) Let us consider a sequence f of real numbers. Suppose f is not polynomially bounded. Let us consider a natural number k . Then $f \notin O(\{n^k\}_{n \in \mathbb{N}})$.
- (21) Let us consider a sequence f of real numbers. Suppose for every natural number k , $f \notin O(\{n^k\}_{n \in \mathbb{N}})$. Then f is not polynomially bounded.
- (22) Let us consider a positive real number a . Then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is positive.

Let us consider a real number a . Now we state the propositions:

- (23) If $1 \leq a$, then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is non-decreasing. The theorem is a consequence of (9).
- (24) If $1 < a$, then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is increasing. The theorem is a consequence of (8).
- (25) Let us consider a natural number a . If $1 < a$, then $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ is not polynomially bounded.

PROOF: Consider k being a natural number such that $\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}} \in O(\{n^k\}_{n \in \mathbb{N}})$. Reconsider $f = \{n^k\}_{n \in \mathbb{N}}$ as an eventually positive sequence of real numbers. Reconsider $t = \{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}}$ as an eventually non-negative sequence of real numbers. $t \in O(f)$ and for every element n of \mathbb{N} such that $1 \leq n$ holds $0 < f(n)$. Consider c being a real number such that $c > 0$ and for every element n of \mathbb{N} such that $n \geq 1$ holds $(\{a^{1 \cdot n + 0}\}_{n \in \mathbb{N}})(n) \leq c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$. For every natural number n such that $n \geq 1$ holds $2^n \leq c \cdot n^k$ by [24, (7)]. There exist natural numbers N, b such that for every natural number n such that $N \leq n$ holds $2^n \leq b \cdot n^k$ by [24, (3)]. \square

3. POLYNOMIAL SEQUENCES

Now we state the proposition:

(26) Let us consider a finite 0-sequence x of \mathbb{R} , and a sequence y of real numbers. Then

- (i) $x \cdot y$ is a finite transfinite sequence of elements of \mathbb{R} , and
- (ii) $\text{dom}(x \cdot y) = \text{dom } x$, and
- (iii) for every object i such that $i \in \text{dom } x$ holds $(x \cdot y)(i) = x(i) \cdot y(i)$.

Let x be a finite 0-sequence of \mathbb{R} and y be a sequence of real numbers. Observe that the functor $x \cdot y$ yields a finite 0-sequence of \mathbb{R} . Now we state the proposition:

(27) Let us consider a finite 0-sequence d of \mathbb{R} , and natural numbers x, i . Suppose $i \in \text{dom } d$. Then $(d \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})(i) = d(i) \cdot x^i$. The theorem is a consequence of (26).

Let c be a finite 0-sequence of \mathbb{R} . The functor $\text{Seq}_{\text{poly}}(c)$ yielding a sequence of real numbers is defined by

(Def. 2) for every natural number x , $it(x) = \sum(c \cdot \{x^{1 \cdot n + 0}\}_{n \in \mathbb{N}})$.

Let us consider a finite 0-sequence d of \mathbb{R} and a natural number k . Now we state the propositions:

(28) Suppose $\text{len } d = k + 1$. Then there exists a real number a and there exists a finite 0-sequence d_1 of \mathbb{R} and there exists a sequence y of real numbers such that $\text{len } d_1 = k$ and $d_1 = d \upharpoonright k$ and $a = d(k)$ and $d = d_1 \hat{\ } \langle a \rangle$ and $\text{Seq}_{\text{poly}}(d) = \text{Seq}_{\text{poly}}(d_1) + y$ and for every natural number i , $y(i) = a \cdot i^k$. PROOF: Consider a being a real number, d_1 being a finite 0-sequence of \mathbb{R} such that $\text{len } d_1 = k$ and $d_1 = d \upharpoonright k$ and $a = d(k)$ and $d = d_1 \hat{\ } \langle a \rangle$. Define $\mathcal{F}(\text{natural number}) = a \cdot i^k$. Consider y being a sequence of real numbers such that for every natural number x , $y(x) = \mathcal{F}(x)$ from [15, Sch. 1]. For every element x of \mathbb{N} , $(\text{Seq}_{\text{poly}}(d))(x) = (\text{Seq}_{\text{poly}}(d_1) + y)(x)$ by (26), [1, (13), (44)], (27). \square

(29) If $\text{len } d = 1$, then there exists a real number a such that $a = d(0)$ and for every natural number x , $(\text{Seq}_{\text{poly}}(d))(x) = a$. The theorem is a consequence of (26).

(30) If $\text{len } d = 1$ and d is non-negative yielding, then $\text{Seq}_{\text{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}})$. The theorem is a consequence of (29).

