

Extended Real-Valued Double Sequence and Its Convergence¹

Noboru Endou
Gifu National College of Technology
Gifu, Japan

Summary. In this article we introduce the convergence of extended real-valued double sequences [16], [17]. It is similar to our previous articles [15], [10]. In addition, we also prove Fatou's lemma and the monotone convergence theorem for double sequences.

MSC: 40A05 40B05 03B35

Keywords: double sequence; Fatou's lemma for double sequence; monotone convergence theorem for double sequence

MML identifier: DBLSEQ_3, version: 8.1.04 5.32.1246

The notation and terminology used in this paper have been introduced in the following articles: [5], [21], [15], [10], [12], [6], [7], [22], [13], [11], [14], [1], [2], [8], [18], [24], [25], [26], [20], [23], [3], [4], and [9].

1. PRELIMINARIES

Let X be a non empty set. One can verify that there exists a function from X into \mathbb{R} which is non-negative and non-positive and there exists a function from X into $\overline{\mathbb{R}}$ which is without $-\infty$, without $+\infty$, non-negative, and non-positive and every function from X into $\overline{\mathbb{R}}$ which is non-negative is also without $-\infty$ and every function from X into $\overline{\mathbb{R}}$ which is non-positive is also without $+\infty$ and there exists a without $+\infty$ function from X into $\overline{\mathbb{R}}$ which is without $-\infty$.

Let f be a function from X into $\overline{\mathbb{R}}$. Let us observe that the functor $-f$ yields a function from X into $\overline{\mathbb{R}}$. Let f be a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Note that $-f$ is without $+\infty$.

¹This work was supported by JSPS KAKENHI 23500029.

Let f be a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Let us observe that $-f$ is without $-\infty$.

Let f be a non-negative function from X into $\overline{\mathbb{R}}$. Note that $-f$ is non-positive.

Let f be a non-positive function from X into $\overline{\mathbb{R}}$. Let us observe that $-f$ is non-negative.

Let A, B be non empty sets and f be a without $-\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. Let us observe that f^T is without $-\infty$.

Let f be a without $+\infty$ function from $A \times B$ into $\overline{\mathbb{R}}$. One can verify that f^T is without $+\infty$.

Let f be a non-negative function from $A \times B$ into $\overline{\mathbb{R}}$. One can check that f^T is non-negative.

Let f be a non-positive function from $A \times B$ into $\overline{\mathbb{R}}$. Note that f^T is non-positive.

Now we state the propositions:

- (1) Let us consider a sequence s of extended reals. Then $(\sum_{\alpha=0}^{\kappa} (-s)(\alpha))_{\kappa \in \mathbb{N}} = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$.

PROOF: Define \mathcal{Q} [natural number] \equiv

$(-\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1) = -(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1)$. For every natural number n , $\mathcal{Q}[n]$ from [1, Sch. 2]. Define \mathcal{P} [natural number] $\equiv (\sum_{\alpha=0}^{\kappa} (-s)(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1) = (-\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

- (2) Let us consider a non empty set X , and a partial function f from X to $\overline{\mathbb{R}}$. Then $--f = f$.
- (3) Let us consider non empty sets X, Y , and a function f from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(-f)^T = -f^T$.

Let s be a non-negative sequence of extended reals. One can verify that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is non-negative.

Let s be a non-positive sequence of extended reals. Let us observe that $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is non-positive.

Now we state the propositions:

- (4) Let us consider a non-negative sequence s of extended reals, and a natural number m . Then $s(m) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$.

PROOF: Define \mathcal{P} [natural number] $\equiv s(\mathbb{S}_1) \leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\mathbb{S}_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [4, (51)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

- (5) Let us consider a non-positive sequence s of extended reals, and a natural number m . Then $s(m) \geq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$. The theorem is a consequence of (4), (1), and (2).

- (6) Let us consider a non empty set X . Then every without $-\infty$, without $+\infty$ function from X into $\overline{\mathbb{R}}$ is a function from X into \mathbb{R} .

Let X be a non empty set and f_1, f_2 be without $-\infty$ functions from X into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Let f_1, f_2 be without $+\infty$ functions from X into $\overline{\mathbb{R}}$. One can verify that the functor $f_1 + f_2$ yields a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Let f_1 be a without $-\infty$ function from X into $\overline{\mathbb{R}}$ and f_2 be a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Let us observe that the functor $f_1 - f_2$ yields a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Let f_1 be a without $+\infty$ function from X into $\overline{\mathbb{R}}$ and f_2 be a without $-\infty$ function from X into $\overline{\mathbb{R}}$. Observe that the functor $f_1 - f_2$ yields a without $+\infty$ function from X into $\overline{\mathbb{R}}$. Now we state the propositions:

- (7) Let us consider a non empty set X , an element x of X , and functions f_1, f_2 from X into $\overline{\mathbb{R}}$. Then
- (i) if f_1 is without $-\infty$ and f_2 is without $-\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and
 - (ii) if f_1 is without $+\infty$ and f_2 is without $+\infty$, then $(f_1 + f_2)(x) = f_1(x) + f_2(x)$, and
 - (iii) if f_1 is without $-\infty$ and f_2 is without $+\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$, and
 - (iv) if f_1 is without $+\infty$ and f_2 is without $-\infty$, then $(f_1 - f_2)(x) = f_1(x) - f_2(x)$.

- (8) Let us consider a non empty set X , and without $-\infty$ functions f_1, f_2 from X into $\overline{\mathbb{R}}$. Then
- (i) $f_1 + f_2 = f_1 - -f_2$, and
 - (ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

- (9) Let us consider a non empty set X , and without $+\infty$ functions f_1, f_2 from X into $\overline{\mathbb{R}}$. Then
- (i) $f_1 + f_2 = f_1 - -f_2$, and
 - (ii) $-(f_1 + f_2) = -f_1 - f_2$.

The theorem is a consequence of (7).

- (10) Let us consider a non empty set X , a without $-\infty$ function f_1 from X into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from X into $\overline{\mathbb{R}}$. Then
- (i) $f_1 - f_2 = f_1 + -f_2$, and
 - (ii) $f_2 - f_1 = f_2 + -f_1$, and
 - (iii) $-(f_1 - f_2) = -f_1 + f_2$, and

$$(iv) \quad -(f_2 - f_1) = -f_2 + f_1.$$

The theorem is a consequence of (8), (2), and (9).

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and n, m be natural numbers. One can check that the functor $f(n, m)$ yields an element of $\overline{\mathbb{R}}$. Now we state the propositions:

(11) Let us consider without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(12) Let us consider without $+\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then $(f_1 + f_2)(n, m) = f_1(n, m) + f_2(n, m)$. The theorem is a consequence of (7).

