

Altitude, Orthocenter of a Triangle and Triangulation

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Summary. We introduce the altitudes of a triangle (the cevians perpendicular to the opposite sides). Using the generalized Ceva's Theorem, we prove the existence and uniqueness of the orthocenter of a triangle [7]. Finally, we formalize in Mizar [1] some formulas [2] to calculate distance using triangulation.

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1. PRELIMINARIES

From now on n denotes a natural number, i denotes an integer, r, s, t denote real numbers, A_1, B_1, C_1, D_1 denote points of \mathcal{E}_T^n , L_1, L_2 denote elements of $\text{Lines}(\mathcal{R}^n)$, and A, B, C denote points of \mathcal{E}_T^2 .

Now we state the propositions:

- (1) If $0 < i \cdot r < r$, then $i = 1$.
- (2) Let us consider an integer i . If $\frac{-3}{2} < i < \frac{1}{2}$, then $i = 0$ or $i = -1$.
- (3) Suppose r is not zero and s is not zero and t is not zero. Then $(\frac{-r}{s}) \cdot (\frac{-t}{-r}) \cdot (\frac{-s}{-t}) = 1$.
- (4) If $0 < r < 2 \cdot \pi$, then $\sin(\frac{r}{2}) \neq 0$. The theorem is a consequence of (1).
- (5) If $-2 \cdot \pi < r < 0$, then $\sin(\frac{r}{2}) \neq 0$. The theorem is a consequence of (4).
- (6) $\tan(2 \cdot \pi - r) = -\tan r$.
- (7) If $A_1 \in \text{Line}(B_1, C_1)$ and $A_1 \neq C_1$, then $\text{Line}(B_1, C_1) = \text{Line}(A_1, C_1)$.

- (8) If $A_1 \neq C_1$ and $A_1 \in \text{Line}(B_1, C_1)$, then $B_1 \in \text{Line}(A_1, C_1)$.
- (9) Suppose $A_1 \neq B_1$ and $A_1 \neq C_1$ and $|(A_1 - B_1, A_1 - C_1)| = 0$ and $L_1 = \text{Line}(A_1, B_1)$ and $L_2 = \text{Line}(A_1, C_1)$. Then $L_1 \perp L_2$.
- (10) If $B_1 \neq C_1$ and $|(B_1 - A_1, B_1 - C_1)| = 0$, then $A_1 \neq C_1$.
- (11) $|(A_1 - B_1, A_1 - C_1)| = |(B_1 - A_1, C_1 - A_1)|$.
- (12) Suppose $B_1 \neq C_1$ and $r = -\left(\frac{|(B_1, C_1)| - |(C_1, C_1)| - |(A_1, B_1)| + |(A_1, C_1)|}{|(B_1 - C_1, B_1 - C_1)|}\right)$ and $D_1 = r \cdot B_1 + (1 - r) \cdot C_1$. Then $|(D_1 - A_1, D_1 - C_1)| = 0$.
- (13) If $A_1 \neq B_1$ and $C_1 = r \cdot A_1 + (1 - r) \cdot B_1$ and $C_1 = B_1$, then $r = 0$.
- (14) (i) $|(B_1, C_1)| - |(C_1, C_1)| - |(A_1, B_1)| + |(A_1, C_1)| = |(C_1 - A_1, B_1 - C_1)|$,
and
(ii) $|(B_1 - C_1, B_1 - C_1)| + |(C_1 - A_1, B_1 - C_1)| = |(B_1 - C_1, B_1 - A_1)|$.
- (15) $|(A_1 - B_1, A_1 - C_1)| = -|(A_1 - B_1, C_1 - A_1)|$.
- (16) $|(B_1 - A_1, C_1 - A_1)| = |(A_1 - B_1, A_1 - C_1)|$.
- (17) $|(B_1 - A_1, C_1 - A_1)| = -|(B_1 - A_1, A_1 - C_1)|$. The theorem is a consequence of (16) and (15).
- (18) Suppose $B_1 \neq C_1$ and $C_1 \neq A_1$ and $A_1 \neq B_1$ and $|(C_1 - A_1, B_1 - C_1)|$ is not zero and $|(B_1 - C_1, A_1 - B_1)|$ is not zero and $|(C_1 - A_1, A_1 - B_1)|$ is not zero and $r = -\left(\frac{|(B_1, C_1)| - |(C_1, C_1)| - |(A_1, B_1)| + |(A_1, C_1)|}{|(B_1 - C_1, B_1 - C_1)|}\right)$ and $s = -\left(\frac{|(C_1, A_1)| - |(A_1, A_1)| - |(B_1, C_1)| + |(B_1, A_1)|}{|(C_1 - A_1, C_1 - A_1)|}\right)$ and $t = -\left(\frac{|(A_1, B_1)| - |(B_1, B_1)| - |(C_1, A_1)| + |(C_1, B_1)|}{|(A_1 - B_1, A_1 - B_1)|}\right)$. Then $\frac{\left(\frac{r}{1-r}\right)^s \cdot t}{1-t} = 1$. The theorem is a consequence of (14), (15), and (3).
- (19) If $C_1 = r \cdot A_1 + (1 - r) \cdot B_1$ and $r = 1$, then $C_1 = A_1$.
- (20) If $C_1 = r \cdot A_1 + (1 - r) \cdot B_1$ and $r = 0$, then $C_1 = B_1$.
- (21) If $|(B_1 - C_1, B_1 - C_1)| = -|(C_1 - A_1, B_1 - C_1)|$, then $|(B_1 - C_1, A_1 - B_1)| = 0$. The theorem is a consequence of (15).
- (22) Suppose $B_1 \neq C_1$ and $r = -\left(\frac{|(B_1, C_1)| - |(C_1, C_1)| - |(A_1, B_1)| + |(A_1, C_1)|}{|(B_1 - C_1, B_1 - C_1)|}\right)$ and $r = 1$. Then $|(B_1 - C_1, A_1 - B_1)| = 0$. The theorem is a consequence of (14) and (21).
- (23) If $A \neq B$ and $A \neq C$, then $|A - B| + |A - C| \neq 0$.
- (24) If A, B, C form a triangle, then $A \notin \text{Line}(B, C)$.
- (25) If $A \neq B$ and $B \neq C$ and $|(B - A, B - C)| = 0$, then $\angle(A, B, C) = \frac{\pi}{2}$ or $\angle(A, B, C) = \left(\frac{3}{2}\right) \cdot \pi$.
- (26) If A, B, C form a triangle, then $\sin\left(\frac{\angle(A, B, C)}{2}\right) > 0$.
- (27) If $\angle(B, A, C) \neq \angle(C, B, A)$, then $\sin\left(\frac{\angle(B, A, C) - \angle(C, B, A)}{2}\right) \neq 0$. The theorem is a consequence of (5) and (4).

