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Product Pre-Measure

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Summary. In this article we formalize in Mizar [5] product pre-measure on product sets of measurable sets. Although there are some approaches to construct product measure [22], [6], [9], [21], [25], we start it from σ-measure because existence of σ-measure on any semialgebras has been proved in [15]. In this approach, we use some theorems for integrals.

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1. Preliminaries

Now we state the proposition:

(1) Let us consider non empty sets A, A_1, A_2, B, B_1, B_2. Then $A_1 \times B_1$ misses $A_2 \times B_2$ and $A \times B = A_1 \times B_1 \cup A_2 \times B_2$ if and only if A_1 misses A_2 and $A = A_1 \cup A_2$ and $B = B_1$ and $B = B_2$ or B_1 misses B_2 and $B = B_1 \cup B_2$ and $A = A_1$ and $A = A_2$.

Let C, D be non empty sets, F be a sequence of D^C, and n be a natural number. One can check that the functor $F(n)$ yields a function from C into D.

(2) Let us consider sets X, Y, A, B, and objects x, y. Suppose $x \in X$ and $y \in Y$. Then $\chi_{A,X}(x) \cdot \chi_{B,Y}(y) = \chi_{A \times B, X \times Y}(x, y)$.

Let A, B be sets. One can verify that $\chi_{A,B}$ is non-negative.

(3) Let us consider a non empty set X, a semialgebra S of sets of X, a pre-measure P of S, an induced measure m of S and P, and an induced σ-measure M of S and m. Then $\mathrm{COM}(M)$ is complete on $\mathrm{COM}(\sigma(\text{the field generated by } S), M)$.

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The functor $\mathrm{Intervals}_{\mathbb{R}}$ yielding a semialgebra of sets of $\mathbb{R}$ is defined by the term

(Def. 1) the set of all I where I is an interval.

Now we state the propositions:

(4) $\mathrm{Halflines} \subseteq \mathrm{Intervals}_{\mathbb{R}}$.

(5) Let us consider a subset I of $\mathbb{R}$. If I is an interval, then $I \in$ the Borel sets.

(6) (i) $\sigma(\mathrm{Intervals}_{\mathbb{R}}) =$ the Borel sets, and

(ii) σ(the field generated by $\mathrm{Intervals}_{\mathbb{R}}$) $=$ the Borel sets.

The theorem is a consequence of (4) and (5).

2. Family of Semialgebras, Fields and Measures

Now we state the propositions:

(7) Let us consider sets X_1, X_2, a non empty family S_1 of subsets of X_1, and a non empty family S_2 of subsets of X_2. Then the set of all $a \times b$ where a is an element of S_1, b is an element of S_2 is a non empty family of subsets of $X_1 \times X_2$.

(8) Let us consider sets X, Y, a family M of subsets of X with the empty element, and a family N of subsets of Y with the empty element. Then the set of all $A \times B$ where A is an element of M, B is an element of N is a family of subsets of $X \times Y$ with the empty element. The theorem is a consequence of (7).

(9) Let us consider a set X, and disjoint valued finite sequences O, T of elements of X. Suppose $\bigcup \mathrm{rng}\, O$ misses $\bigcup \mathrm{rng}\, T$. Then $O \frown T$ is a disjoint valued finite sequence of elements of X.

(10) Let us consider sets X_1, X_2, a semiring S_1 of X_1, and a semiring S_2 of X_2. Then the set of all $A \times B$ where A is an element of S_1, B is an element of S_2 is a semiring of $X_1 \times X_2$.

(11) Let us consider sets X_1, X_2, a semialgebra S_1 of sets of X_1, and a semialgebra S_2 of sets of X_2. Then the set of all $A \times B$ where A is an element of S_1, B is an element of S_2 is a semialgebra of sets of $X_1 \times X_2$. The theorem is a consequence of (10).

(12) Let us consider sets X_1, X_2, a field O of subsets of X_1, and a field T of subsets of X_2. Then the set of all $A \times B$ where A is an element of O, B is an element of T is a semialgebra of sets of $X_1 \times X_2$. The theorem is a consequence of (11).

Let n be a non zero natural number and X be a non-empty, n-element finite sequence.

A family of semialgebras of X is an n-element finite sequence and is defined by

(Def. 2) for every natural number i such that $i \in \operatorname{Seg} n$ holds $it(i)$ is a semialgebra of sets of $X(i)$.

Let us observe that a family of semialgebras of X is a $\cap$-closed yielding family of semirings of X. Now we state the proposition:

(13) Let us consider a non zero natural number n, a non-empty, n-element finite sequence X, a family S of semialgebras of X, and a natural number i. If $i \in \operatorname{Seg} n$, then $X(i) \in S(i)$.