(13) Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a without $+\infty$ function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then

$$(i) \quad (f_1 - f_2)(n, m) = f_1(n, m) - f_2(n, m), \text{ and}$$

$$(ii) \quad (f_2 - f_1)(n, m) = f_2(n, m) - f_1(n, m).$$

The theorem is a consequence of (7).

(14) Let us consider non empty sets X, Y , and without $-\infty$ functions f_1, f_2 from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^T = f_1^T + f_2^T$. The theorem is a consequence of (7).

(15) Let us consider non empty sets X, Y , and without $+\infty$ functions f_1, f_2 from $X \times Y$ into $\overline{\mathbb{R}}$. Then $(f_1 + f_2)^T = f_1^T + f_2^T$. The theorem is a consequence of (7).

(16) Let us consider non empty sets X, Y , a without $-\infty$ function f_1 from $X \times Y$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from $X \times Y$ into $\overline{\mathbb{R}}$. Then

$$(i) \quad (f_1 - f_2)^T = f_1^T - f_2^T, \text{ and}$$

$$(ii) \quad (f_2 - f_1)^T = f_2^T - f_1^T.$$

The theorem is a consequence of (7).

One can verify that every sequence of extended reals which is convergent to $+\infty$ is also convergent and every sequence of extended reals which is convergent to $-\infty$ is also convergent and every sequence of extended reals which is convergent to a finite limit is also convergent and there exists a sequence of extended reals which is convergent and there exists a without $-\infty$ sequence of extended reals which is convergent and there exists a without $+\infty$ sequence of extended reals which is convergent.

Now we state the proposition:

(17) Let us consider a convergent sequence s of extended reals. Then

- (i) s is convergent to a finite limit iff $-s$ is convergent to a finite limit, and
- (ii) s is convergent to $+\infty$ iff $-s$ is convergent to $-\infty$, and
- (iii) s is convergent to $-\infty$ iff $-s$ is convergent to $+\infty$, and
- (iv) $-s$ is convergent, and
- (v) $\lim(-s) = -\lim s$.

The theorem is a consequence of (2).

Let us consider without $-\infty$ sequences s_1, s_2 of extended reals. Now we state the propositions:

(18) Suppose s_1 is convergent to $+\infty$ and s_2 is convergent to $+\infty$. Then

- (i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and
- (ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

(19) Suppose s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit. Then

- (i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and
- (ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Now we state the proposition:

(20) Let us consider without $+\infty$ sequences s_1, s_2 of extended reals. Suppose s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit. Then

- (i) $s_1 + s_2$ is convergent to $+\infty$ and convergent, and
- (ii) $\lim(s_1 + s_2) = +\infty$.

The theorem is a consequence of (7).

Let us consider without $-\infty$ sequences s_1, s_2 of extended reals. Now we state the propositions:

(21) Suppose s_1 is convergent to $-\infty$ and s_2 is convergent to $-\infty$. Then

- (i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and
- (ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(22) Suppose s_1 is convergent to $-\infty$ and s_2 is convergent to a finite limit. Then

- (i) $s_1 + s_2$ is convergent to $-\infty$ and convergent, and
- (ii) $\lim(s_1 + s_2) = -\infty$.

The theorem is a consequence of (7).

(23) Suppose s_1 is convergent to a finite limit and s_2 is convergent to a finite limit. Then

- (i) $s_1 + s_2$ is convergent to a finite limit and convergent, and
- (ii) $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (7).

Now we state the propositions:

(24) Let us consider without $+\infty$ sequences s_1, s_2 of extended reals. Then

- (i) if s_1 is convergent to $+\infty$ and s_2 is convergent to $+\infty$, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and
- (ii) if s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit, then $s_1 + s_2$ is convergent to $+\infty$ and convergent and $\lim(s_1 + s_2) = +\infty$, and
- (iii) if s_1 is convergent to $-\infty$ and s_2 is convergent to $-\infty$, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and
- (iv) if s_1 is convergent to $-\infty$ and s_2 is convergent to a finite limit, then $s_1 + s_2$ is convergent to $-\infty$ and convergent and $\lim(s_1 + s_2) = -\infty$, and
- (v) if s_1 is convergent to a finite limit and s_2 is convergent to a finite limit, then $s_1 + s_2$ is convergent to a finite limit and convergent and $\lim(s_1 + s_2) = \lim s_1 + \lim s_2$.

The theorem is a consequence of (17), (21), (10), (9), (2), (22), (18), (19), and (23).

(25) Let us consider a without $-\infty$ sequence s_1 of extended reals, and a without $+\infty$ sequence s_2 of extended reals. Then

- (i) if s_1 is convergent to $+\infty$ and s_2 is convergent to $-\infty$, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and
- (ii) if s_1 is convergent to $+\infty$ and s_2 is convergent to a finite limit, then $s_1 - s_2$ is convergent to $+\infty$ and convergent and $s_2 - s_1$ is convergent to $-\infty$ and convergent and $\lim(s_1 - s_2) = +\infty$ and $\lim(s_2 - s_1) = -\infty$, and
- (iii) if s_1 is convergent to $-\infty$ and s_2 is convergent to a finite limit, then $s_1 - s_2$ is convergent to $-\infty$ and convergent and $s_2 - s_1$ is convergent to $+\infty$ and convergent and $\lim(s_1 - s_2) = -\infty$ and $\lim(s_2 - s_1) = +\infty$, and

- (iv) if s_1 is convergent to a finite limit and s_2 is convergent to a finite limit, then $s_1 - s_2$ is convergent to a finite limit and convergent and $s_2 - s_1$ is convergent to a finite limit and convergent and $\lim(s_1 - s_2) = \lim s_1 - \lim s_2$ and $\lim(s_2 - s_1) = \lim s_2 - \lim s_1$.

The theorem is a consequence of (17), (24), (18), (10), (19), (22), (23), and (2).

2. SUBSEQUENCES OF CONVERGENT EXTENDED REAL-VALUED SEQUENCES

Let us consider sequences s_1, s_2 of extended reals. Now we state the propositions:

- (26) Suppose s_2 is a subsequence of s_1 and s_1 is convergent to a finite limit. Then

- (i) s_2 is convergent to a finite limit, and
- (ii) $\lim s_1 = \lim s_2$.

PROOF: Consider g being a real number such that $\lim s_1 = g$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - \lim s_1| < p$ and s_1 is convergent to a finite limit. Reconsider $L = \lim s_1$ as an extended real number. There exists a real number g such that for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|(s_2(m) - g \text{ qua extended real})| < p$ by [19, (14)], [7, (15)]. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_2(m) - L| < p$ by [19, (14)], [7, (15)]. \square

- (27) Suppose s_2 is a subsequence of s_1 and s_1 is convergent to $+\infty$. Then

- (i) s_2 is convergent to $+\infty$, and
- (ii) $\lim s_2 = +\infty$.

