

# Extended Real-Valued Double Sequence and Its Convergence<sup>1</sup>

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**Summary.** In this article we introduce the convergence of extended real-valued double sequences [16], [17]. It is similar to our previous articles [15], [10]. In addition, we also prove Fatou's lemma and the monotone convergence theorem for double sequences.

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The notation and terminology used in this paper have been introduced in the following articles: [5], [21], [15], [10], [12], [6], [7], [22], [13], [11], [14], [1], [2], [8], [18], [24], [25], [26], [20], [23], [3], [4], and [9].

## 1. PRELIMINARIES

Let  $X$  be a non empty set. One can verify that there exists a function from  $X$  into  $\mathbb{R}$  which is non-negative and non-positive and there exists a function from  $X$  into  $\overline{\mathbb{R}}$  which is without  $-\infty$ , without  $+\infty$ , non-negative, and non-positive and every function from  $X$  into  $\overline{\mathbb{R}}$  which is non-negative is also without  $-\infty$  and every function from  $X$  into  $\overline{\mathbb{R}}$  which is non-positive is also without  $+\infty$  and there exists a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$  which is without  $-\infty$ .

Let  $f$  be a function from  $X$  into  $\overline{\mathbb{R}}$ . Let us observe that the functor  $-f$  yields a function from  $X$  into  $\overline{\mathbb{R}}$ . Let  $f$  be a without  $-\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Note that  $-f$  is without  $+\infty$ .

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Let  $f$  be a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Let us observe that  $-f$  is without  $-\infty$ .

Let  $f$  be a non-negative function from  $X$  into  $\overline{\mathbb{R}}$ . Note that  $-f$  is non-positive.

Let  $f$  be a non-positive function from  $X$  into  $\overline{\mathbb{R}}$ . Let us observe that  $-f$  is non-negative.

Let  $A, B$  be non empty sets and  $f$  be a without  $-\infty$  function from  $A \times B$  into  $\overline{\mathbb{R}}$ . Let us observe that  $f^T$  is without  $-\infty$ .

Let  $f$  be a without  $+\infty$  function from  $A \times B$  into  $\overline{\mathbb{R}}$ . One can verify that  $f^T$  is without  $+\infty$ .

Let  $f$  be a non-negative function from  $A \times B$  into  $\overline{\mathbb{R}}$ . One can check that  $f^T$  is non-negative.

Let  $f$  be a non-positive function from  $A \times B$  into  $\overline{\mathbb{R}}$ . Note that  $f^T$  is non-positive.

Now we state the propositions:

- (1) Let us consider a sequence  $s$  of extended reals. Then  $(\sum_{\alpha=0}^{\kappa} (-s)(\alpha))_{\kappa \in \mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ .

PROOF: Define  $\mathcal{Q}$ [natural number]  $\equiv$

$(-\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1)$ . For every natural number  $n$ ,  $\mathcal{Q}[n]$  from [1, Sch. 2]. Define  $\mathcal{P}$ [natural number]  $\equiv (\sum_{\alpha=0}^{\kappa} (-s)(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1) = (-\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1)$ . For every natural number  $n$  such that  $\mathcal{P}[n]$  holds  $\mathcal{P}[n + 1]$ . For every natural number  $n$ ,  $\mathcal{P}[n]$  from [1, Sch. 2].  $\square$

- (2) Let us consider a non empty set  $X$ , and a partial function  $f$  from  $X$  to  $\overline{\mathbb{R}}$ . Then  $--f = f$ .
- (3) Let us consider non empty sets  $X, Y$ , and a function  $f$  from  $X \times Y$  into  $\overline{\mathbb{R}}$ . Then  $(-f)^T = -f^T$ .

Let  $s$  be a non-negative sequence of extended reals. One can verify that  $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$  is non-negative.

Let  $s$  be a non-positive sequence of extended reals. Let us observe that  $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$  is non-positive.

Now we state the propositions:

- (4) Let us consider a non-negative sequence  $s$  of extended reals, and a natural number  $m$ . Then  $s(m) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$ .

PROOF: Define  $\mathcal{P}$ [natural number]  $\equiv s(\mathbb{S}_1) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$  by [4, (51)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2].  $\square$

- (5) Let us consider a non-positive sequence  $s$  of extended reals, and a natural number  $m$ . Then  $s(m) \geq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$ . The theorem is a consequence of (4), (1), and (2).

(6) Let us consider a non empty set  $X$ . Then every without  $-\infty$ , without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$  is a function from  $X$  into  $\mathbb{R}$ .

Let  $X$  be a non empty set and  $f_1, f_2$  be without  $-\infty$  functions from  $X$  into  $\overline{\mathbb{R}}$ . One can verify that the functor  $f_1 + f_2$  yields a without  $-\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Let  $f_1, f_2$  be without  $+\infty$  functions from  $X$  into  $\overline{\mathbb{R}}$ . One can verify that the functor  $f_1 + f_2$  yields a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Let  $f_1$  be a without  $-\infty$  function from  $X$  into  $\overline{\mathbb{R}}$  and  $f_2$  be a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Let us observe that the functor  $f_1 - f_2$  yields a without  $-\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Let  $f_1$  be a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$  and  $f_2$  be a without  $-\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Observe that the functor  $f_1 - f_2$  yields a without  $+\infty$  function from  $X$  into  $\overline{\mathbb{R}}$ . Now we state the propositions:

(7) Let us consider a non empty set  $X$ , an element  $x$  of  $X$ , and functions  $f_1, f_2$  from  $X$  into  $\overline{\mathbb{R}}$ . Then

- (i) if  $f_1$  is without  $-\infty$  and  $f_2$  is without  $-\infty$ , then  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ , and
- (ii) if  $f_1$  is without  $+\infty$  and  $f_2$  is without  $+\infty$ , then  $(f_1 + f_2)(x) = f_1(x) + f_2(x)$ , and
- (iii) if  $f_1$  is without  $-\infty$  and  $f_2$  is without  $+\infty$ , then  $(f_1 - f_2)(x) = f_1(x) - f_2(x)$ , and
- (iv) if  $f_1$  is without  $+\infty$  and  $f_2$  is without  $-\infty$ , then  $(f_1 - f_2)(x) = f_1(x) - f_2(x)$ .

(8) Let us consider a non empty set  $X$ , and without  $-\infty$  functions  $f_1, f_2$  from  $X$  into  $\overline{\mathbb{R}}$ . Then

- (i)  $f_1 + f_2 = f_1 - -f_2$ , and
- (ii)  $-(f_1 + f_2) = -f_1 - f_2$ .

The theorem is a consequence of (7).

(9) Let us consider a non empty set  $X$ , and without  $+\infty$  functions  $f_1, f_2$  from  $X$  into  $\overline{\mathbb{R}}$ . Then

- (i)  $f_1 + f_2 = f_1 - -f_2$ , and
- (ii)  $-(f_1 + f_2) = -f_1 - f_2$ .

The theorem is a consequence of (7).

(10) Let us consider a non empty set  $X$ , a without  $-\infty$  function  $f_1$  from  $X$  into  $\overline{\mathbb{R}}$ , and a without  $+\infty$  function  $f_2$  from  $X$  into  $\overline{\mathbb{R}}$ . Then

- (i)  $f_1 - f_2 = f_1 + -f_2$ , and
- (ii)  $f_2 - f_1 = f_2 + -f_1$ , and
- (iii)  $-(f_1 - f_2) = -f_1 + f_2$ , and

$$(iv) \quad -(f_2 - f_1) = -f_2 + f_1.$$

The theorem is a consequence of (8), (2), and (9).