(28) If A, B, C form a triangle, then $\sin \angle(A, B, C) \neq 0$.

Let us assume that A, C, B form a triangle and $\angle(A, C, B) < \pi$. Now we state the propositions:

(29) $\angle(A, C, B) = \pi - (\angle(C, B, A) + \angle(B, A, C))$.

(30) $\angle(B, A, C) + \angle(C, B, A) = \pi - \angle(A, C, B)$. The theorem is a consequence of (29).

Let us assume that A, B, C form a triangle. Now we state the propositions:

(31) $\angle(B, A, C) - \angle(C, B, A) \neq \pi$.

(32) $\angle(B, A, C) - \angle(C, B, A) \neq -\pi$.

Let us assume that A, B, C form a triangle. Now we state the propositions:

(33) $(-2) \cdot \pi < \angle(B, A, C) - \angle(C, B, A) < 2 \cdot \pi$.

(34) $-\pi < \frac{\angle(B, A, C) - \angle(C, B, A)}{2} < \pi$. The theorem is a consequence of (33).

Let us assume that A, B, C form a triangle and $\angle(B, A, C) < \pi$. Now we state the propositions:

(35) $-\pi < \angle(B, A, C) - \angle(C, B, A) < \pi$.

(36) $-(\frac{\pi}{2}) < \frac{\angle(B, A, C) - \angle(C, B, A)}{2} < \frac{\pi}{2}$. The theorem is a consequence of (35).

2. ORTHOCENTER

From now on D denotes a point of \mathcal{E}_T^2 and a, b, c, d denote real numbers.

Let A, B, C be points of \mathcal{E}_T^2 . Assume $B \neq C$. The functor $\text{Altit } \Delta(A, B, C)$ yielding an element of $\text{Lines}(\mathcal{R}^2)$ is defined by

(Def. 1) there exist elements L_1, L_2 of $\text{Lines}(\mathcal{R}^2)$ such that $it = L_1$ and $L_2 = \text{Line}(B, C)$ and $A \in L_1$ and $L_1 \perp L_2$.