- (28) Suppose s_2 is a subsequence of s_1 and s_1 is convergent to $-\infty$. Then

- (i) s_2 is convergent to $-\infty$, and
- (ii) $\lim s_2 = -\infty$.

3. CONVERGENCY FOR EXTENDED REAL-VALUED DOUBLE SEQUENCES

Let us consider a function R from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . Now we state the propositions:

- (29) Suppose the lim in the first coordinate of R is convergent. Then the first coordinate major iterated lim of $R = \lim(\text{the lim in the first coordinate of } R)$.
- (30) Suppose the lim in the second coordinate of R is convergent. Then the second coordinate major iterated lim of $R = \lim(\text{the lim in the second coordinate of } R)$.

Let E be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that E is P-convergent to a finite limit if and only if

- (Def. 1) there exists a real number p such that for every real number e such that $0 < e$ there exists a natural number N such that for every natural numbers n, m such that $n \geq N$ and $m \geq N$ holds $|E(n, m) - (p \text{ qua extended real})| < e$.

We say that E is P-convergent to $+\infty$ if and only if

- (Def. 2) for every real number g such that $0 < g$ there exists a natural number N such that for every natural numbers n, m such that $n \geq N$ and $m \geq N$ holds $g \leq E(n, m)$.

We say that E is P-convergent to $-\infty$ if and only if

- (Def. 3) for every real number g such that $g < 0$ there exists a natural number N such that for every natural numbers n, m such that $n \geq N$ and $m \geq N$ holds $E(n, m) \leq g$.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that f is convergent in the first coordinate to $+\infty$ if and only if

- (Def. 4) for every element m of \mathbb{N} , $\text{curry}'(f, m)$ is convergent to $+\infty$.

We say that f is convergent in the first coordinate to $-\infty$ if and only if

- (Def. 5) for every element m of \mathbb{N} , $\text{curry}'(f, m)$ is convergent to $-\infty$.

We say that f is convergent in the first coordinate to a finite limit if and only if

- (Def. 6) for every element m of \mathbb{N} , $\text{curry}'(f, m)$ is convergent to a finite limit.

We say that f is convergent in the first coordinate if and only if

- (Def. 7) for every element m of \mathbb{N} , $\text{curry}'(f, m)$ is convergent.

We say that f is convergent in the second coordinate to $+\infty$ if and only if

- (Def. 8) for every element m of \mathbb{N} , $\text{curry}(f, m)$ is convergent to $+\infty$.

We say that f is convergent in the second coordinate to $-\infty$ if and only if

- (Def. 9) for every element m of \mathbb{N} , $\text{curry}(f, m)$ is convergent to $-\infty$.

We say that f is convergent in the second coordinate to a finite limit if and only if

(Def. 10) for every element m of \mathbb{N} , $\text{curry}(f, m)$ is convergent to a finite limit.

We say that f is convergent in the second coordinate if and only if

(Def. 11) for every element m of \mathbb{N} , $\text{curry}(f, m)$ is convergent.

Now we state the propositions:

(31) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

(i) if f is convergent in the first coordinate to $+\infty$ or convergent in the first coordinate to $-\infty$ or convergent in the first coordinate to a finite limit, then f is convergent in the first coordinate, and

(ii) if f is convergent in the second coordinate to $+\infty$ or convergent in the second coordinate to $-\infty$ or convergent in the second coordinate to a finite limit, then f is convergent in the second coordinate.

(32) Let us consider non empty sets X, Y, Z , a function F from $X \times Y$ into Z , and an element x of X . Then $\text{curry}(F, x) = \text{curry}'(F^T, x)$.

(33) Let us consider non empty sets X, Y, Z , a function F from $X \times Y$ into Z , and an element y of Y . Then $\text{curry}'(F, y) = \text{curry}(F^T, y)$.

(34) Let us consider non empty sets X, Y , a function F from $X \times Y$ into $\overline{\mathbb{R}}$, and an element x of X . Then $\text{curry}(-F, x) = -\text{curry}(F, x)$.

(35) Let us consider non empty sets X, Y , a function F from $X \times Y$ into $\overline{\mathbb{R}}$, and an element y of Y . Then $\text{curry}'(-F, y) = -\text{curry}'(F, y)$.

Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(36) (i) f is convergent in the first coordinate to $+\infty$ iff f^T is convergent in the second coordinate to $+\infty$, and

(ii) f is convergent in the second coordinate to $+\infty$ iff f^T is convergent in the first coordinate to $+\infty$, and

(iii) f is convergent in the first coordinate to $-\infty$ iff f^T is convergent in the second coordinate to $-\infty$, and

(iv) f is convergent in the second coordinate to $-\infty$ iff f^T is convergent in the first coordinate to $-\infty$, and

(v) f is convergent in the first coordinate to a finite limit iff f^T is convergent in the second coordinate to a finite limit, and

(vi) f is convergent in the second coordinate to a finite limit iff f^T is convergent in the first coordinate to a finite limit.

The theorem is a consequence of (33) and (32).

(37) (i) f is convergent in the first coordinate to $+\infty$ iff $-f$ is convergent in the first coordinate to $-\infty$, and

- (ii) f is convergent in the first coordinate to $-\infty$ iff $-f$ is convergent in the first coordinate to $+\infty$, and
- (iii) f is convergent in the first coordinate to a finite limit iff $-f$ is convergent in the first coordinate to a finite limit, and
- (iv) f is convergent in the second coordinate to $+\infty$ iff $-f$ is convergent in the second coordinate to $-\infty$, and
- (v) f is convergent in the second coordinate to $-\infty$ iff $-f$ is convergent in the second coordinate to $+\infty$, and
- (vi) f is convergent in the second coordinate to a finite limit iff $-f$ is convergent in the second coordinate to a finite limit.

The theorem is a consequence of (35), (17), (2), and (34).

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functors: the lim in the first coordinate of f and the lim in the second coordinate of f yielding sequences of extended reals are defined by conditions

(Def. 12) for every element m of \mathbb{N} , the lim in the first coordinate of $f(m) = \text{lim curry}'(f, m)$,

(Def. 13) for every element n of \mathbb{N} , the lim in the second coordinate of $f(n) = \text{lim curry}(f, n)$,

respectively. Now we state the proposition:

(38) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

- (i) the lim in the first coordinate of $f =$ the lim in the second coordinate of f^T , and
- (ii) the lim in the second coordinate of $f =$ the lim in the first coordinate of f^T .

The theorem is a consequence of (33) and (32).

Let X, Y be non empty sets, F be a without $+\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$, and x be an element of X . Let us observe that $\text{curry}(F, x)$ is without $+\infty$.