Let  $f$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and  $n, m$  be natural numbers. One can check that the functor  $f(n, m)$  yields an element of  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (11) Let us consider without  $-\infty$  functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then  $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$ . The theorem is a consequence of (7).
- (12) Let us consider without  $+\infty$  functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then  $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$ . The theorem is a consequence of (7).
- (13) Let us consider a without  $-\infty$  function  $f_1$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , a without  $+\infty$  function  $f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then
- (i)  $(f_1 - f_2)(n, m) = f_1(n, m) - f_2(n, m)$ , and
- (ii)  $(f_2 - f_1)(n, m) = f_2(n, m) - f_1(n, m)$ .

The theorem is a consequence of (7).

- (14) Let us consider non empty sets  $X, Y$ , and without  $-\infty$  functions  $f_1, f_2$  from  $X \times Y$  into  $\overline{\mathbb{R}}$ . Then  $(f_1 + f_2)^T = f_1^T + f_2^T$ . The theorem is a consequence of (7).
- (15) Let us consider non empty sets  $X, Y$ , and without  $+\infty$  functions  $f_1, f_2$  from  $X \times Y$  into  $\overline{\mathbb{R}}$ . Then  $(f_1 + f_2)^T = f_1^T + f_2^T$ . The theorem is a consequence of (7).
- (16) Let us consider non empty sets  $X, Y$ , a without  $-\infty$  function  $f_1$  from  $X \times Y$  into  $\overline{\mathbb{R}}$ , and a without  $+\infty$  function  $f_2$  from  $X \times Y$  into  $\overline{\mathbb{R}}$ . Then
- (i)  $(f_1 - f_2)^T = f_1^T - f_2^T$ , and
- (ii)  $(f_2 - f_1)^T = f_2^T - f_1^T$ .

The theorem is a consequence of (7).

One can verify that every sequence of extended reals which is convergent to  $+\infty$  is also convergent and every sequence of extended reals which is convergent to  $-\infty$  is also convergent and every sequence of extended reals which is convergent to a finite limit is also convergent and there exists a sequence of extended reals which is convergent and there exists a without  $-\infty$  sequence of extended reals which is convergent and there exists a without  $+\infty$  sequence of extended reals which is convergent.

Now we state the proposition:

- (17) Let us consider a convergent sequence  $s$  of extended reals. Then

- (i)  $s$  is convergent to a finite limit iff  $-s$  is convergent to a finite limit, and
- (ii)  $s$  is convergent to  $+\infty$  iff  $-s$  is convergent to  $-\infty$ , and
- (iii)  $s$  is convergent to  $-\infty$  iff  $-s$  is convergent to  $+\infty$ , and
- (iv)  $-s$  is convergent, and
- (v)  $\lim(-s) = -\lim s$ .

The theorem is a consequence of (2).

Let us consider without  $-\infty$  sequences  $s_1, s_2$  of extended reals. Now we state the propositions:

- (18) Suppose  $s_1$  is convergent to  $+\infty$  and  $s_2$  is convergent to  $+\infty$ . Then
  - (i)  $s_1 + s_2$  is convergent to  $+\infty$  and convergent, and
  - (ii)  $\lim(s_1 + s_2) = +\infty$ .

The theorem is a consequence of (7).

- (19) Suppose  $s_1$  is convergent to  $+\infty$  and  $s_2$  is convergent to a finite limit. Then
  - (i)  $s_1 + s_2$  is convergent to  $+\infty$  and convergent, and
  - (ii)  $\lim(s_1 + s_2) = +\infty$ .

The theorem is a consequence of (7).

Now we state the proposition:

- (20) Let us consider without  $+\infty$  sequences  $s_1, s_2$  of extended reals. Suppose  $s_1$  is convergent to  $+\infty$  and  $s_2$  is convergent to a finite limit. Then
  - (i)  $s_1 + s_2$  is convergent to  $+\infty$  and convergent, and
  - (ii)  $\lim(s_1 + s_2) = +\infty$ .

The theorem is a consequence of (7).

Let us consider without  $-\infty$  sequences  $s_1, s_2$  of extended reals. Now we state the propositions:

- (21) Suppose  $s_1$  is convergent to  $-\infty$  and  $s_2$  is convergent to  $-\infty$ . Then
  - (i)  $s_1 + s_2$  is convergent to  $-\infty$  and convergent, and
  - (ii)  $\lim(s_1 + s_2) = -\infty$ .

The theorem is a consequence of (7).

- (22) Suppose  $s_1$  is convergent to  $-\infty$  and  $s_2$  is convergent to a finite limit. Then
  - (i)  $s_1 + s_2$  is convergent to  $-\infty$  and convergent, and
  - (ii)  $\lim(s_1 + s_2) = -\infty$ .

The theorem is a consequence of (7).

(23) Suppose  $s_1$  is convergent to a finite limit and  $s_2$  is convergent to a finite limit. Then

- (i)  $s_1 + s_2$  is convergent to a finite limit and convergent, and
- (ii)  $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$ .

The theorem is a consequence of (7).

Now we state the propositions:

(24) Let us consider without  $+\infty$  sequences  $s_1, s_2$  of extended reals. Then

- (i) if  $s_1$  is convergent to  $+\infty$  and  $s_2$  is convergent to  $+\infty$ , then  $s_1 + s_2$  is convergent to  $+\infty$  and convergent and  $\lim(s_1 + s_2) = +\infty$ , and
- (ii) if  $s_1$  is convergent to  $+\infty$  and  $s_2$  is convergent to a finite limit, then  $s_1 + s_2$  is convergent to  $+\infty$  and convergent and  $\lim(s_1 + s_2) = +\infty$ , and
- (iii) if  $s_1$  is convergent to  $-\infty$  and  $s_2$  is convergent to  $-\infty$ , then  $s_1 + s_2$  is convergent to  $-\infty$  and convergent and  $\lim(s_1 + s_2) = -\infty$ , and
- (iv) if  $s_1$  is convergent to  $-\infty$  and  $s_2$  is convergent to a finite limit, then  $s_1 + s_2$  is convergent to  $-\infty$  and convergent and  $\lim(s_1 + s_2) = -\infty$ , and
- (v) if  $s_1$  is convergent to a finite limit and  $s_2$  is convergent to a finite limit, then  $s_1 + s_2$  is convergent to a finite limit and convergent and  $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$ .

The theorem is a consequence of (17), (21), (10), (9), (2), (22), (18), (19), and (23).

(25) Let us consider a without  $-\infty$  sequence  $s_1$  of extended reals, and a without  $+\infty$  sequence  $s_2$  of extended reals. Then

- (i) if  $s_1$  is convergent to  $+\infty$  and  $s_2$  is convergent to  $-\infty$ , then  $s_1 - s_2$  is convergent to  $+\infty$  and convergent and  $s_2 - s_1$  is convergent to  $-\infty$  and convergent and  $\lim(s_1 - s_2) = +\infty$  and  $\lim(s_2 - s_1) = -\infty$ , and
- (ii) if  $s_1$  is convergent to  $+\infty$  and  $s_2$  is convergent to a finite limit, then  $s_1 - s_2$  is convergent to  $+\infty$  and convergent and  $s_2 - s_1$  is convergent to  $-\infty$  and convergent and  $\lim(s_1 - s_2) = +\infty$  and  $\lim(s_2 - s_1) = -\infty$ , and
- (iii) if  $s_1$  is convergent to  $-\infty$  and  $s_2$  is convergent to a finite limit, then  $s_1 - s_2$  is convergent to  $-\infty$  and convergent and  $s_2 - s_1$  is convergent to  $+\infty$  and convergent and  $\lim(s_1 - s_2) = -\infty$  and  $\lim(s_2 - s_1) = +\infty$ , and

- (iv) if  $s_1$  is convergent to a finite limit and  $s_2$  is convergent to a finite limit, then  $s_1 - s_2$  is convergent to a finite limit and convergent and  $s_2 - s_1$  is convergent to a finite limit and convergent and  $\lim(s_1 - s_2) = \lim s_1 - \lim s_2$  and  $\lim(s_2 - s_1) = \lim s_2 - \lim s_1$ .