Let us assume that $B \neq C$. Now we state the propositions:

(37) $A \in \text{Altit } \Delta(A, B, C)$.

(38) $\text{Altit } \Delta(A, B, C)$ is a line.

(39) $\text{Altit } \Delta(A, B, C) = \text{Altit } \Delta(A, C, B)$.

Now we state the propositions:

(40) If $B \neq C$ and $D \in \text{Altit } \Delta(A, B, C)$, then

$$\text{Altit } \Delta(D, B, C) = \text{Altit } \Delta(A, B, C).$$

(41) If $B \neq C$ and $D \in \text{Line}(B, C)$ and $D \neq C$, then $\text{Altit } \Delta(A, B, C) = \text{Altit } \Delta(A, D, C)$. The theorem is a consequence of (7).

Let A, B, C be points of \mathcal{E}_T^2 . Assume $B \neq C$. The functor $\text{FootAltit } \Delta(A, B, C)$ yielding a point of \mathcal{E}_T^2 is defined by

(Def. 2) there exists a point P of \mathcal{E}_T^2 such that $it = P$ and $\text{AltIt} \triangle(A, B, C) \cap \text{Line}(B, C) = \{P\}$.

Let us assume that $B \neq C$. Now we state the propositions:

- (42) $\text{FootAltIt} \triangle(A, B, C) = \text{FootAltIt} \triangle(A, C, B)$. The theorem is a consequence of (39).
- (43) (i) $\text{FootAltIt} \triangle(A, B, C) \in \text{Line}(B, C)$, and
(ii) $\text{FootAltIt} \triangle(A, B, C) \in \text{AltIt} \triangle(A, B, C)$.

Now we state the propositions:

- (44) If $B \neq C$ and $A \notin \text{Line}(B, C)$, then $\text{AltIt} \triangle(A, B, C) = \text{Line}(A, \text{FootAltIt} \triangle(A, B, C))$. The theorem is a consequence of (43).
- (45) If $B \neq C$ and $A \in \text{Line}(B, C)$, then $\text{FootAltIt} \triangle(A, B, C) = A$.
- (46) If $B \neq C$ and $\text{FootAltIt} \triangle(A, B, C) = A$, then $A \in \text{Line}(B, C)$.

Let us assume that $B \neq C$. Now we state the propositions:

- (47) $|(A - \text{FootAltIt} \triangle(A, B, C), B - C)| = 0$. The theorem is a consequence of (44) and (45).
- (48) $|(A - \text{FootAltIt} \triangle(A, B, C), B - \text{FootAltIt} \triangle(A, B, C))| = 0$. The theorem is a consequence of (43), (44), and (45).
- (49) $|(A - \text{FootAltIt} \triangle(A, B, C), C - \text{FootAltIt} \triangle(A, B, C))| = 0$. The theorem is a consequence of (42) and (48).

Now we state the propositions:

- (50) If $B \neq C$ and $B = \text{FootAltIt} \triangle(A, B, C)$, then $|(B - A, B - C)| = 0$. The theorem is a consequence of (49), (11), and (43).
- (51) If $B \neq C$ and $D \in \text{Line}(B, C)$ and $D \neq C$, then $\text{FootAltIt} \triangle(A, B, C) = \text{FootAltIt} \triangle(A, D, C)$. The theorem is a consequence of (7) and (41).
- (52) If $B \neq C$ and $|(B - A, B - C)| = 0$, then $B = \text{FootAltIt} \triangle(A, B, C)$. The theorem is a consequence of (9) and (45).
- (53) If $B \neq C$ and $B \neq A$ and $\angle(A, B, C) = \frac{\pi}{2}$, then $\text{FootAltIt} \triangle(A, B, C) = B$. The theorem is a consequence of (11) and (52).
- (54) If A, B, C form a triangle, then $A \neq \text{FootAltIt} \triangle(A, B, C)$. The theorem is a consequence of (43).
- (55) If A, B, C form a triangle and $|(B - A, B - C)| \neq 0$, then $\text{FootAltIt} \triangle(A, B, C), B, A$ form a triangle.