Let y be an element of Y . One can verify that $\text{curry}'(F, y)$ is without $+\infty$.

Let F be a without $-\infty$ function from $X \times Y$ into $\overline{\mathbb{R}}$ and x be an element of X . Let us note that $\text{curry}(F, x)$ is without $-\infty$.

Let y be an element of Y . Observe that $\text{curry}'(F, y)$ is without $-\infty$.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The partial sums in the second coordinate of f yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 14) for every natural numbers n, m , $it(n, 0) = f(n, 0)$ and $it(n, m + 1) = it(n, m) + f(n, m + 1)$.

The partial sums in the first coordinate of f yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by

(Def. 15) for every natural numbers n, m , $it(0, m) = f(0, m)$ and $it(n + 1, m) = it(n, m) + f(n + 1, m)$.

Let f be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the second coordinate of f is without $-\infty$.

Let f be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the second coordinate of f is without $+\infty$.

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the second coordinate of f is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the second coordinate of f is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a without $-\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us note that the partial sums in the first coordinate of f is without $-\infty$.

Let f be a without $+\infty$ function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Observe that the partial sums in the first coordinate of f is without $+\infty$.

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Let us observe that the partial sums in the first coordinate of f is non-negative as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a non-positive function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. One can check that the partial sums in the first coordinate of f is non-positive as a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. The functor $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ is defined by the term

(Def. 16) the partial sums in the second coordinate of the partial sums in the first coordinate of f .

Now we state the propositions:

(39) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then

- (i) (the partial sums in the first coordinate of f)(n, m) = (the partial sums in the second coordinate of f^T)(m, n), and
- (ii) (the partial sums in the second coordinate of f)(n, m) = (the partial sums in the first coordinate of f^T)(m, n).

PROOF: Define \mathcal{P} [natural number] \equiv (the partial sums in the first coordinate of f)($\$1, m$) = (the partial sums in the second coordinate of f^T)($m, \$1$). For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. Define \mathcal{Q} [natural number] \equiv (the partial sums in the second coordinate of f)($n, \$1$) = (the partial sums in the first

coordinate of f^T)($\$1, n$). For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k + 1]$. For every natural number k , $\mathcal{Q}[k]$ from [1, Sch. 2]. \square

(40) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

- (i) (the partial sums in the first coordinate of f) T = the partial sums in the second coordinate of f^T , and
- (ii) (the partial sums in the second coordinate of f) T = the partial sums in the first coordinate of f^T .

The theorem is a consequence of (39).

(41) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, an extended real-valued function g , and a natural number n . Suppose for every natural number k , (the partial sums in the first coordinate of f)(n, k) = $g(k)$. Then

- (i) for every natural number k , $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, k) = (\sum_{\alpha=0}^{\kappa} g(\alpha))_{\kappa \in \mathbb{N}}(k)$, and
- (ii) (the lim in the second coordinate of $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$)(n) = $\sum g$.

(42) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

- (i) the partial sums in the second coordinate of $-f$ = $-($ the partial sums in the second coordinate of f), and
- (ii) the partial sums in the first coordinate of $-f$ = $-($ the partial sums in the first coordinate of f).

PROOF: For every element z of $\mathbb{N} \times$

\mathbb{N} , $(-($ the partial sums in the second coordinate of f))(z) = (the partial sums in the second coordinate of $-f$)(z) by [9, (87)]. For every element z of $\mathbb{N} \times \mathbb{N}$,

$(-($ the partial sums in the first coordinate of f))(z) = (the partial sums in the first coordinate of $-f$)(z) by [9, (87)]. \square

(43) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and elements m, n of \mathbb{N} . Then

- (i) (the partial sums in the first coordinate of f)(m, n) = $(\sum_{\alpha=0}^{\kappa} (\text{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(m)$, and
- (ii) (the partial sums in the second coordinate of f)(m, n) = $(\sum_{\alpha=0}^{\kappa} (\text{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(n)$.

PROOF: Define \mathcal{P} [natural number] \equiv (the partial sums in the first coordinate of f)($\$1, n$) = $(\sum_{\alpha=0}^{\kappa} (\text{curry}'(f, n))(\alpha))_{\kappa \in \mathbb{N}}(\$1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. Define \mathcal{Q} [natural number] \equiv (the partial sums in the second coordinate of f)($m, \$1$) = $(\sum_{\alpha=0}^{\kappa} (\text{curry}(f, m))(\alpha))_{\kappa \in \mathbb{N}}(\$1)$. For

every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k + 1]$. For every natural number k , $\mathcal{Q}[k]$ from [1, Sch. 2]. \square

- (44) Let us consider without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then
- (i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of f_1) + (the partial sums in the second coordinate of f_2), and
 - (ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of f_1) + (the partial sums in the first coordinate of f_2).

The theorem is a consequence of (11).

- (45) Let us consider without $+\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then
- (i) the partial sums in the second coordinate of $f_1 + f_2 =$ (the partial sums in the second coordinate of f_1) + (the partial sums in the second coordinate of f_2), and
 - (ii) the partial sums in the first coordinate of $f_1 + f_2 =$ (the partial sums in the first coordinate of f_1) + (the partial sums in the first coordinate of f_2).

The theorem is a consequence of (10), (9), (2), (42), (44), and (8).

- (46) Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then
- (i) the partial sums in the second coordinate of $f_1 - f_2 =$ (the partial sums in the second coordinate of f_1) – (the partial sums in the second coordinate of f_2), and
 - (ii) the partial sums in the first coordinate of $f_1 - f_2 =$ (the partial sums in the first coordinate of f_1) – (the partial sums in the first coordinate of f_2), and
 - (iii) the partial sums in the second coordinate of $f_2 - f_1 =$ (the partial sums in the second coordinate of f_2) – (the partial sums in the second coordinate of f_1), and
 - (iv) the partial sums in the first coordinate of $f_2 - f_1 =$ (the partial sums in the first coordinate of f_2) – (the partial sums in the first coordinate of f_1).

The theorem is a consequence of (10), (44), (42), and (45).

- (47) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then
- (i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n+1, m) =$ (the partial sums in the second coordinate of f)($n + 1, m$) + $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and

- (ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of f)($n, m + 1$) = (the partial sums in the first coordinate of f)($n, m + 1$) + (the partial sums in the first coordinate of the partial sums in the second coordinate of f)(n, m).