The theorem is a consequence of (17), (24), (18), (10), (19), (22), (23), and (2).

## 2. SUBSEQUENCES OF CONVERGENT EXTENDED REAL-VALUED SEQUENCES

Let us consider sequences  $s_1, s_2$  of extended reals. Now we state the propositions:

- (26) Suppose  $s_2$  is a subsequence of  $s_1$  and  $s_1$  is convergent to a finite limit. Then

- (i)  $s_2$  is convergent to a finite limit, and
- (ii)  $\lim s_1 = \lim s_2$ .

PROOF: Consider  $g$  being a real number such that  $\lim s_1 = g$  and for every real number  $p$  such that  $0 < p$  there exists a natural number  $n$  such that for every natural number  $m$  such that  $n \leq m$  holds  $|s_1(m) - \lim s_1| < p$  and  $s_1$  is convergent to a finite limit. Reconsider  $L = \lim s_1$  as an extended real number. There exists a real number  $g$  such that for every real number  $p$  such that  $0 < p$  there exists a natural number  $n$  such that for every natural number  $m$  such that  $n \leq m$  holds  $|(s_2(m) - g \text{ qua extended real})| < p$  by [19, (14)], [7, (15)]. For every real number  $p$  such that  $0 < p$  there exists a natural number  $n$  such that for every natural number  $m$  such that  $n \leq m$  holds  $|s_2(m) - L| < p$  by [19, (14)], [7, (15)].  $\square$

- (27) Suppose  $s_2$  is a subsequence of  $s_1$  and  $s_1$  is convergent to  $+\infty$ . Then

- (i)  $s_2$  is convergent to  $+\infty$ , and
- (ii)  $\lim s_2 = +\infty$ .

- (28) Suppose  $s_2$  is a subsequence of  $s_1$  and  $s_1$  is convergent to  $-\infty$ . Then

- (i)  $s_2$  is convergent to  $-\infty$ , and
- (ii)  $\lim s_2 = -\infty$ .

## 3. CONVERGENCY FOR EXTENDED REAL-VALUED DOUBLE SEQUENCES

Let us consider a function  $R$  from  $\mathbb{N} \times \mathbb{N}$  into  $\mathbb{R}$ . Now we state the propositions:

- (29) Suppose the lim in the first coordinate of  $R$  is convergent. Then the first coordinate major iterated lim of  $R = \lim(\text{the lim in the first coordinate of } R)$ .
- (30) Suppose the lim in the second coordinate of  $R$  is convergent. Then the second coordinate major iterated lim of  $R = \lim(\text{the lim in the second coordinate of } R)$ .

Let  $E$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . We say that  $E$  is P-convergent to a finite limit if and only if

- (Def. 1) there exists a real number  $p$  such that for every real number  $e$  such that  $0 < e$  there exists a natural number  $N$  such that for every natural numbers  $n, m$  such that  $n \geq N$  and  $m \geq N$  holds  $|E(n, m) - (p \text{ qua extended real})| < e$ .

We say that  $E$  is P-convergent to  $+\infty$  if and only if

- (Def. 2) for every real number  $g$  such that  $0 < g$  there exists a natural number  $N$  such that for every natural numbers  $n, m$  such that  $n \geq N$  and  $m \geq N$  holds  $g \leq E(n, m)$ .

We say that  $E$  is P-convergent to  $-\infty$  if and only if

- (Def. 3) for every real number  $g$  such that  $g < 0$  there exists a natural number  $N$  such that for every natural numbers  $n, m$  such that  $n \geq N$  and  $m \geq N$  holds  $E(n, m) \leq g$ .

Let  $f$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . We say that  $f$  is convergent in the first coordinate to  $+\infty$  if and only if

- (Def. 4) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(f, m)$  is convergent to  $+\infty$ .

We say that  $f$  is convergent in the first coordinate to  $-\infty$  if and only if

- (Def. 5) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(f, m)$  is convergent to  $-\infty$ .

We say that  $f$  is convergent in the first coordinate to a finite limit if and only if

- (Def. 6) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(f, m)$  is convergent to a finite limit.

We say that  $f$  is convergent in the first coordinate if and only if

- (Def. 7) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(f, m)$  is convergent.

We say that  $f$  is convergent in the second coordinate to  $+\infty$  if and only if

- (Def. 8) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}(f, m)$  is convergent to  $+\infty$ .

We say that  $f$  is convergent in the second coordinate to  $-\infty$  if and only if

- (Def. 9) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}(f, m)$  is convergent to  $-\infty$ .



We say that  $f$  is convergent in the second coordinate to a finite limit if and only if

(Def. 10) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}(f, m)$  is convergent to a finite limit.

We say that  $f$  is convergent in the second coordinate if and only if

(Def. 11) for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}(f, m)$  is convergent.

Now we state the propositions:

(31) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then

(i) if  $f$  is convergent in the first coordinate to  $+\infty$  or convergent in the first coordinate to  $-\infty$  or convergent in the first coordinate to a finite limit, then  $f$  is convergent in the first coordinate, and

(ii) if  $f$  is convergent in the second coordinate to  $+\infty$  or convergent in the second coordinate to  $-\infty$  or convergent in the second coordinate to a finite limit, then  $f$  is convergent in the second coordinate.

(32) Let us consider non empty sets  $X, Y, Z$ , a function  $F$  from  $X \times Y$  into  $Z$ , and an element  $x$  of  $X$ . Then  $\text{curry}(F, x) = \text{curry}'(F^T, x)$ .

(33) Let us consider non empty sets  $X, Y, Z$ , a function  $F$  from  $X \times Y$  into  $Z$ , and an element  $y$  of  $Y$ . Then  $\text{curry}'(F, y) = \text{curry}(F^T, y)$ .

(34) Let us consider non empty sets  $X, Y$ , a function  $F$  from  $X \times Y$  into  $\overline{\mathbb{R}}$ , and an element  $x$  of  $X$ . Then  $\text{curry}(-F, x) = -\text{curry}(F, x)$ .

(35) Let us consider non empty sets  $X, Y$ , a function  $F$  from  $X \times Y$  into  $\overline{\mathbb{R}}$ , and an element  $y$  of  $Y$ . Then  $\text{curry}'(-F, y) = -\text{curry}'(F, y)$ .

Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Now we state the propositions:

(36) (i)  $f$  is convergent in the first coordinate to  $+\infty$  iff  $f^T$  is convergent in the second coordinate to  $+\infty$ , and

(ii)  $f$  is convergent in the second coordinate to  $+\infty$  iff  $f^T$  is convergent in the first coordinate to  $+\infty$ , and

(iii)  $f$  is convergent in the first coordinate to  $-\infty$  iff  $f^T$  is convergent in the second coordinate to  $-\infty$ , and

(iv)  $f$  is convergent in the second coordinate to  $-\infty$  iff  $f^T$  is convergent in the first coordinate to  $-\infty$ , and

(v)  $f$  is convergent in the first coordinate to a finite limit iff  $f^T$  is convergent in the second coordinate to a finite limit, and

(vi)  $f$  is convergent in the second coordinate to a finite limit iff  $f^T$  is convergent in the first coordinate to a finite limit.