(31) Let us consider a natural number k , a real number a , and a sequence y of real numbers. Suppose $0 \leq a$ and for every natural number i , $y(i) = a \cdot i^k$. Then $y \in O(\{n^k\}_{n \in \mathbb{N}})$.

(32) Let us consider natural numbers k, n . If $k \leq n$, then $O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^n\}_{n \in \mathbb{N}})$.

PROOF: Consider i being a natural number such that $n = k + i$. Define $\mathcal{P}[\text{natural number}] \equiv O(\{n^k\}_{n \in \mathbb{N}}) \subseteq O(\{n^{(k+\$1)}\}_{n \in \mathbb{N}})$. For every natural number x such that $\mathcal{P}[x]$ holds $\mathcal{P}[x + 1]$. For every natural number x , $\mathcal{P}[x]$ from [1, Sch. 2]. \square

(33) Let us consider a natural number k , and a non-negative yielding finite 0-sequence c of \mathbb{R} . Suppose $\text{len } c = k + 1$. Then $\text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every non-negative yielding finite 0-sequence c of \mathbb{R} such that $\text{len } c = \$1 + 1$ holds $\text{Seq}_{\text{poly}}(c) \in O(\{n^{\$1}\}_{n \in \mathbb{N}})$. $\mathcal{P}[0]$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by (28), [7, (47)], [1, (13), (39)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

(34) Let us consider a natural number k , and a finite 0-sequence c of \mathbb{R} . Then there exists a finite 0-sequence d of \mathbb{R} such that

- (i) $\text{len } d = \text{len } c$, and
- (ii) for every natural number i such that $i \in \text{dom } d$ holds $d(i) = |c(i)|$.

PROOF: Define $\mathcal{F}(\text{natural number}) = |c(\$1)| (\in \mathbb{R})$. Consider d being a finite 0-sequence of \mathbb{R} such that $\text{len } d = \text{len } c$ and for every natural number j such that $j \in \text{len } c$ holds $d(j) = \mathcal{F}(j)$ from [18, Sch. 1]. \square

(35) Let us consider a finite 0-sequence c of \mathbb{R} , and a finite 0-sequence d of \mathbb{R} . Suppose $\text{len } d = \text{len } c$ and for every natural number i such that $i \in \text{dom } d$ holds $d(i) = |c(i)|$. Let us consider a natural number n . Then $(\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(d))(n)$.

PROOF: $\text{dom}(d \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}}) = \text{dom } d$. For every natural number i such that $i \in \text{dom}(c \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})$ holds $(c \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})(i) \leq (d \cdot \{x^{1-n+0}\}_{n \in \mathbb{N}})(i)$ by (26), (27), [19, (4)]. \square

(36) Let us consider a natural number k , and a finite 0-sequence c of \mathbb{R} . Suppose $\text{len } c = k + 1$ and $\text{Seq}_{\text{poly}}(c)$ is eventually nonnegative. Then $\text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.

PROOF: Consider d being a finite 0-sequence of \mathbb{R} such that $\text{len } d = \text{len } c$ and for every natural number i such that $i \in \text{dom } d$ holds $d(i) = |c(i)|$. For every natural number i such that $i \in \text{dom } d$ holds $0 \leq d(i)$ by [6, (46)]. For every real number r such that $r \in \text{rng } d$ holds $0 \leq r$. $\text{Seq}_{\text{poly}}(d) \in O(\{n^k\}_{n \in \mathbb{N}})$. Consider t being an element of $\mathbb{R}^{\mathbb{N}}$ such that $\text{Seq}_{\text{poly}}(d) = t$ and there exists a real number c and there exists an element N of \mathbb{N} such that $c > 0$ and for every element n of \mathbb{N} such that $n \geq N$ holds $t(n) \leq c \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geq 0$. Consider N_1 being a natural number such that for every natural number n such that $N_1 \leq n$ holds $0 \leq (\text{Seq}_{\text{poly}}(c))(n)$.

Consider a being a real number, N_2 being an element of \mathbb{N} such that $a > 0$ and for every element n of \mathbb{N} such that $n \geq N_2$ holds $t(n) \leq a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $t(n) \geq 0$. Set $N = N_1 + N_2$. For every element n of \mathbb{N} such that $n \geq N$ holds $(\text{Seq}_{\text{poly}}(c))(n) \leq a \cdot \{n^k\}_{n \in \mathbb{N}}(n)$ and $(\text{Seq}_{\text{poly}}(c))(n) \geq 0$ by [1, (11)], (35). \square