PROOF: Set $R_1 = (\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$. Set $C_1 =$ the partial sums in the first coordinate of the partial sums in the second coordinate of f . Set $R_2 =$ the partial sums in the first coordinate of f . Set $C_2 =$ the partial sums in the second coordinate of f . Define \mathcal{P} [natural number] $\equiv R_1(n + 1, \$1) = C_2(n + 1, \$1) + R_1(n, \$1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. Define \mathcal{Q} [natural number] $\equiv C_1(\$1, m + 1) = R_2(\$1, m + 1) + C_1(\$1, m)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k + 1]$. For every natural number k , $\mathcal{Q}[k]$ from [1, Sch. 2]. \square

- (48) Let us consider a without $+\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then
 - (i) $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n + 1, m) =$ (the partial sums in the second coordinate of f)($n + 1, m$) + $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}(n, m)$, and
 - (ii) (the partial sums in the first coordinate of the partial sums in the second coordinate of f)($n, m + 1$) = (the partial sums in the first coordinate of f)($n, m + 1$) + (the partial sums in the first coordinate of the partial sums in the second coordinate of f)(n, m).

The theorem is a consequence of (2), (42), and (47).

- (49) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ or without $+\infty$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}} =$ the partial sums in the first coordinate of the partial sums in the second coordinate of f .
- (50) Let us consider without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$. The theorem is a consequence of (44).
- (51) Let us consider without $+\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} (f_1 + f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} + (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$. The theorem is a consequence of (45).
- (52) Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a without $+\infty$ function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then
 - (i) $(\sum_{\alpha=0}^{\kappa} (f_1 - f_2)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}}$, and
 - (ii) $(\sum_{\alpha=0}^{\kappa} (f_2 - f_1)(\alpha))_{\kappa \in \mathbb{N}} = (\sum_{\alpha=0}^{\kappa} f_2(\alpha))_{\kappa \in \mathbb{N}} - (\sum_{\alpha=0}^{\kappa} f_1(\alpha))_{\kappa \in \mathbb{N}}$.

The theorem is a consequence of (46).

- (53) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element k of \mathbb{N} . Then

- (i) $\text{curry}'(\text{the partial sums in the first coordinate of } f, k) = (\sum_{\alpha=0}^{\kappa}(\text{curry}'(f, k))(\alpha))_{\kappa \in \mathbb{N}}$, and
- (ii) $\text{curry}(\text{the partial sums in the second coordinate of } f, k) = (\sum_{\alpha=0}^{\kappa}(\text{curry}(f, k))(\alpha))_{\kappa \in \mathbb{N}}$.

The theorem is a consequence of (43).

(54) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose f is without $-\infty$ or without $+\infty$. Then

- (i) $\text{curry}((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0) = \text{curry}(\text{the partial sums in the second coordinate of } f, 0)$, and
- (ii) $\text{curry}'((\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}, 0) = \text{curry}'(\text{the partial sums in the first coordinate of } f, 0)$.

(55) Let us consider non empty sets C, D , without $-\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element c of C . Then $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$. The theorem is a consequence of (7).

(56) Let us consider non empty sets C, D , without $-\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element d of D . Then $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$. The theorem is a consequence of (7).

(57) Let us consider non empty sets C, D , without $+\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element c of C . Then $\text{curry}(F_1 + F_2, c) = \text{curry}(F_1, c) + \text{curry}(F_2, c)$. The theorem is a consequence of (7).

(58) Let us consider non empty sets C, D , without $+\infty$ functions F_1, F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element d of D . Then $\text{curry}'(F_1 + F_2, d) = \text{curry}'(F_1, d) + \text{curry}'(F_2, d)$. The theorem is a consequence of (7).

(59) Let us consider non empty sets C, D , a without $-\infty$ function F_1 from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element c of C . Then

- (i) $\text{curry}(F_1 - F_2, c) = \text{curry}(F_1, c) - \text{curry}(F_2, c)$, and
- (ii) $\text{curry}(F_2 - F_1, c) = \text{curry}(F_2, c) - \text{curry}(F_1, c)$.

The theorem is a consequence of (7).

(60) Let us consider non empty sets C, D , a without $-\infty$ function F_1 from $C \times D$ into $\overline{\mathbb{R}}$, a without $+\infty$ function F_2 from $C \times D$ into $\overline{\mathbb{R}}$, and an element d of D . Then

- (i) $\text{curry}'(F_1 - F_2, d) = \text{curry}'(F_1, d) - \text{curry}'(F_2, d)$, and
- (ii) $\text{curry}'(F_2 - F_1, d) = \text{curry}'(F_2, d) - \text{curry}'(F_1, d)$.

The theorem is a consequence of (7).

4. NON-NEGATIVE EXTENDED REAL-VALUED DOUBLE SEQUENCES

Now we state the propositions:

- (61) Let us consider a non-negative sequence s of extended reals. Suppose $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}$ is not convergent to $+\infty$. Let us consider a natural number n . Then $s(n)$ is a real number.
- (62) Let us consider a non-negative sequence s of extended reals. Suppose s is non-decreasing. Then s is convergent to $+\infty$ or convergent to a finite limit.

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and n be an element of \mathbb{N} . Let us observe that $\text{curry}(f, n)$ is non-negative and $\text{curry}'(f, n)$ is non-negative.

Now we state the propositions:

- (63) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element m of \mathbb{N} . Then $\text{curry}(\text{the partial sums in the second coordinate of } f, m)$ is non-decreasing.
 PROOF: Set $P = \text{curry}(\text{the partial sums in the second coordinate of } f, m)$. For every natural numbers n, j such that $j \leq n$ holds $P(j) \leq P(n)$ by [4, (51)], [1, (13), (20)]. \square
- (64) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an element n of \mathbb{N} . Then $\text{curry}'(\text{the partial sums in the first coordinate of } f, n)$ is non-decreasing. The theorem is a consequence of (63), (40), and (33).

Let f be a non-negative function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and m be an element of \mathbb{N} . One can check that $\text{curry}(\text{the partial sums in the second coordinate of } f, m)$ is non-decreasing and $\text{curry}'(\text{the partial sums in the first coordinate of } f, m)$ is non-decreasing.

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

- (65) (i) if f is convergent in the first coordinate, then the lim in the first coordinate of f is non-negative, and
 (ii) if f is convergent in the second coordinate, then the lim in the second coordinate of f is non-negative.
- (66) (i) the partial sums in the first coordinate of f is convergent in the first coordinate, and
 (ii) the partial sums in the second coordinate of f is convergent in the second coordinate.

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, an element m of \mathbb{N} , and a natural number n .

Let us assume that $\text{curry}'(\text{the partial sums in the first coordinate of } f, m)$ is not convergent to $+\infty$. Now we state the propositions:

(67) $f(n, m)$ is a real number.

(68) $f(m, n)$ is a real number.

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers n, m . Now we state the propositions:

(69) Suppose for every natural number i such that $i \leq n$ holds $f(i, m)$ is a real number. Then (the partial sums in the first coordinate of f)(n, m) $< +\infty$.