The theorem is a consequence of (33) and (32).

(37) (i)  $f$  is convergent in the first coordinate to  $+\infty$  iff  $-f$  is convergent in the first coordinate to  $-\infty$ , and

- (ii)  $f$  is convergent in the first coordinate to  $-\infty$  iff  $-f$  is convergent in the first coordinate to  $+\infty$ , and
- (iii)  $f$  is convergent in the first coordinate to a finite limit iff  $-f$  is convergent in the first coordinate to a finite limit, and
- (iv)  $f$  is convergent in the second coordinate to  $+\infty$  iff  $-f$  is convergent in the second coordinate to  $-\infty$ , and
- (v)  $f$  is convergent in the second coordinate to  $-\infty$  iff  $-f$  is convergent in the second coordinate to  $+\infty$ , and
- (vi)  $f$  is convergent in the second coordinate to a finite limit iff  $-f$  is convergent in the second coordinate to a finite limit.

The theorem is a consequence of (35), (17), (2), and (34).

Let  $f$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . The functors: the lim in the first coordinate of  $f$  and the lim in the second coordinate of  $f$  yielding sequences of extended reals are defined by conditions

(Def. 12) for every element  $m$  of  $\mathbb{N}$ , the lim in the first coordinate of  $f(m) = \text{lim curry}'(f, m)$ ,

(Def. 13) for every element  $n$  of  $\mathbb{N}$ , the lim in the second coordinate of  $f(n) = \text{lim curry}(f, n)$ ,

respectively. Now we state the proposition:

(38) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then

- (i) the lim in the first coordinate of  $f =$  the lim in the second coordinate of  $f^T$ , and
- (ii) the lim in the second coordinate of  $f =$  the lim in the first coordinate of  $f^T$ .

The theorem is a consequence of (33) and (32).

Let  $X, Y$  be non empty sets,  $F$  be a without  $+\infty$  function from  $X \times Y$  into  $\overline{\mathbb{R}}$ , and  $x$  be an element of  $X$ . Let us observe that  $\text{curry}(F, x)$  is without  $+\infty$ .

Let  $y$  be an element of  $Y$ . One can verify that  $\text{curry}'(F, y)$  is without  $+\infty$ .

Let  $F$  be a without  $-\infty$  function from  $X \times Y$  into  $\overline{\mathbb{R}}$  and  $x$  be an element of  $X$ . Let us note that  $\text{curry}(F, x)$  is without  $-\infty$ .

Let  $y$  be an element of  $Y$ . Observe that  $\text{curry}'(F, y)$  is without  $-\infty$ .

Let  $f$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . The partial sums in the second coordinate of  $f$  yielding a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  is defined by

(Def. 14) for every natural numbers  $n, m$ ,  $it(n, 0) = f(n, 0)$  and  $it(n, m + 1) = it(n, m) + f(n, m + 1)$ .

The partial sums in the first coordinate of  $f$  yielding a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  is defined by

(Def. 15) for every natural numbers  $n, m$ ,  $it(0, m) = f(0, m)$  and  $it(n + 1, m) = it(n, m) + f(n + 1, m)$ .

Let  $f$  be a without  $-\infty$  function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Let us note that the partial sums in the second coordinate of  $f$  is without  $-\infty$ .

Let  $f$  be a without  $+\infty$  function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Observe that the partial sums in the second coordinate of  $f$  is without  $+\infty$ .

Let  $f$  be a non-negative function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Let us observe that the partial sums in the second coordinate of  $f$  is non-negative as a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ .

Let  $f$  be a non-positive function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . One can check that the partial sums in the second coordinate of  $f$  is non-positive as a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ .

Let  $f$  be a without  $-\infty$  function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Let us note that the partial sums in the first coordinate of  $f$  is without  $-\infty$ .

Let  $f$  be a without  $+\infty$  function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Observe that the partial sums in the first coordinate of  $f$  is without  $+\infty$ .

Let  $f$  be a non-negative function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Let us observe that the partial sums in the first coordinate of  $f$  is non-negative as a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ .

Let  $f$  be a non-positive function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . One can check that the partial sums in the first coordinate of  $f$  is non-positive as a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ .

Let  $f$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . The functor  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  yielding a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  is defined by the term

(Def. 16) the partial sums in the second coordinate of the partial sums in the first coordinate of  $f$ .

Now we state the propositions:

(39) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then

- (i) (the partial sums in the first coordinate of  $f$ )( $n, m$ ) = (the partial sums in the second coordinate of  $f^T$ )( $m, n$ ), and
- (ii) (the partial sums in the second coordinate of  $f$ )( $n, m$ ) = (the partial sums in the first coordinate of  $f^T$ )( $m, n$ ).

PROOF: Define  $\mathcal{P}$ [natural number]  $\equiv$  (the partial sums in the first coordinate of  $f$ )( $\$1, m$ ) = (the partial sums in the second coordinate of  $f^T$ )( $m, \$1$ ). For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2]. Define  $\mathcal{Q}$ [natural number]  $\equiv$  (the partial sums in the second coordinate of  $f$ )( $n, \$1$ ) = (the partial sums in the first

coordinate of  $f^T)(\$_1, n)$ . For every natural number  $k$  such that  $\mathcal{Q}[k]$  holds  $\mathcal{Q}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{Q}[k]$  from [1, Sch. 2].  $\square$

(40) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then

- (i) (the partial sums in the first coordinate of  $f)^T =$  the partial sums in the second coordinate of  $f^T$ , and
- (ii) (the partial sums in the second coordinate of  $f)^T =$  the partial sums in the first coordinate of  $f^T$ .

The theorem is a consequence of (39).

(41) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , an extended real-valued function  $g$ , and a natural number  $n$ . Suppose for every natural number  $k$ , (the partial sums in the first coordinate of  $f)(n, k) = g(k)$ . Then

- (i) for every natural number  $k$ ,  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, k) = (\sum_{\alpha=0}^{\kappa} g(\alpha))_{\kappa \in \mathbb{N}}(k)$ , and
- (ii) (the lim in the second coordinate of  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}})(n) = \sum g$ .

(42) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then

- (i) the partial sums in the second coordinate of  $-f =$   
 -(the partial sums in the second coordinate of  $f$ ), and
- (ii) the partial sums in the first coordinate of  $-f =$   
 -(the partial sums in the first coordinate of  $f$ ).