PROOF: Set $p = \text{FootAltIt} \triangle(A, B, C)$. Consider P being a point of \mathcal{E}_T^2 such that $\text{FootAltIt} \triangle(A, B, C) = P$ and $\text{AltIt} \triangle(A, B, C) \cap \text{Line}(B, C) = \{P\}$. Consider L_1, L_2 being elements of $\text{Lines}(\mathcal{R}^2)$ such that $\text{AltIt} \triangle(A, B, C) = L_1$ and $L_2 = \text{Line}(B, C)$ and $A \in L_1$ and $L_1 \perp L_2$. $P \neq B$. $p \neq A$. p, B, A are mutually different. $P \in \text{Line}(B, C)$. $B, C \in \text{Line}(B, P)$. $\angle(p, B, A) \neq \pi$

by [11, (11)], [12, (12)], (50), (8). $\angle(B, A, p) \neq \pi$ by [11, (11)], [12, (12)].
 $\angle(A, p, B) \neq \pi$ by [11, (11)], [12, (12)], (8), (54). \square

Let A, B, C be points of \mathcal{E}_T^2 . Assume $B \neq C$. The functor $|\text{AltIt} \triangle(A, B, C)|$ yielding a real number is defined by the term

(Def. 3) $|A - \text{FootAltIt} \triangle(A, B, C)|$.

Let us assume that $B \neq C$. Now we state the propositions:

(56) $0 \leq |\text{AltIt} \triangle(A, B, C)|$.

(57) $|\text{AltIt} \triangle(A, B, C)| = |\text{AltIt} \triangle(A, C, B)|$. The theorem is a consequence of (42).

Now we state the propositions:

(58) If $B \neq C$ and $|(B - A, B - C)| = 0$, then $|\text{FootAltIt} \triangle(A, B, C) - A| = |A - B|$. The theorem is a consequence of (52).

(59) Suppose $B \neq C$ and $r = -\left(\frac{|(B,C)| - |(C,C)| - |(A,B)| + |(A,C)|}{|(B-C, B-C)|}\right)$ and $D = r \cdot B + (1 - r) \cdot C$ and $D \neq C$. Then $D = \text{FootAltIt} \triangle(A, B, C)$.

PROOF: $|(D - A, D - C)| = 0$. $D = \text{FootAltIt} \triangle(A, D, C)$. $D \in \text{Line}(B, C)$ by [6, (4)]. \square

(60) Suppose $B \neq C$ and $r = -\left(\frac{|(B,C)| - |(C,C)| - |(A,B)| + |(A,C)|}{|(B-C, B-C)|}\right)$ and $D = r \cdot B + (1 - r) \cdot C$ and $D = C$. Then $C = \text{FootAltIt} \triangle(A, B, C)$. The theorem is a consequence of (13), (14), (15), (52), and (42).

(61) Suppose A, B, C form a triangle and $|(C - A, B - C)|$ is not zero and $|(B - C, A - B)|$ is not zero and $|(C - A, A - B)|$ is not zero. Then $\text{Line}(A, \text{FootAltIt} \triangle(A, B, C)), \text{Line}(C, \text{FootAltIt} \triangle(C, A, B)), \text{Line}(B, \text{FootAltIt} \triangle(B, C, A))$ are concurrent. The theorem is a consequence of (60), (17), (47), (59), (18), and (22).

(62) If A, B, C form a triangle and $|(C - A, B - C)|$ is zero, then $\text{FootAltIt} \triangle(A, B, C) = C$ and $\text{FootAltIt} \triangle(B, C, A) = C$. The theorem is a consequence of (15), (52), and (42).

(63) Suppose A, B, C form a triangle and $C \in \text{AltIt} \triangle(A, B, C)$ and $C \in \text{AltIt} \triangle(B, C, A)$. Then $\text{AltIt} \triangle(A, B, C) \cap \text{AltIt} \triangle(B, C, A)$ is a point.

PROOF: Consider L_1, L_2 being elements of $\text{Lines}(\mathcal{R}^2)$ such that $\text{AltIt} \triangle(A, B, C) = L_1$ and $L_2 = \text{Line}(B, C)$ and $A \in L_1$ and $L_1 \perp L_2$. Consider L_3, L_4 being elements of $\text{Lines}(\mathcal{R}^2)$ such that $\text{AltIt} \triangle(B, C, A) = L_3$ and $L_4 = \text{Line}(C, A)$ and $B \in L_3$ and $L_3 \perp L_4$. $L_1 \not\parallel L_3$ by [9, (41)], [6, (16)], [8, (108)], [12, (13)]. L_1 is not a point and L_3 is not a point. \square

(64) Suppose B, C, A form a triangle and $C \in \text{AltIt} \triangle(B, C, A)$ and $C \in \text{AltIt} \triangle(C, A, B)$. Then $\text{AltIt} \triangle(B, C, A) \cap \text{AltIt} \triangle(C, A, B)$ is a point.