PROOF: Define \mathcal{P} [natural number] \equiv if $\$1 \leq n$, then (the partial sums in the first coordinate of f)($\$1, m$) $< +\infty$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [4, (51)], [1, (13)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

(70) Suppose for every natural number i such that $i \leq m$ holds $f(n, i)$ is a real number. Then (the partial sums in the second coordinate of f)(n, m) $< +\infty$.

PROOF: Define \mathcal{P} [natural number] \equiv if $\$1 \leq m$, then (the partial sums in the second coordinate of f)($n, \$1$) $< +\infty$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [4, (51)], [1, (13)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

Now we state the proposition:

(71) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to $-\infty$. Then there exists an element m of \mathbb{N} such that curry' (the partial sums in the first coordinate of f, m) is convergent to $-\infty$. The theorem is a consequence of (54).

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a natural number m . Now we state the propositions:

(72) for every element k of \mathbb{N} such that $k \leq m$ holds curry (the partial sums in the second coordinate of f, k) is not convergent to $+\infty$ if and only if for every element k of \mathbb{N} such that $k \leq m$ holds $\lim \text{curry}$ (the partial sums in the second coordinate of f, k) $< +\infty$. The theorem is a consequence of (62).

(73) for every element k of \mathbb{N} such that $k \leq m$ holds curry' (the partial sums in the first coordinate of f, k) is not convergent to $+\infty$ if and only if for every element k of \mathbb{N} such that $k \leq m$ holds $\lim \text{curry}'$ (the partial sums in the first coordinate of f, k) $< +\infty$. The theorem is a consequence of (62).

(74) $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element k of \mathbb{N} such that $k \leq m$ and curry (the partial sums in the second coordinate

of f, k is convergent to $+\infty$. The theorem is a consequence of (72), (65), and (4).

- (75) $(\sum_{\alpha=0}^{\kappa}(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = +\infty$ if and only if there exists an element k of \mathbb{N} such that $k \leq m$ and $\text{curry}'(\text{the partial sums in the first coordinate of } f, k)$ is convergent to $+\infty$. The theorem is a consequence of (38), (40), (74), and (32).

Now we state the proposition:

- (76) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then

- (i) (the partial sums in the first coordinate of $f)(n, m) \geq f(n, m)$, and
- (ii) (the partial sums in the second coordinate of $f)(n, m) \geq f(n, m)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq n$, then (the partial sums in the first coordinate of $f)(\$1, m) \geq f(\$1, m)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [4, (51)]. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv$ if $\$1 \leq m$, then (the partial sums in the second coordinate of $f)(n, \$1) \geq f(n, \$1)$. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [4, (51)]. For every natural number k , $\mathcal{Q}[k]$ from [1, Sch. 2]. \square

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and an element m of \mathbb{N} . Now we state the propositions:

- (77) Suppose there exists an element k of \mathbb{N} such that $k \leq m$ and $\text{curry}(\text{the partial sums in the second coordinate of } f, k)$ is convergent to $+\infty$. Then
- (i) $\text{curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m)$ is convergent to $+\infty$, and
 - (ii) $\lim \text{curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m) = +\infty$.

PROOF: For every real number g such that $0 < g$ there exists a natural number N such that for every natural number n such that $N \leq n$ holds $g \leq (\text{curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m))(n)$ by [26, (7)], (76). \square

- (78) Suppose there exists an element k of \mathbb{N} such that $k \leq m$ and $\text{curry}'(\text{the partial sums in the first coordinate of } f, k)$ is convergent to $+\infty$. Then
- (i) $\text{curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m)$ is convergent to $+\infty$, and
 - (ii) $\lim \text{curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m) = +\infty$.

The theorem is a consequence of (40), (32), and (77).

Now we state the propositions:

(79) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit if and only if the partial sums in the first coordinate of f is convergent in the first coordinate to a finite limit. The theorem is a consequence of (54), (47), (7), and (23).

(80) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the first coordinate to a finite limit. Let us consider an element m of \mathbb{N} . Then $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = \text{lim curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, m)$.

PROOF: The partial sums in the first coordinate of f is convergent in the first coordinate to a finite limit. Define $\mathcal{P}[\text{natural number}] \equiv$ for every element k of \mathbb{N} such that $k \leq \$_1$ holds $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(k) = \text{lim curry}'(\text{the partial sums in the first coordinate of the partial sums in the second coordinate of } f, k)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$ by [1, (13)], [14, (7)], (47), [4, (51)]. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(81) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit if and only if the partial sums in the second coordinate of f is convergent in the second coordinate to a finite limit. The theorem is a consequence of (36), (40), and (79).

(82) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose $(\sum_{\alpha=0}^{\kappa} f(\alpha))_{\kappa \in \mathbb{N}}$ is convergent in the second coordinate to a finite limit. Let us consider an element m of \mathbb{N} . Then $(\sum_{\alpha=0}^{\kappa} (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(\alpha))_{\kappa \in \mathbb{N}}(m) = \text{lim curry}(\text{the partial sums in the second coordinate of the partial sums in the first coordinate of } f, m)$. The theorem is a consequence of (36), (40), (38), (80), and (32).

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence s of extended reals. Now we state the propositions:

(83) Suppose for every element m of \mathbb{N} , $s(m) = \text{lim inf curry}'(f, m)$. Then $\sum s \leq \text{lim inf}(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$.

PROOF: For every element m of \mathbb{N} and for every elements N, n of \mathbb{N}

such that $n \geq N$ holds (the inferior realsequence $\text{curry}'(f, m))(N) \leq f(n, m)$ by [26, (7), (8)]. Define \mathcal{F} (element of \mathbb{N}) = the inferior realsequence $\text{curry}'(f, \$_1)$. Define \mathcal{G} (element of \mathbb{N} , element of \mathbb{N}) = (the inferior realsequence $\text{curry}'(f, \$_2))(\$_1)$. Consider g being a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ such that for every element n of \mathbb{N} and for every element m of \mathbb{N} , $g(n, m) = \mathcal{G}(n, m)$ from [5, Sch. 4]. For every element m of \mathbb{N} and for every elements N, n of \mathbb{N} such that $n \geq N$ holds (the partial sums in the second coordinate of g)(N, m) \leq (the partial sums in the second coordinate of f)(n, m). For every element m of \mathbb{N} and for every elements N, n of \mathbb{N} such that $n \geq N$ holds (the partial sums in the second coordinate of g)(N, m) \leq (the inferior realsequence the lim in the second coordinate of the partial sums in the second coordinate of f)(n) by [26, (37), (23)]. Define \mathcal{Q} [natural number] \equiv for every element m of \mathbb{N} such that $m = \$_1$ holds $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) = \lim \text{curry}'(\text{the partial sums in the second coordinate of } g, m)$. For every element m of \mathbb{N} , $\text{curry}'(\text{the partial sums in the second coordinate of } g, m)$ is convergent by [26, (7), (37)]. For every natural number k such that $\mathcal{Q}[k]$ holds $\mathcal{Q}[k+1]$ by [26, (37)], [4, (51), (52)], [14, (11)]. For every natural number k , $\mathcal{Q}[k]$ from [1, Sch. 2]. For every natural number m , $(\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m) \leq \liminf(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$ by [26, (37), (38)]. For every object m such that $m \in \text{dom } s$ holds $0 \leq s(m)$ by [4, (51), (52)], [26, (23)]. \square

- (84) Suppose for every element m of \mathbb{N} , $s(m) = \liminf \text{curry}(f, m)$. Then $\sum s \leq \liminf(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)$. The theorem is a consequence of (32), (83), (38), and (40).