PROOF: For every element  $z$  of  $\mathbb{N} \times$

$\mathbb{N}$ ,  $(-(\text{the partial sums in the second coordinate of } f))(z) =$  (the partial sums in the second coordinate of  $-f)(z)$  by [9, (87)]. For every element  $z$  of  $\mathbb{N} \times \mathbb{N}$ ,

$(-(\text{the partial sums in the first coordinate of } f))(z) =$  (the partial sums in the first coordinate of  $-f)(z)$  by [9, (87)].  $\square$

(43) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and elements  $m, n$  of  $\mathbb{N}$ . Then

- (i) (the partial sums in the first coordinate of  $f)(m, n) = (\sum_{\alpha=0}^{\kappa} (\text{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(m)$ , and
- (ii) (the partial sums in the second coordinate of  $f)(m, n) = (\sum_{\alpha=0}^{\kappa} (\text{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(n)$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv$  (the partial sums in the first coordinate of  $f)(\$_1, n) = (\sum_{\alpha=0}^{\kappa} (\text{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2]. Define  $\mathcal{Q}[\text{natural number}] \equiv$  (the partial sums in the second coordinate of  $f)(m, \$_1) = (\sum_{\alpha=0}^{\kappa} (\text{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$ . For

every natural number  $k$  such that  $\mathcal{Q}[k]$  holds  $\mathcal{Q}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{Q}[k]$  from [1, Sch. 2].  $\square$

- (44) Let us consider without  $-\infty$  functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then
- (i) the partial sums in the second coordinate of  $f_1 + f_2 =$  (the partial sums in the second coordinate of  $f_1$ ) + (the partial sums in the second coordinate of  $f_2$ ), and
  - (ii) the partial sums in the first coordinate of  $f_1 + f_2 =$  (the partial sums in the first coordinate of  $f_1$ ) + (the partial sums in the first coordinate of  $f_2$ ).

The theorem is a consequence of (11).

- (45) Let us consider without  $+\infty$  functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then
- (i) the partial sums in the second coordinate of  $f_1 + f_2 =$  (the partial sums in the second coordinate of  $f_1$ ) + (the partial sums in the second coordinate of  $f_2$ ), and
  - (ii) the partial sums in the first coordinate of  $f_1 + f_2 =$  (the partial sums in the first coordinate of  $f_1$ ) + (the partial sums in the first coordinate of  $f_2$ ).

The theorem is a consequence of (10), (9), (2), (42), (44), and (8).

- (46) Let us consider a without  $-\infty$  function  $f_1$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and a without  $+\infty$  function  $f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then
- (i) the partial sums in the second coordinate of  $f_1 - f_2 =$  (the partial sums in the second coordinate of  $f_1$ ) - (the partial sums in the second coordinate of  $f_2$ ), and
  - (ii) the partial sums in the first coordinate of  $f_1 - f_2 =$  (the partial sums in the first coordinate of  $f_1$ ) - (the partial sums in the first coordinate of  $f_2$ ), and
  - (iii) the partial sums in the second coordinate of  $f_2 - f_1 =$  (the partial sums in the second coordinate of  $f_2$ ) - (the partial sums in the second coordinate of  $f_1$ ), and
  - (iv) the partial sums in the first coordinate of  $f_2 - f_1 =$  (the partial sums in the first coordinate of  $f_2$ ) - (the partial sums in the first coordinate of  $f_1$ ).

The theorem is a consequence of (10), (44), (42), and (45).

- (47) Let us consider a without  $-\infty$  function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then
- (i)  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) =$  (the partial sums in the second coordinate of  $f$ )( $n + 1, m$ ) +  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$ , and

- (ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of  $f$ )( $n, m + 1$ ) = (the partial sums in the first coordinate of  $f$ )( $n, m + 1$ ) + (the partial sums in the first coordinate of the partial sums in the second coordinate of  $f$ )( $n, m$ ).

PROOF: Set  $R_1 = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ . Set  $C_1 =$  the partial sums in the first coordinate of the partial sums in the second coordinate of  $f$ . Set  $R_2 =$  the partial sums in the first coordinate of  $f$ . Set  $C_2 =$  the partial sums in the second coordinate of  $f$ . Define  $\mathcal{P}$ [natural number]  $\equiv R_1(n + 1, \$1) = C_2(n + 1, \$1) + R_1(n, \$1)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2]. Define  $\mathcal{Q}$ [natural number]  $\equiv C_1(\$1, m + 1) = R_2(\$1, m + 1) + C_1(\$1, m)$ . For every natural number  $k$  such that  $\mathcal{Q}[k]$  holds  $\mathcal{Q}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{Q}[k]$  from [1, Sch. 2].  $\square$

- (48) Let us consider a without  $+\infty$  function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then
  - (i)  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n + 1, m) =$  (the partial sums in the second coordinate of  $f$ )( $n + 1, m$ ) +  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$ , and
  - (ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of  $f$ )( $n, m + 1$ ) = (the partial sums in the first coordinate of  $f$ )( $n, m + 1$ ) + (the partial sums in the first coordinate of the partial sums in the second coordinate of  $f$ )( $n, m$ ).

The theorem is a consequence of (2), (42), and (47).

- (49) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose  $f$  is without  $-\infty$  or without  $+\infty$ . Then  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  = the partial sums in the first coordinate of the partial sums in the second coordinate of  $f$ .
- (50) Let us consider without  $-\infty$  functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then  $(\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$ . The theorem is a consequence of (44).
- (51) Let us consider without  $+\infty$  functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then  $(\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$ . The theorem is a consequence of (45).
- (52) Let us consider a without  $-\infty$  function  $f_1$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and a without  $+\infty$  function  $f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then
  - (i)  $(\sum_{\alpha=0}^{\kappa} (f_1 - f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$ , and
  - (ii)  $(\sum_{\alpha=0}^{\kappa} (f_2 - f_1)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}$ .

The theorem is a consequence of (46).

- (53) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and an element  $k$  of  $\mathbb{N}$ . Then

- (i)  $\text{curry}'(\text{the partial sums in the first coordinate of } f, k) = (\sum_{\alpha=0}^{\kappa}(\text{curry}'(f, k))(\alpha))_{\kappa \in \mathbb{N}}$ , and
- (ii)  $\text{curry}(\text{the partial sums in the second coordinate of } f, k) = (\sum_{\alpha=0}^{\kappa}(\text{curry}(f, k))(\alpha))_{\kappa \in \mathbb{N}}$ .

The theorem is a consequence of (43).

(54) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose  $f$  is without  $-\infty$  or without  $+\infty$ . Then

- (i)  $\text{curry}((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0) = \text{curry}(\text{the partial sums in the second coordinate of } f, 0)$ , and
- (ii)  $\text{curry}'((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0) = \text{curry}'(\text{the partial sums in the first coordinate of } f, 0)$ .

(55) Let us consider non empty sets  $C, D$ , without  $-\infty$  functions  $F_1, F_2$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , and an element  $c$  of  $C$ . Then  $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$ . The theorem is a consequence of (7).

(56) Let us consider non empty sets  $C, D$ , without  $-\infty$  functions  $F_1, F_2$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , and an element  $d$  of  $D$ . Then  $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$ . The theorem is a consequence of (7).

(57) Let us consider non empty sets  $C, D$ , without  $+\infty$  functions  $F_1, F_2$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , and an element  $c$  of  $C$ . Then  $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$ . The theorem is a consequence of (7).

(58) Let us consider non empty sets  $C, D$ , without  $+\infty$  functions  $F_1, F_2$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , and an element  $d$  of  $D$ . Then  $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$ . The theorem is a consequence of (7).

(59) Let us consider non empty sets  $C, D$ , a without  $-\infty$  function  $F_1$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , a without  $+\infty$  function  $F_2$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , and an element  $c$  of  $C$ . Then

- (i)  $\text{curry}(F_1 - F_2, c) = \text{curry}(F_1, c) - \text{curry}(F_2, c)$ , and
- (ii)  $\text{curry}(F_2 - F_1, c) = \text{curry}(F_2, c) - \text{curry}(F_1, c)$ .

The theorem is a consequence of (7).

(60) Let us consider non empty sets  $C, D$ , a without  $-\infty$  function  $F_1$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , a without  $+\infty$  function  $F_2$  from  $C \times D$  into  $\overline{\mathbb{R}}$ , and an element  $d$  of  $D$ . Then

- (i)  $\text{curry}'(F_1 - F_2, d) = \text{curry}'(F_1, d) - \text{curry}'(F_2, d)$ , and
- (ii)  $\text{curry}'(F_2 - F_1, d) = \text{curry}'(F_2, d) - \text{curry}'(F_1, d)$ .