- (37) Let us consider natural numbers k, n . If $0 < n$, then $n \cdot \{n^k\}_{n \in \mathbb{N}}(n) = \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$.
- (38) Let us consider a finite 0-sequence c of \mathbb{R} . Suppose $\text{len } c = 0$. Let us consider a natural number x . Then $(\text{Seq}_{\text{poly}}(c))(x) = 0$.
- (39) Let us consider an eventually nonnegative sequence f of real numbers, and a natural number k . Suppose $f \in O(\{n^k\}_{n \in \mathbb{N}})$. Then there exists a natural number N such that for every natural number n such that $N \leq n$ holds $f(n) \leq \{n^{(k+1)}\}_{n \in \mathbb{N}}(n)$. The theorem is a consequence of (37).
- (40) Let us consider a finite 0-sequence c of \mathbb{R} . Then there exists a finite 0-sequence a_1 of \mathbb{R} such that

- (i) $a_1 = |c|$, and
- (ii) for every natural number n , $(\text{Seq}_{\text{poly}}(c))(n) \leq (\text{Seq}_{\text{poly}}(a_1))(n)$.

PROOF: Reconsider $a_1 = |c|$ as a finite 0-sequence of \mathbb{R} . Set $m_1 = c \cdot \{n^{1-n+0}\}_{n \in \mathbb{N}}$. Set $m_2 = a_1 \cdot \{n^{1-n+0}\}_{n \in \mathbb{N}}$. For every natural number x such that $x \in \text{dom } m_1$ holds $m_1(x) \leq m_2(x)$ by [19, (4)]. \square

- (41) Let us consider finite 0-sequences c, a_1 of \mathbb{R} . Suppose $a_1 = |c|$. Let us consider a natural number n . Then $|(\text{Seq}_{\text{poly}}(c))(n)| \leq (\text{Seq}_{\text{poly}}(a_1))(n)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every finite 0-sequences c, a_1 of \mathbb{R} such that $\text{len } c = \$_1$ and $a_1 = |c|$ for every natural number x , $|(\text{Seq}_{\text{poly}}(c))(x)| \leq (\text{Seq}_{\text{poly}}(a_1))(x)$. $\mathcal{P}[0]$ by (26), [6, (44)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by (28), [7, (47)], [15, (7)], [6, (56), (65)]. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

- (42) Let us consider a real number a . Suppose $0 < a$. Let us consider a natural number k , and a non-negative yielding finite 0-sequence d of \mathbb{R} . Suppose $\text{len } d = k$. Then there exists a natural number N such that for every natural number x such that $N \leq x$ for every natural number i such that $i \in \text{dom } d$ holds $d(i) \cdot x^i \cdot k \leq a \cdot x^k$.

PROOF: For every natural number i such that $i \in \text{dom } d$ holds $0 \leq d(i)$ by [7, (3)]. \square

- (43) Let us consider a natural number k , a finite 0-sequence d of \mathbb{R} , a real number a , and a sequence y of real numbers. Suppose $0 < a$ and $\text{len } d = k$ and for every natural number x , $y(x) = a \cdot x^k$. Then there exists a natural number N such that for every natural number x such that $N \leq x$ holds

$|(\text{Seq}_{\text{poly}}(d))(x)| \leq y(x)$. The theorem is a consequence of (38), (42), (26), (27), and (41).

- (44) Let us consider a natural number k , and a finite 0-sequence d of \mathbb{R} . Suppose $\text{len } d = k+1$ and $0 < d(k)$. Then $\text{Seq}_{\text{poly}}(d)$ is eventually nonnegative. PROOF: Consider a being a real number, d_1 being a finite 0-sequence of \mathbb{R} , y being a sequence of real numbers such that $\text{len } d_1 = k$ and $d_1 = d \upharpoonright k$ and $a = d(k)$ and $d = d_1 \hat{\ } \langle a \rangle$ and $\text{Seq}_{\text{poly}}(d) = \text{Seq}_{\text{poly}}(d_1) + y$ and for every natural number i , $y(i) = a \cdot i^k$. Consider N being a natural number such that for every natural number i such that $N \leq i$ holds $|(\text{Seq}_{\text{poly}}(d_1))(i)| \leq y(i)$. For every natural number i such that $N \leq i$ holds $0 \leq (\text{Seq}_{\text{poly}}(d))(i)$ by [19, (4)], [15, (7)]. \square

Let us consider a natural number k and a finite 0-sequence c of \mathbb{R} .

Let us assume that $\text{len } c = k+1$ and $0 < c(k)$. Now we state the propositions:

- (45) $\text{Seq}_{\text{poly}}(c) \in O(\{n^k\}_{n \in \mathbb{N}})$.
 (46) $\text{Seq}_{\text{poly}}(c)$ is polynomially bounded. The theorem is a consequence of (36) and (44).

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