Now we state the proposition:

- (85) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a sequence s of extended reals, and natural numbers n, m . Then
 - (i) if for every natural numbers i, j , $f(i, j) \leq s(i)$, then (the partial sums in the first coordinate of f)(n, m) $\leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(n)$, and
 - (ii) if for every natural numbers i, j , $f(i, j) \leq s(j)$, then (the partial sums in the second coordinate of f)(n, m) $\leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(m)$.

PROOF: Define \mathcal{P} [natural number] \equiv (the partial sums in the second coordinate of f)($n, \$_1$) $\leq (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number k , $\mathcal{P}[k]$ from [1, Sch. 2]. \square

Let us consider a sequence s of extended reals and an extended real number r . Now we state the propositions:

(86) If for every natural number n , $s(n) \leq r$, then $\limsup s \leq r$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = r$. Consider f being a function from \mathbb{N} into $\overline{\mathbb{R}}$ such that for every element n of \mathbb{N} , $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number n , $f(n) = r$. For every natural number n , $s(n) \leq f(n)$. \square

(87) If for every natural number n , $r \leq s(n)$, then $r \leq \liminf s$.

PROOF: Define $\mathcal{F}(\text{element of } \mathbb{N}) = r$. Consider f being a function from \mathbb{N} into $\overline{\mathbb{R}}$ such that for every element n of \mathbb{N} , $f(n) = \mathcal{F}(n)$ from [7, Sch. 4]. For every natural number n , $f(n) = r$. For every natural number n , $f(n) \leq s(n)$. \square

Now we state the proposition:

(88) Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then

- (i) for every natural numbers i_1, i_2, j such that $i_1 \leq i_2$ holds (the partial sums in the first coordinate of $f)(i_1, j) \leq$ (the partial sums in the first coordinate of $f)(i_2, j)$, and
- (ii) for every natural numbers i, j_1, j_2 such that $j_1 \leq j_2$ holds (the partial sums in the second coordinate of $f)(i, j_1) \leq$ (the partial sums in the second coordinate of $f)(i, j_2)$.

Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and natural numbers i, j, k .

Now we state the propositions:

(89) Suppose for every element m of \mathbb{N} , $\text{curry}'(f, m)$ is non-decreasing and $i \leq j$. Then (the partial sums in the second coordinate of $f)(i, k) \leq$ (the partial sums in the second coordinate of $f)(j, k)$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ (the partial sums in the second coordinate of $f)(i, \$_1) \leq$ (the partial sums in the second coordinate of $f)(j, \$_1)$. For every natural number n such that $\mathcal{P}[n]$ holds $\mathcal{P}[n + 1]$ by [26, (7)]. For every natural number n , $\mathcal{P}[n]$ from [1, Sch. 2]. \square

(90) Suppose for every element n of \mathbb{N} , $\text{curry}(f, n)$ is non-decreasing and $i \leq j$. Then (the partial sums in the first coordinate of $f)(k, i) \leq$ (the partial sums in the first coordinate of $f)(k, j)$. The theorem is a consequence of (32), (89), and (39).

Let us consider a non-negative function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and a sequence s of extended reals. Now we state the propositions:

(91) Suppose for every element m of \mathbb{N} , $\text{curry}'(f, m)$ is non-decreasing and $s(m) = \lim \text{curry}'(f, m)$. Then

- (i) the \lim in the second coordinate of the partial sums in the second coordinate of f is non-decreasing, and

(ii) $\sum s = \lim(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$.

PROOF: $\sum s \leq \liminf(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)$. For every natural numbers $n, m, f(n, m) \leq s(m)$ by [26, (37)], [6, (3)]. For every natural numbers n, m such that $m \leq n$ holds (the lim in the second coordinate of the partial sums in the second coordinate of $f)(m) \leq (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(n)$ by [26, (37)], (89), [26, (38)]. For every natural number $n, (\text{the lim in the second coordinate of the partial sums in the second coordinate of } f)(n) \leq \sum s$ by [26, (37)], [4, (39)], (87), [26, (41)]. $\limsup(\text{the lim in the second coordinate of the partial sums in the second coordinate of } f) \leq \sum s$. \square

(92) Suppose for every element m of \mathbb{N} , $\text{curry}(f, m)$ is non-decreasing and $s(m) = \lim \text{curry}(f, m)$. Then

(i) the lim in the first coordinate of the partial sums in the first coordinate of f is non-decreasing, and

(ii) $\sum s = \lim(\text{the lim in the first coordinate of the partial sums in the first coordinate of } f)$.

The theorem is a consequence of (32), (91), (33), and (40).

5. PRINGSHEIM SENSE CONVERGENCE FOR EXTENDED REAL-VALUED DOUBLE SEQUENCES

Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Now we state the propositions:

(93) If f is P-convergent to $+\infty$, then f is not P-convergent to $-\infty$ and f is not P-convergent to a finite limit.

(94) If f is P-convergent to $-\infty$, then f is not P-convergent to $+\infty$ and f is not P-convergent to a finite limit.

Let f be a function from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. We say that f is P-convergent if and only if

(Def. 17) f is P-convergent to a finite limit or P-convergent to $+\infty$ or P-convergent to $-\infty$.

Assume f is P-convergent. The functor P-lim f yielding an extended real is defined by

(Def. 18) there exists a real number p such that $it = p$ and for every real number e such that $0 < e$ there exists a natural number N such that for every natural numbers n, m such that $n \geq N$ and $m \geq N$ holds $|f(n, m) - it| < e$

and f is P-convergent to a finite limit or $it = +\infty$ and f is P-convergent to $+\infty$ or $it = -\infty$ and f is P-convergent to $-\infty$.