PROOF: Consider L_1, L_2 being elements of $\text{Lines}(\mathcal{R}^2)$ such that

Altitude $\Delta(B, C, A) = L_1$ and $L_2 = \text{Line}(C, A)$ and $B \in L_1$ and $L_1 \perp L_2$. Consider L_3, L_4 being elements of $\text{Lines}(\mathcal{R}^2)$ such that $\text{Altitude} \Delta(C, A, B) = L_3$ and $L_4 = \text{Line}(A, B)$ and $C \in L_3$ and $L_3 \perp L_4$. $L_1 \nparallel L_3$ by [8, (71), (111)], [6, (16)], [9, (41)]. L_1 is not a point and L_3 is not a point. \square

- (65) Suppose C, A, B form a triangle and $C \in \text{Altitude} \Delta(C, A, B)$ and $C \in \text{Altitude} \Delta(A, B, C)$. Then $\text{Altitude} \Delta(C, A, B) \cap \text{Altitude} \Delta(A, B, C)$ is a point.

PROOF: Consider L_1, L_2 being elements of $\text{Lines}(\mathcal{R}^2)$ such that $\text{Altitude} \Delta(C, A, B) = L_1$ and $L_2 = \text{Line}(A, B)$ and $C \in L_1$ and $L_1 \perp L_2$. Consider L_3, L_4 being elements of $\text{Lines}(\mathcal{R}^2)$ such that $\text{Altitude} \Delta(A, B, C) = L_3$ and $L_4 = \text{Line}(B, C)$ and $A \in L_3$ and $L_3 \perp L_4$. $L_1 \nparallel L_3$ by [8, (71), (111)], [6, (16)], [9, (41)]. L_1 is not a point and L_3 is not a point. \square

- (66) Suppose A, B, C form a triangle and $|(C - A, B - C)| = 0$. Then

- (i) $\text{Altitude} \Delta(A, B, C) \cap \text{Altitude} \Delta(B, C, A) = \{C\}$, and
- (ii) $\text{Altitude} \Delta(B, C, A) \cap \text{Altitude} \Delta(C, A, B) = \{C\}$, and
- (iii) $\text{Altitude} \Delta(C, A, B) \cap \text{Altitude} \Delta(A, B, C) = \{C\}$.

PROOF: $A \notin \text{Line}(B, C)$ and $B \notin \text{Line}(C, A)$. $\text{FootAltitude} \Delta(A, B, C) = C$ and $\text{FootAltitude} \Delta(B, C, A) = C$. $\text{Altitude} \Delta(A, B, C) = \text{Line}(A, C)$ and $\text{Altitude} \Delta(B, C, A) = \text{Line}(B, C)$. $C \in \text{Altitude} \Delta(C, A, B)$. $\text{Altitude} \Delta(A, B, C) \cap \text{Altitude} \Delta(B, C, A) = \{C\}$ by [6, (22)], (63). $\text{Altitude} \Delta(B, C, A) \cap \text{Altitude} \Delta(C, A, B) = \{C\}$ by [12, (15)], (37), (64), [6, (22)]. $\text{Altitude} \Delta(C, A, B) \cap \text{Altitude} \Delta(A, B, C) = \{C\}$ by [12, (15)], (37), (65), [6, (22)]. \square

- (67) Suppose A, B, C form a triangle. Then there exists a point P of \mathcal{E}_T^2 such that

- (i) $\text{Altitude} \Delta(A, B, C) \cap \text{Altitude} \Delta(B, C, A) = \{P\}$, and
- (ii) $\text{Altitude} \Delta(B, C, A) \cap \text{Altitude} \Delta(C, A, B) = \{P\}$, and
- (iii) $\text{Altitude} \Delta(C, A, B) \cap \text{Altitude} \Delta(A, B, C) = \{P\}$.

The theorem is a consequence of (66), (61), (24), (44), and (38).

Let A, B, C be points of \mathcal{E}_T^2 . Assume A, B, C form a triangle. The functor Orthocenter $\Delta(A, B, C)$ yielding a point of \mathcal{E}_T^2 is defined by

- (Def. 4) $\text{Altitude} \Delta(A, B, C) \cap \text{Altitude} \Delta(B, C, A) = \{it\}$ and $\text{Altitude} \Delta(B, C, A) \cap \text{Altitude} \Delta(C, A, B) = \{it\}$ and $\text{Altitude} \Delta(C, A, B) \cap \text{Altitude} \Delta(A, B, C) = \{it\}$.