The theorem is a consequence of (7).

## 4. NON-NEGATIVE EXTENDED REAL-VALUED DOUBLE SEQUENCES

Now we state the propositions:

- (61) Let us consider a non-negative sequence  $s$  of extended reals. Suppose  $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$  is not convergent to  $+\infty$ . Let us consider a natural number  $n$ . Then  $s(n)$  is a real number.
- (62) Let us consider a non-negative sequence  $s$  of extended reals. Suppose  $s$  is non-decreasing. Then  $s$  is convergent to  $+\infty$  or convergent to a finite limit.

Let  $f$  be a non-negative function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and  $n$  be an element of  $\mathbb{N}$ . Let us observe that  $\text{curry}(f, n)$  is non-negative and  $\text{curry}'(f, n)$  is non-negative.

Now we state the propositions:

- (63) Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and an element  $m$  of  $\mathbb{N}$ . Then  $\text{curry}(\text{the partial sums in the second coordinate of } f, m)$  is non-decreasing.  
 PROOF: Set  $P = \text{curry}(\text{the partial sums in the second coordinate of } f, m)$ . For every natural numbers  $n, j$  such that  $j \leq n$  holds  $P(j) \leq P(n)$  by [4, (51)], [1, (13), (20)].  $\square$
- (64) Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and an element  $n$  of  $\mathbb{N}$ . Then  $\text{curry}'(\text{the partial sums in the first coordinate of } f, n)$  is non-decreasing. The theorem is a consequence of (63), (40), and (33).

Let  $f$  be a non-negative function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and  $m$  be an element of  $\mathbb{N}$ . One can check that  $\text{curry}(\text{the partial sums in the second coordinate of } f, m)$  is non-decreasing and  $\text{curry}'(\text{the partial sums in the first coordinate of } f, m)$  is non-decreasing.

Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (65) (i) if  $f$  is convergent in the first coordinate, then the lim in the first coordinate of  $f$  is non-negative, and  
 (ii) if  $f$  is convergent in the second coordinate, then the lim in the second coordinate of  $f$  is non-negative.
- (66) (i) the partial sums in the first coordinate of  $f$  is convergent in the first coordinate, and  
 (ii) the partial sums in the second coordinate of  $f$  is convergent in the second coordinate.

Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , an element  $m$  of  $\mathbb{N}$ , and a natural number  $n$ .

Let us assume that  $\text{curry}'(\text{the partial sums in the first coordinate of } f, m)$  is not convergent to  $+\infty$ . Now we state the propositions:



(67)  $f(n, m)$  is a real number.

(68)  $f(m, n)$  is a real number.

Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and natural numbers  $n, m$ . Now we state the propositions:

(69) Suppose for every natural number  $i$  such that  $i \leq n$  holds  $f(i, m)$  is a real number. Then (the partial sums in the first coordinate of  $f$ )( $n, m$ )  $< +\infty$ .

PROOF: Define  $\mathcal{P}$ [natural number]  $\equiv$  if  $\$1 \leq n$ , then (the partial sums in the first coordinate of  $f$ )( $\$1, m$ )  $< +\infty$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$  by [4, (51)], [1, (13)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2].  $\square$

(70) Suppose for every natural number  $i$  such that  $i \leq m$  holds  $f(n, i)$  is a real number. Then (the partial sums in the second coordinate of  $f$ )( $n, m$ )  $< +\infty$ .

PROOF: Define  $\mathcal{P}$ [natural number]  $\equiv$  if  $\$1 \leq m$ , then (the partial sums in the second coordinate of  $f$ )( $n, \$1$ )  $< +\infty$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$  by [4, (51)], [1, (13)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2].  $\square$

Now we state the proposition:

(71) Let us consider a without  $-\infty$  function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  is convergent in the first coordinate to  $-\infty$ . Then there exists an element  $m$  of  $\mathbb{N}$  such that  $\text{curry}'$ (the partial sums in the first coordinate of  $f, m$ ) is convergent to  $-\infty$ . The theorem is a consequence of (54).

Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and a natural number  $m$ . Now we state the propositions:

(72) for every element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  holds  $\text{curry}$ (the partial sums in the second coordinate of  $f, k$ ) is not convergent to  $+\infty$  if and only if for every element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  holds  $\lim \text{curry}$ (the partial sums in the second coordinate of  $f, k$ )  $< +\infty$ . The theorem is a consequence of (62).

(73) for every element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  holds  $\text{curry}'$ (the partial sums in the first coordinate of  $f, k$ ) is not convergent to  $+\infty$  if and only if for every element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  holds  $\lim \text{curry}'$ (the partial sums in the first coordinate of  $f, k$ )  $< +\infty$ . The theorem is a consequence of (62).

(74)  $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$  if and only if there exists an element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  and  $\text{curry}$ (the partial sums in the second coordinate

of  $f, k$  is convergent to  $+\infty$ . The theorem is a consequence of (72), (65), and (4).

- (75)  $(\sum_{\alpha=0}^{\kappa}(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$  if and only if there exists an element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  and  $\text{curry}'(\text{the partial sums in the first coordinate of } f, k)$  is convergent to  $+\infty$ . The theorem is a consequence of (38), (40), (74), and (32).

Now we state the proposition:

- (76) Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then
  - (i) (the partial sums in the first coordinate of  $f)(n, m) \geq f(n, m)$ , and
  - (ii) (the partial sums in the second coordinate of  $f)(n, m) \geq f(n, m)$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $\$1 \leq n$ , then (the partial sums in the first coordinate of  $f)(\$1, m) \geq f(\$1, m)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k+1]$  by [4, (51)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2]. Define  $\mathcal{Q}[\text{natural number}] \equiv$  if  $\$1 \leq m$ , then (the partial sums in the second coordinate of  $f)(n, \$1) \geq f(n, \$1)$ . For every natural number  $k$  such that  $\mathcal{Q}[k]$  holds  $\mathcal{Q}[k+1]$  by [4, (51)]. For every natural number  $k$ ,  $\mathcal{Q}[k]$  from [1, Sch. 2].  $\square$

Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and an element  $m$  of  $\mathbb{N}$ . Now we state the propositions:

- (77) Suppose there exists an element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  and  $\text{curry}(\text{the partial sums in the second coordinate of } f, k)$  is convergent to  $+\infty$ . Then
  - (i)  $\text{curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m)$  is convergent to  $+\infty$ , and
  - (ii)  $\lim \text{curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m) = +\infty$ .

PROOF: For every real number  $g$  such that  $0 < g$  there exists a natural number  $N$  such that for every natural number  $n$  such that  $N \leq n$  holds  $g \leq (\text{curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m))(n)$  by [26, (7)], (76).  $\square$

- (78) Suppose there exists an element  $k$  of  $\mathbb{N}$  such that  $k \leq m$  and  $\text{curry}'(\text{the partial sums in the first coordinate of } f, k)$  is convergent to  $+\infty$ . Then
  - (i)  $\text{curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m)$  is convergent to  $+\infty$ , and
  - (ii)  $\lim \text{curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m) = +\infty$ .

The theorem is a consequence of (40), (32), and (77).

Now we state the propositions:

(79) Let us consider a without  $-\infty$  function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  is convergent in the first coordinate to a finite limit if and only if the partial sums in the first coordinate of  $f$  is convergent in the first coordinate to a finite limit. The theorem is a consequence of (54), (47), (7), and (23).