Let us consider a non-empty, 1-element finite sequence X and a family S of semialgebras of X. Now we state the propositions:

(14) the set of all $\prod\langle s\rangle$ where s is an element of $S(1)$ is a semialgebra of sets of the set of all $\langle x\rangle$ where x is an element of $X(1)$. The theorem is a consequence of (13).

(15) SemiringProduct(S) is a semialgebra of sets of $\prod X$. The theorem is a consequence of (14).

(16) Let us consider sets X_1, X_2, a semialgebra S_1 of sets of X_1, and a semialgebra S_2 of sets of X_2. Then the set of all $s_1 \times s_2$ where s_1 is an element of S_1, s_2 is an element of S_2 is a semialgebra of sets of $X_1 \times X_2$.

(17) Let us consider a non zero natural number n, a non-empty, n-element finite sequence X, and a family S of semialgebras of X. Then SemiringProduct (S) is a semialgebra of sets of $\prod X$.

Proof: Define $\mathcal{P}$[non zero natural number] $\equiv$ for every non-empty, $\$_1$-element finite sequence X for every family S of semialgebras of X, SemiringProduct(S) is a semialgebra of sets of $\prod X$. $\mathcal{P}[1]$. For every non zero natural number k, $\mathcal{P}[k]$ from [3, Sch. 10]. □

(18) Let us consider a non zero natural number n, a non-empty, n-element finite sequence X_8, a non-empty, 1-element finite sequence X_1, a family S_4 of semialgebras of X_8, and a family S_1 of semialgebras of X_1. Then SemiringProduct($S_4 \frown S_1$) is a semialgebra of sets of $\prod(X_8 \frown X_1)$. The theorem is a consequence of (17), (16), and (13).

Let n be a non zero natural number and X be a non-empty, n-element finite sequence.

A family of fields of X is an n-element finite sequence and is defined by

(Def. 3) for every natural number i such that $i \in \operatorname{Seg} n$ holds $it(i)$ is a field of subsets of $X(i)$.

Let S be a family of fields of X and i be a natural number. Assume $i \in \operatorname{Seg} n$. Observe that the functor $S(i)$ yields a field of subsets of $X(i)$.

Observe that a family of fields of X is a family of semialgebras of X.

Let us consider a non-empty, 1-element finite sequence X and a family S of fields of X. Now we state the propositions:

(19) the set of all $\prod\langle s\rangle$ where s is an element of $S(1)$ is a field of subsets of the set of all $\langle x\rangle$ where x is an element of $X(1)$. The theorem is a consequence of (14).

(20) SemiringProduct(S) is a field of subsets of $\prod X$. The theorem is a consequence of (19).

Let n be a non zero natural number, X be a non-empty, n-element finite sequence, and S be a family of fields of X.

A family of measures of S is an n-element finite sequence and is defined by

(Def. 4) for every natural number i such that $i \in \operatorname{Seg} n$ holds $it(i)$ is a measure on $S(i)$.

3. Product of Two Measures

Let X_1, X_2 be sets, S_1 be a field of subsets of X_1, and S_2 be a field of subsets of X_2. The functor MeasRect(S_1, S_2) yielding a semialgebra of sets of $X_1 \times X_2$ is defined by the term

(Def. 5) the set of all $A \times B$ where A is an element of S_1, B is an element of S_2.

Now we state the proposition:

(21) Let us consider a set X, and a field F of subsets of X. Then there exists a semialgebra S of sets of X such that

(i) $F = S$, and

(ii) $F =$ the field generated by S.

Let X_1, X_2 be sets, S_1 be a field of subsets of X_1, S_2 be a field of subsets of X_2, m_1 be a measure on S_1, and m_2 be a measure on S_2. The functor ProdpreMeas(m_1, m_2) yielding a non-negative, zeroed function from MeasRect (S_1, S_2) into $\overline{\mathbb{R}}$ is defined by

(Def. 6) for every element C of MeasRect(S_1, S_2), there exists an element A of S_1 and there exists an element B of S_2 such that $C = A \times B$ and $it(C) = m_1(A) \cdot m_2(B)$.

Now we state the propositions:

(22) Let us consider sets X_1, X_2, a field S_1 of subsets of X_1, a field S_2 of subsets of X_2, a measure m_1 on S_1, a measure m_2 on S_2, and sets A, B.

Suppose $A \in S_1$ and $B \in S_2$. Then $(\text{ProdpreMeas}(m_1, m_2))(A \times B) = m_1(A) \cdot m_2(B)$.

(23) Let us consider sets X_1, X_2, a non empty family S_1 of subsets of X_1, a non empty family S_2 of subsets of X_2, a non empty family S of subsets of $X_1 \times X_2$, and a finite sequence H of elements of S. Suppose $S =$ the set of all $A \times B$ where A is an element of S_1, B is an element of S_2. Then there exists a finite sequence F of elements of S_1 and there exists a finite sequence G of elements of S_2 such that $\operatorname{len} H = \operatorname{len} F$ and $\operatorname{len} H = \operatorname{len} G$ and for every natural number k such that $k \in \operatorname{dom} H$ and $H(k) \neq \emptyset$ holds $H(k) = F(k) \times G(k)$.