3. TRIANGULATION

Let us assume that $B \neq A$. Now we state the propositions:

- (68) $(\sin \angle(B, A, C) + \sin \angle(C, B, A)) \cdot (|C - B| - |C - A|) = (\sin \angle(B, A, C) - \sin \angle(C, B, A)) \cdot (|C - B| + |C - A|)$.

$$(69) \quad \sin\left(\frac{\angle(B,A,C)+\angle(C,B,A)}{2}\right) \cdot \cos\left(\frac{\angle(B,A,C)-\angle(C,B,A)}{2}\right) \cdot (|C - B| - |C - A|) = \sin\left(\frac{\angle(B,A,C)-\angle(C,B,A)}{2}\right) \cdot \cos\left(\frac{\angle(B,A,C)+\angle(C,B,A)}{2}\right) \cdot (|C - B| + |C - A|). \text{ The theorem is a consequence of (68).}$$

Now we state the proposition:

$$(70) \quad \text{Suppose } A, B, C \text{ form a triangle and } \angle(B, A, C) - \angle(C, B, A) \neq \pi \text{ and } \angle(B, A, C) - \angle(C, B, A) \neq -\pi. \text{ Then } \cos\left(\frac{\angle(B,A,C)-\angle(C,B,A)}{2}\right) \neq 0. \text{ The theorem is a consequence of (2).}$$

Let us assume that A, C, B form a triangle and $\angle(A, C, B) < \pi$. Now we state the propositions:

$$(71) \quad \tan\left(\frac{\angle(B,A,C)-\angle(C,B,A)}{2}\right) = \cot\left(\frac{\angle(A,C,B)}{2}\right) \cdot \left(\frac{|C-B|-|C-A|}{|C-B|+|C-A|}\right).$$

PROOF: $\angle(B, A, C) - \angle(C, B, A) \neq \pi$ and $\angle(B, A, C) - \angle(C, B, A) \neq -\pi$. Set $\alpha = \frac{\angle(B,A,C)-\angle(C,B,A)}{2}$. Set $\beta = \frac{\angle(B,A,C)+\angle(C,B,A)}{2}$. $\angle(A, C, B) = \pi - (\angle(C, B, A) + \angle(B, A, C))$. Set $\alpha_1 = \frac{\angle(A,C,B)}{2}$. $\sin \alpha_1 \neq 0$. $|C-B|+|C-A| \neq 0$ by [11, (42)]. $\sin \beta \cdot \cos \alpha \cdot (|C - B| - |C - A|) = \sin \alpha \cdot \cos \beta \cdot (|C - B| + |C - A|)$. $(|C - B| - |C - A|) \cdot \cos \alpha_1 \cdot 1 = (|C - B| + |C - A|) \cdot \sin \alpha_1 \cdot \left(\frac{\sin \alpha}{\cos \alpha}\right)$. \square

$$(72) \quad \frac{\angle(B,A,C)-\angle(C,B,A)}{2} = \arctan\left(\cot\left(\frac{\angle(A,C,B)}{2}\right) \cdot \left(\frac{|C-B|-|C-A|}{|C-B|+|C-A|}\right)\right). \text{ The theorem is a consequence of (71) and (36).}$$

$$(73) \quad \angle(B, A, C) - \angle(C, B, A) = 2 \cdot \arctan\left(\cot\left(\frac{\angle(A,C,B)}{2}\right) \cdot \left(\frac{|C-B|-|C-A|}{|C-B|+|C-A|}\right)\right). \text{ The theorem is a consequence of (72).}$$

$$(74) \quad \text{(i) } \angle(B, A, C) = \arctan\left(\cot\left(\frac{\angle(A,C,B)}{2}\right) \cdot \left(\frac{|C-B|-|C-A|}{|C-B|+|C-A|}\right)\right) + \left(\frac{\pi}{2}\right) - \left(\frac{\angle(A,C,B)}{2}\right), \text{ and}$$

$$\text{(ii) } \angle(C, B, A) = \left(\frac{\pi}{2}\right) - \left(\frac{\angle(A,C,B)}{2}\right) - \arctan\left(\cot\left(\frac{\angle(A,C,B)}{2}\right) \cdot \left(\frac{|C-B|-|C-A|}{|C-B|+|C-A|}\right)\right).$$