Now we state the propositions:

- (95) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number r . Suppose for every natural numbers $n, m, f(n, m) = r$. Then
 - (i) f is P-convergent to a finite limit, and
 - (ii) $\text{P-lim } f = r$.
- (96) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leq n_2$ and $m_1 \leq m_2$ holds $f(n_1, m_1) \leq f(n_2, m_2)$. Then
 - (i) f is P-convergent, and
 - (ii) $\text{P-lim } f = \sup \text{rng } f$.
- (97) Let us consider functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers $n, m, f_1(n, m) \leq f_2(n, m)$. Then $\sup \text{rng } f_1 \leq \sup \text{rng } f_2$.
- (98) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and natural numbers n, m . Then $f(n, m) \leq \sup \text{rng } f$.

Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$ and an extended real number K . Now we state the propositions:

- (99) If for every natural numbers $n, m, f(n, m) \leq K$, then $\sup \text{rng } f \leq K$.
- (100) If $K \neq +\infty$ and for every natural numbers $n, m, f(n, m) \leq K$, then $\sup \text{rng } f < +\infty$.

Now we state the propositions:

- (101) Let us consider a without $-\infty$ function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Then $\sup \text{rng } f \neq +\infty$ if and only if there exists a real number K such that $0 < K$ and for every natural numbers $n, m, f(n, m) \leq K$.
- (102) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and an extended real c . Suppose for every natural numbers $n, m, f(n, m) = c$. Then
 - (i) f is P-convergent, and
 - (ii) $\text{P-lim } f = c$, and
 - (iii) $\text{P-lim } f = \sup \text{rng } f$.
- (103) Let us consider a function f from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and without $-\infty$ functions f_1, f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$. Suppose for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leq n_2$ and $m_1 \leq m_2$ holds $f_1(n_1, m_1) \leq f_1(n_2, m_2)$ and for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leq n_2$ and $m_1 \leq m_2$ holds $f_2(n_1, m_1) \leq f_2(n_2, m_2)$ and for every natural numbers $n, m, f_1(n, m) + f_2(n, m) = f(n, m)$. Then

- (i) f is P-convergent, and
- (ii) $\text{P-lim } f = \sup \text{rng } f$, and
- (iii) $\text{P-lim } f = \text{P-lim } f_1 + \text{P-lim } f_2$, and
- (iv) $\sup \text{rng } f = \sup \text{rng } f_1 + \sup \text{rng } f_2$.

The theorem is a consequence of (96) and (101).

Let us consider a without $-\infty$ function f_1 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, a function f_2 from $\mathbb{N} \times \mathbb{N}$ into $\overline{\mathbb{R}}$, and a real number c . Now we state the propositions:

- (104) Suppose $0 \leq c$ and for every natural numbers n, m , $f_2(n, m) = c \cdot f_1(n, m)$. Then
- (i) $\sup \text{rng } f_2 = c \cdot \sup \text{rng } f_1$, and
 - (ii) f_2 is without $-\infty$.

The theorem is a consequence of (102) and (101).

- (105) Suppose $0 \leq c$ and for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leq n_2$ and $m_1 \leq m_2$ holds $f_1(n_1, m_1) \leq f_1(n_2, m_2)$ and for every natural numbers n, m , $f_2(n, m) = c \cdot f_1(n, m)$. Then
- (i) for every natural numbers n_1, m_1, n_2, m_2 such that $n_1 \leq n_2$ and $m_1 \leq m_2$ holds $f_2(n_1, m_1) \leq f_2(n_2, m_2)$, and
 - (ii) f_2 is without $-\infty$ and P-convergent, and
 - (iii) $\text{P-lim } f_2 = \sup \text{rng } f_2$, and
 - (iv) $\text{P-lim } f_2 = c \cdot \text{P-lim } f_1$.

The theorem is a consequence of (96) and (104).

REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The ordinal numbers. *Formalized Mathematics*, 1(1):91–96, 1990.
- [3] Józef Białas. Infimum and supremum of the set of real numbers. Measure theory. *Formalized Mathematics*, 2(1):163–171, 1991.
- [4] Józef Białas. Series of positive real numbers. Measure theory. *Formalized Mathematics*, 2(1):173–183, 1991.
- [5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [9] Czesław Byliński. Some basic properties of sets. *Formalized Mathematics*, 1(1):47–53, 1990.
- [10] Noboru Endou. Double series and sums. *Formalized Mathematics*, 22(1):57–68, 2014. doi:10.2478/forma-2014-0006.

- [11] Noboru Endou and Yasunari Shidama. Integral of measurable function. *Formalized Mathematics*, 14(2):53–70, 2006. doi:10.2478/v10037-006-0008-x.
- [12] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Basic properties of extended real numbers. *Formalized Mathematics*, 9(3):491–494, 2001.
- [13] Noboru Endou, Katsumi Wasaki, and Yasunari Shidama. Definitions and basic properties of measurable functions. *Formalized Mathematics*, 9(3):495–500, 2001.
- [14] Noboru Endou, Keiko Narita, and Yasunari Shidama. The Lebesgue monotone convergence theorem. *Formalized Mathematics*, 16(2):167–175, 2008. doi:10.2478/v10037-008-0023-1.
- [15] Noboru Endou, Hiroyuki Okazaki, and Yasunari Shidama. Double sequences and limits. *Formalized Mathematics*, 21(3):163–170, 2013. doi:10.2478/forma-2013-0018.
- [16] Gerald B. Folland. *Real Analysis: Modern Techniques and Their Applications*. Wiley, 2 edition, 1999.
- [17] D.J.H. Garling. *A Course in Mathematical Analysis: Volume 1, Foundations and Elementary Real Analysis*, volume 1. Cambridge University Press, 2013.
- [18] Andrzej Kondracki. Basic properties of rational numbers. *Formalized Mathematics*, 1(5):841–845, 1990.
- [19] Jarosław Kotowicz. Monotone real sequences. Subsequences. *Formalized Mathematics*, 1(3):471–475, 1990.
- [20] Jarosław Kotowicz. Convergent sequences and the limit of sequences. *Formalized Mathematics*, 1(2):273–275, 1990.
- [21] Adam Naumowicz. Conjugate sequences, bounded complex sequences and convergent complex sequences. *Formalized Mathematics*, 6(2):265–268, 1997.
- [22] Michał J. Trybulec. Integers. *Formalized Mathematics*, 1(3):501–505, 1990.
- [23] Zinaida Trybulec. Properties of subsets. *Formalized Mathematics*, 1(1):67–71, 1990.
- [24] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.
- [25] Edmund Woronowicz. Relations defined on sets. *Formalized Mathematics*, 1(1):181–186, 1990.
- [26] Hiroshi Yamazaki, Noboru Endou, Yasunari Shidama, and Hiroyuki Okazaki. Inferior limit, superior limit and convergence of sequences of extended real numbers. *Formalized Mathematics*, 15(4):231–236, 2007. doi:10.2478/v10037-007-0026-3.

Received July 1, 2015