(80) Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  is convergent in the first coordinate to a finite limit. Let us consider an element  $m$  of  $\mathbb{N}$ . Then  $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = \text{lim curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m)$ .

PROOF: The partial sums in the first coordinate of  $f$  is convergent in the first coordinate to a finite limit. Define  $\mathcal{P}[\text{natural number}] \equiv$  for every element  $k$  of  $\mathbb{N}$  such that  $k \leq \$_1$  holds  $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(k) = \text{lim curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, k)$ . For every natural number  $n$  such that  $\mathcal{P}[n]$  holds  $\mathcal{P}[n + 1]$  by [1, (13)], [14, (7)], (47), [4, (51)]. For every natural number  $n$ ,  $\mathcal{P}[n]$  from [1, Sch. 2].  $\square$

(81) Let us consider a without  $-\infty$  function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  is convergent in the second coordinate to a finite limit if and only if the partial sums in the second coordinate of  $f$  is convergent in the second coordinate to a finite limit. The theorem is a consequence of (36), (40), and (79).

(82) Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose  $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$  is convergent in the second coordinate to a finite limit. Let us consider an element  $m$  of  $\mathbb{N}$ . Then  $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = \text{lim curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m)$ . The theorem is a consequence of (36), (40), (38), (80), and (32).

Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and a sequence  $s$  of extended reals. Now we state the propositions:

(83) Suppose for every element  $m$  of  $\mathbb{N}$ ,  $s(m) = \text{lim inf curry}'(f, m)$ . Then  $\sum s \leq \text{lim inf}(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$ .

PROOF: For every element  $m$  of  $\mathbb{N}$  and for every elements  $N, n$  of  $\mathbb{N}$

such that  $n \geq N$  holds (the inferior realsequence  $\text{curry}'(f, m)(N) \leq f(n, m)$  by [26, (7), (8)]. Define  $\mathcal{F}$ (element of  $\mathbb{N}$ ) = the inferior realsequence  $\text{curry}'(f, \$_1)$ . Define  $\mathcal{G}$ (element of  $\mathbb{N}$ , element of  $\mathbb{N}$ ) = (the inferior realsequence  $\text{curry}'(f, \$_2))(\$_1)$ . Consider  $g$  being a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  such that for every element  $n$  of  $\mathbb{N}$  and for every element  $m$  of  $\mathbb{N}$ ,  $g(n, m) = \mathcal{G}(n, m)$  from [5, Sch. 4]. For every element  $m$  of  $\mathbb{N}$  and for every elements  $N, n$  of  $\mathbb{N}$  such that  $n \geq N$  holds (the partial sums in the second coordinate of  $g)(N, m) \leq$  (the partial sums in the second coordinate of  $f)(n, m)$ . For every element  $m$  of  $\mathbb{N}$  and for every elements  $N, n$  of  $\mathbb{N}$  such that  $n \geq N$  holds (the partial sums in the second coordinate of  $g)(N, m) \leq$  (the inferior realsequence the lim in the second coordinate of the partial sums in the second coordinate of  $f)(n)$  by [26, (37), (23)]. Define  $\mathcal{Q}$ [natural number]  $\equiv$  for every element  $m$  of  $\mathbb{N}$  such that  $m = \$_1$  holds  $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) = \lim \text{curry}'(\text{the partial sums in the second coordinate of } g, m)$ . For every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(\text{the partial sums in the second coordinate of } g, m)$  is convergent by [26, (7), (37)]. For every natural number  $k$  such that  $\mathcal{Q}[k]$  holds  $\mathcal{Q}[k+1]$  by [26, (37)], [4, (51), (52)], [14, (11)]. For every natural number  $k$ ,  $\mathcal{Q}[k]$  from [1, Sch. 2]. For every natural number  $m$ ,  $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) \leq \liminf(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$  by [26, (37), (38)]. For every object  $m$  such that  $m \in \text{dom } s$  holds  $0 \leq s(m)$  by [4, (51), (52)], [26, (23)].  $\square$

- (84) Suppose for every element  $m$  of  $\mathbb{N}$ ,  $s(m) = \liminf \text{curry}(f, m)$ . Then  $\sum s \leq \liminf(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)$ . The theorem is a consequence of (32), (83), (38), and (40).

Now we state the proposition:

- (85) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , a sequence  $s$  of extended reals, and natural numbers  $n, m$ . Then
  - (i) if for every natural numbers  $i, j$ ,  $f(i, j) \leq s(i)$ , then (the partial sums in the first coordinate of  $f)(n, m) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)$ , and
  - (ii) if for every natural numbers  $i, j$ ,  $f(i, j) \leq s(j)$ , then (the partial sums in the second coordinate of  $f)(n, m) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$ .

PROOF: Define  $\mathcal{P}$ [natural number]  $\equiv$  (the partial sums in the second coordinate of  $f)(n, \$_1) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 2].  $\square$

Let us consider a sequence  $s$  of extended reals and an extended real number  $r$ . Now we state the propositions:

(86) If for every natural number  $n$ ,  $s(n) \leq r$ , then  $\limsup s \leq r$ .

PROOF: Define  $\mathcal{F}(\text{element of } \mathbb{N}) = r$ . Consider  $f$  being a function from  $\mathbb{N}$  into  $\overline{\mathbb{R}}$  such that for every element  $n$  of  $\mathbb{N}$ ,  $f(n) = \mathcal{F}(n)$  from [7, Sch. 4]. For every natural number  $n$ ,  $f(n) = r$ . For every natural number  $n$ ,  $s(n) \leq f(n)$ .  $\square$

(87) If for every natural number  $n$ ,  $r \leq s(n)$ , then  $r \leq \liminf s$ .

PROOF: Define  $\mathcal{F}(\text{element of } \mathbb{N}) = r$ . Consider  $f$  being a function from  $\mathbb{N}$  into  $\overline{\mathbb{R}}$  such that for every element  $n$  of  $\mathbb{N}$ ,  $f(n) = \mathcal{F}(n)$  from [7, Sch. 4]. For every natural number  $n$ ,  $f(n) = r$ . For every natural number  $n$ ,  $f(n) \leq s(n)$ .  $\square$

Now we state the proposition:

(88) Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then

- (i) for every natural numbers  $i_1, i_2, j$  such that  $i_1 \leq i_2$  holds (the partial sums in the first coordinate of  $f)(i_1, j) \leq$  (the partial sums in the first coordinate of  $f)(i_2, j)$ , and
- (ii) for every natural numbers  $i, j_1, j_2$  such that  $j_1 \leq j_2$  holds (the partial sums in the second coordinate of  $f)(i, j_1) \leq$  (the partial sums in the second coordinate of  $f)(i, j_2)$ .

Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and natural numbers  $i, j, k$ . Now we state the propositions:

(89) Suppose for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(f, m)$  is non-decreasing and  $i \leq j$ . Then (the partial sums in the second coordinate of  $f)(i, k) \leq$  (the partial sums in the second coordinate of  $f)(j, k)$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv$  (the partial sums in the second coordinate of  $f)(i, \$_1) \leq$  (the partial sums in the second coordinate of  $f)(j, \$_1)$ . For every natural number  $n$  such that  $\mathcal{P}[n]$  holds  $\mathcal{P}[n + 1]$  by [26, (7)]. For every natural number  $n$ ,  $\mathcal{P}[n]$  from [1, Sch. 2].  $\square$

(90) Suppose for every element  $n$  of  $\mathbb{N}$ ,  $\text{curry}(f, n)$  is non-decreasing and  $i \leq j$ . Then (the partial sums in the first coordinate of  $f)(k, i) \leq$  (the partial sums in the first coordinate of  $f)(k, j)$ . The theorem is a consequence of (32), (89), and (39).