The theorem is a consequence of (73) and (30).

$$(75) \quad |B - C| = \frac{|A-B| \cdot \sin \angle(B,A,C)}{\sin(\angle(B,A,C) + \angle(C,B,A))}.$$

PROOF: $|B - C| = \frac{|A-B| \cdot \sin \angle(B,A,C)}{\sin \angle(A,C,B)}$ by [11, (6), (43)], (28). $\angle(A, C, B) = \pi - (\angle(C, B, A) + \angle(B, A, C))$. \square

$$(76) \quad |A - C| = \frac{|A-B| \cdot \sin \angle(C,B,A)}{\sin(\angle(B,A,C) + \angle(C,B,A))}.$$

PROOF: $|A - C| = \frac{|A-B| \cdot \sin \angle(C,B,A)}{\sin \angle(A,C,B)}$ by [11, (6)], (28). $\angle(A, C, B) = \pi - (\angle(C, B, A) + \angle(B, A, C))$ by [11, (20)], [10, (47)]. \square

Now we state the propositions:

$$(77) \quad \text{Suppose } A, C, B \text{ form a triangle and } \angle(C, A, B) = \frac{\pi}{2}.$$

Then $|\text{Alt } \triangle(C, A, B)| = |A - B| \cdot \tan \angle(A, B, C)$. The theorem is a consequence of (11) and (58).

$$(78) \quad \text{Suppose } A, B, C \text{ form a triangle and } \angle(C, A, B) = \left(\frac{3}{2}\right) \cdot \pi.$$

Then $|\text{Alt} \triangle(C, A, B)| = |A - B| \cdot \tan \angle(C, B, A)$. The theorem is a consequence of (11) and (58).

(79) Suppose A, C, B form a triangle and $|(A - C, A - B)| = 0$. Then $|\text{Alt} \triangle(C, A, B)| = |A - B| \cdot |\tan \angle(A, B, C)|$. The theorem is a consequence of (11), (77), (56), (6), and (78).

(80) Suppose $B \neq C$ and $\text{FootAlt} \triangle(A, B, C)$, B, A form a triangle. Then
 (i) $|A - B| \cdot \sin \angle(A, B, \text{FootAlt} \triangle(A, B, C)) = |\text{FootAlt} \triangle(A, B, C) - A|$, or

(ii) $|A - B| \cdot (-\sin \angle(A, B, \text{FootAlt} \triangle(A, B, C))) = |\text{FootAlt} \triangle(A, B, C) - A|$.

The theorem is a consequence of (48).

(81) Suppose A, B, C form a triangle and $|(B - A, B - C)| \neq 0$. Then

(i) $|A - B| \cdot \sin \angle(A, B, \text{FootAlt} \triangle(A, B, C)) = |\text{FootAlt} \triangle(A, B, C) - A|$, or

(ii) $|A - B| \cdot (-\sin \angle(A, B, \text{FootAlt} \triangle(A, B, C))) = |\text{FootAlt} \triangle(A, B, C) - A|$.

The theorem is a consequence of (80) and (55).

(82) Suppose A, C, B form a triangle and $\angle(A, C, B) < \pi$ and $|(A - C, A - B)| \neq 0$. Then $|\text{Alt} \triangle(C, A, B)| = |A - B| \cdot \left| \left(\frac{\sin \angle(C, B, A)}{\sin(\angle(B, A, C) + \angle(C, B, A))} \right) \cdot \sin \angle(C, A, \text{FootAlt} \triangle(C, A, B)) \right|$. The theorem is a consequence of (76), (55), and (80).

(83) Suppose $0 < \angle(B, A, D) < \pi$ and $0 < \angle(D, A, C) < \pi$ and D, A, C are mutually different and B, A, D are mutually different. Then $\angle(A, C, D) + \angle(D, B, A) = 2 \cdot \pi - (\angle(B, A, C) + \angle(A, D, B) + \angle(C, D, A))$.

PROOF: $\angle(B, A, D) + \angle(D, A, C) = \angle(B, A, C)$ by [5, (2)], [11, (4)]. $\angle(A, C, D) = \pi - (\angle(C, D, A) + \angle(D, A, C))$ by [10, (47)]. $\angle(D, B, A) = \pi - (\angle(A, D, B) + \angle(B, A, D))$ by [10, (47)]. \square

(84) Suppose A, C, B form a triangle and $\angle(A, C, B) < \pi$ and A, D, B form a triangle and $\angle(A, D, B) < \pi$ and $a = \angle(C, B, A)$ and $b = \angle(B, A, C)$ and $c = \angle(D, B, A)$ and $d = \angle(C, A, D)$. Then $|D - C|^2 = |A - B|^2 \cdot \left(\left(\frac{\sin a}{\sin(a+b)} \right)^2 + \left(\frac{\sin c}{\sin(b+d+c)} \right)^2 - 2 \cdot \left(\frac{\sin a}{\sin(b+a)} \right) \cdot \left(\frac{\sin c}{\sin(b+d+c)} \right) \cdot \cos d \right)$.

PROOF: Set $e = b + d$. $\sin(e + c) = \sin(\angle(B, A, D) + \angle(D, B, A))$ by [14, (79)]. \square

(85) Suppose $\sin(2 \cdot s) \cdot \cos d = \cos(2 \cdot t)$. Then $(r \cdot \cos s)^2 + (r \cdot \sin s)^2 - 2 \cdot (r \cdot \cos s) \cdot (r \cdot \sin s) \cdot \cos d = 2 \cdot r^2 \cdot (\sin t)^2$.

(86) Let us consider real numbers R, ϑ . Suppose $D \neq C$ and $0 \leq R$ and A, C, B form a triangle and $\angle(A, C, B) < \pi$ and A, D, B form a triangle

and $\angle(A, D, B) < \pi$ and $a = \angle(C, B, A)$ and $b = \angle(B, A, C)$ and $c = \angle(D, B, A)$ and $d = \angle(C, A, D)$ and $R \cdot \cos s = \frac{\sin a}{\sin(a+b)}$ and $R \cdot \sin s = \frac{\sin c}{\sin(b+d+c)}$ and $0 < \vartheta < \pi$ and $\sin(2 \cdot s) \cdot \cos d = \cos(2 \cdot \vartheta)$. Then $|D - C| = |A - B| \cdot \sqrt{2} \cdot R \cdot \sin \vartheta$.

PROOF: $|D - C|^2 = |A - B|^2 \cdot ((R \cdot \cos s)^2 + (R \cdot \sin s)^2 - 2 \cdot (R \cdot \cos s) \cdot (R \cdot \sin s) \cdot \cos d)$. $|D - C| \neq -|A - B| \cdot \sqrt{2} \cdot R \cdot \sin \vartheta$ by [13, (25)], [11, (42)]. \square

- (87) Suppose A, C, B form a triangle and $\angle(A, C, B) < \pi$ and D, A, C form a triangle and $\angle(A, D, C) = \frac{\pi}{2}$. Then $|D - C| = \left(\frac{|A-B| \cdot \sin \angle(C, B, A)}{\sin(\angle(B, A, C) + \angle(C, B, A))}\right) \cdot \sin \angle(C, A, D)$. The theorem is a consequence of (76).
- (88) Suppose B, C, A form a triangle and $\angle(B, C, A) < \pi$ and D, C, A form a triangle and $\angle(C, D, A) = \frac{\pi}{2}$. Then $|D - C| = \left(\frac{|A-B| \cdot \sin \angle(A, B, C)}{\sin(\angle(A, B, C) + \angle(C, A, B))}\right) \cdot \sin \angle(D, A, C)$. The theorem is a consequence of (75).
- (89) Suppose A, C, B form a triangle and $\angle(A, C, B) < \pi$ and D, A, C form a triangle and $\angle(A, D, C) = \frac{\pi}{2}$ and $A \in \mathcal{L}(B, D)$ and $A \neq D$. Then $|D - C| = \left(\frac{|A-B| \cdot \sin \angle(C, B, A)}{\sin(\angle(C, A, D) - \angle(C, B, A))}\right) \cdot \sin \angle(C, A, D)$. The theorem is a consequence of (87).
- (90) Suppose B, C, A form a triangle and $\angle(B, C, A) < \pi$ and D, C, A form a triangle and $\angle(C, D, A) = \frac{\pi}{2}$ and $A \in \mathcal{L}(D, B)$ and $A \neq D$. Then $|D - C| = \left(\frac{|A-B| \cdot \sin \angle(A, B, C)}{\sin(\angle(D, A, C) - \angle(A, B, C))}\right) \cdot \sin \angle(D, A, C)$.
 PROOF: $\sin(\angle(C, A, B) + \angle(A, B, C)) = \sin(\angle(D, A, C) - \angle(A, B, C))$ by [4, (1)], [3, (8)]. \square

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