Let us consider a non-negative function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and a sequence  $s$  of extended reals. Now we state the propositions:

(91) Suppose for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}'(f, m)$  is non-decreasing and  $s(m) = \lim \text{curry}'(f, m)$ . Then

- (i) the  $\lim$  in the second coordinate of the partial sums in the second coordinate of  $f$  is non-decreasing, and

- (ii)  $\sum s = \lim(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$ .

PROOF:  $\sum s \leq \liminf(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$ . For every natural numbers  $n, m, f(n, m) \leq s(m)$  by [26, (37)], [6, (3)]. For every natural numbers  $n, m$  such that  $m \leq n$  holds (the lim in the second coordinate of the partial sums in the second coordinate of  $f)(m) \leq (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(n)$  by [26, (37)], (89), [26, (38)]. For every natural number  $n, (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(n) \leq \sum s$  by [26, (37)], [4, (39)], (87), [26, (41)].  $\limsup(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f) \leq \sum s$ .  $\square$

- (92) Suppose for every element  $m$  of  $\mathbb{N}$ ,  $\text{curry}(f, m)$  is non-decreasing and  $s(m) = \lim \text{curry}(f, m)$ . Then
  - (i) the lim in the first coordinate of the partial sums in the first coordinate of  $f$  is non-decreasing, and
  - (ii)  $\sum s = \lim(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)$ .

The theorem is a consequence of (32), (91), (33), and (40).

### 5. PRINGSHEIM SENSE CONVERGENCE FOR EXTENDED REAL-VALUED DOUBLE SEQUENCES

Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Now we state the propositions:

- (93) If  $f$  is P-convergent to  $+\infty$ , then  $f$  is not P-convergent to  $-\infty$  and  $f$  is not P-convergent to a finite limit.
- (94) If  $f$  is P-convergent to  $-\infty$ , then  $f$  is not P-convergent to  $+\infty$  and  $f$  is not P-convergent to a finite limit.

Let  $f$  be a function from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . We say that  $f$  is P-convergent if and only if

- (Def. 17)  $f$  is P-convergent to a finite limit or P-convergent to  $+\infty$  or P-convergent to  $-\infty$ .

Assume  $f$  is P-convergent. The functor P-lim  $f$  yielding an extended real is defined by

- (Def. 18) there exists a real number  $p$  such that  $it = p$  and for every real number  $e$  such that  $0 < e$  there exists a natural number  $N$  such that for every natural numbers  $n, m$  such that  $n \geq N$  and  $m \geq N$  holds  $|f(n, m) - it| < e$

and  $f$  is P-convergent to a finite limit or  $it = +\infty$  and  $f$  is P-convergent to  $+\infty$  or  $it = -\infty$  and  $f$  is P-convergent to  $-\infty$ .

Now we state the propositions:

- (95) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and a real number  $r$ . Suppose for every natural numbers  $n, m, f(n, m) = r$ . Then
  - (i)  $f$  is P-convergent to a finite limit, and
  - (ii)  $\text{P-lim } f = r$ .
- (96) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose for every natural numbers  $n_1, m_1, n_2, m_2$  such that  $n_1 \leq n_2$  and  $m_1 \leq m_2$  holds  $f(n_1, m_1) \leq f(n_2, m_2)$ . Then
  - (i)  $f$  is P-convergent, and
  - (ii)  $\text{P-lim } f = \sup \text{rng } f$ .
- (97) Let us consider functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose for every natural numbers  $n, m, f_1(n, m) \leq f_2(n, m)$ . Then  $\sup \text{rng } f_1 \leq \sup \text{rng } f_2$ .
- (98) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and natural numbers  $n, m$ . Then  $f(n, m) \leq \sup \text{rng } f$ .

Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$  and an extended real number  $K$ . Now we state the propositions:

- (99) If for every natural numbers  $n, m, f(n, m) \leq K$ , then  $\sup \text{rng } f \leq K$ .
- (100) If  $K \neq +\infty$  and for every natural numbers  $n, m, f(n, m) \leq K$ , then  $\sup \text{rng } f < +\infty$ .

Now we state the propositions:

- (101) Let us consider a without  $-\infty$  function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Then  $\sup \text{rng } f \neq +\infty$  if and only if there exists a real number  $K$  such that  $0 < K$  and for every natural numbers  $n, m, f(n, m) \leq K$ .
- (102) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and an extended real  $c$ . Suppose for every natural numbers  $n, m, f(n, m) = c$ . Then
  - (i)  $f$  is P-convergent, and
  - (ii)  $\text{P-lim } f = c$ , and
  - (iii)  $\text{P-lim } f = \sup \text{rng } f$ .
- (103) Let us consider a function  $f$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and without  $-\infty$  functions  $f_1, f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ . Suppose for every natural numbers  $n_1, m_1, n_2, m_2$  such that  $n_1 \leq n_2$  and  $m_1 \leq m_2$  holds  $f_1(n_1, m_1) \leq f_1(n_2, m_2)$  and for every natural numbers  $n_1, m_1, n_2, m_2$  such that  $n_1 \leq n_2$  and  $m_1 \leq m_2$  holds  $f_2(n_1, m_1) \leq f_2(n_2, m_2)$  and for every natural numbers  $n, m, f_1(n, m) + f_2(n, m) = f(n, m)$ . Then

- (i)  $f$  is P-convergent, and
- (ii)  $\text{P-lim } f = \sup \text{rng } f$ , and
- (iii)  $\text{P-lim } f = \text{P-lim } f_1 + \text{P-lim } f_2$ , and
- (iv)  $\sup \text{rng } f = \sup \text{rng } f_1 + \sup \text{rng } f_2$ .

The theorem is a consequence of (96) and (101).

Let us consider a without  $-\infty$  function  $f_1$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , a function  $f_2$  from  $\mathbb{N} \times \mathbb{N}$  into  $\overline{\mathbb{R}}$ , and a real number  $c$ . Now we state the propositions:

(104) Suppose  $0 \leq c$  and for every natural numbers  $n, m$ ,  $f_2(n, m) = c \cdot f_1(n, m)$ . Then

- (i)  $\sup \text{rng } f_2 = c \cdot \sup \text{rng } f_1$ , and
- (ii)  $f_2$  is without  $-\infty$ .

The theorem is a consequence of (102) and (101).

(105) Suppose  $0 \leq c$  and for every natural numbers  $n_1, m_1, n_2, m_2$  such that  $n_1 \leq n_2$  and  $m_1 \leq m_2$  holds  $f_1(n_1, m_1) \leq f_1(n_2, m_2)$  and for every natural numbers  $n, m$ ,  $f_2(n, m) = c \cdot f_1(n, m)$ . Then

- (i) for every natural numbers  $n_1, m_1, n_2, m_2$  such that  $n_1 \leq n_2$  and  $m_1 \leq m_2$  holds  $f_2(n_1, m_1) \leq f_2(n_2, m_2)$ , and
- (ii)  $f_2$  is without  $-\infty$  and P-convergent, and
- (iii)  $\text{P-lim } f_2 = \sup \text{rng } f_2$ , and
- (iv)  $\text{P-lim } f_2 = c \cdot \text{P-lim } f_1$ .

The theorem is a consequence of (96) and (104).

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