PROOF: For every natural number k such that $k \in \operatorname{dom} H$ there exists an element A of S_1 and there exists an element B of S_2 such that $H(k) = A \times B$. Define $\mathcal{P}[\text{natural number}, \text{set}] \equiv$ there exists an element B of S_2 such that $H(\$_1) = \$_2 \times B$. Consider F being a finite sequence of elements of S_1 such that $\operatorname{dom} F = \operatorname{Seg} \operatorname{len} H$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} H$ holds $\mathcal{P}[k, F(k)]$ from [4, Sch. 5]. Define $\mathcal{Q}[\text{natural number}, \text{set}] \equiv$ there exists an element A of S_1 such that $H(\$_1) = A \times \$_2$. For every natural number k such that $k \in \operatorname{Seg} \operatorname{len} H$ there exists an element B of S_2 such that $\mathcal{Q}[k, B]$. Consider G being a finite sequence of elements of S_2 such that $\operatorname{dom} G = \operatorname{Seg} \operatorname{len} H$ and for every natural number k such that $k \in \operatorname{Seg} \operatorname{len} H$ holds $\mathcal{Q}[k, G(k)]$ from [4, Sch. 5]. $\square$

(24) Let us consider a set X, a non empty, semi-diff-closed, $\cap$-closed family S of subsets of X, and elements E_1, E_2 of S. Then there exist disjoint valued finite sequences O, T, F of elements of S such that

(i) $\bigcup \operatorname{rng} O = E_1 \setminus E_2$, and

(ii) $\bigcup \operatorname{rng} T = E_2 \setminus E_1$, and

(iii) $\bigcup \operatorname{rng} F = E_1 \cap E_2$, and

(iv) $(O \frown T) \frown F$ is a disjoint valued finite sequence of elements of S.

The theorem is a consequence of (9).

(25) Let us consider sets X_1, X_2, a field S_1 of subsets of X_1, a field S_2 of subsets of X_2, a measure m_1 on S_1, a measure m_2 on S_2, and elements E_1, E_2 of $\text{MeasRect}(S_1, S_2)$. Suppose E_1 misses E_2 and $E_1 \cup E_2 \in \text{MeasRect}(S_1, S_2)$. Then $(\text{ProdpreMeas}(m_1, m_2))(E_1 \cup E_2) = (\text{ProdpreMeas}(m_1, m_2))(E_1) + (\text{ProdpreMeas}(m_1, m_2))(E_2)$. The theorem is a consequence of (1) and (22).

(26) Let us consider a non empty set X, a non empty family S of subsets of X, a function f from $\mathbb{N}$ into S, and a sequence F of partial functions from X into $\overline{\mathbb{R}}$. Suppose f is disjoint valued and for every natural number

n, $F(n) = \chi_{f(n),X}$. Let us consider an object x. Suppose $x \in X$. Then $\chi_{\bigcup f,X}(x) = (\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}})(x)$.

(27) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and a real number r. Suppose $\operatorname{dom} f \in S$ and $0 \leqslant r$ and for every object x such that $x \in \operatorname{dom} f$ holds $f(x) = r$. Then $\int f \, \mathrm{d}M = r \cdot M(\operatorname{dom} f)$.

Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, a partial function f from X to $\overline{\mathbb{R}}$, and an element A of S. Now we state the propositions:

(28) Suppose there exists an element E of S such that $E = \operatorname{dom} f$ and f is measurable on E and for every object x such that $x \in \operatorname{dom} f \setminus A$ holds $f(x) = 0$ and f is non-negative. Then $\int f \, \mathrm{d}M = \int f \restriction A \, \mathrm{d}M$. The theorem is a consequence of (27).

(29) If f is integrable on M and for every object x such that $x \in \operatorname{dom} f \setminus A$ holds $f(x) = 0$, then $\int f \, \mathrm{d}M = \int f \restriction A \, \mathrm{d}M$. The theorem is a consequence of (27).

(30) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_2 on S_2, a function D from $\mathbb{N}$ into S_1, a function E from $\mathbb{N}$ into S_2, an element A of S_1, an element B of S_2, a sequence F of partial functions from X_2 into $\overline{\mathbb{R}}$, a sequence R of $\mathbb{R}^{X_1}$, and an element x of X_1. Suppose for every natural number n, $R(n) = \chi_{D(n),X_1}$ and for every natural number n, $F(n) = R(n)(x) \cdot \chi_{E(n),X_2}$ and for every natural number n, $E(n) \subseteq B$. Then there exists a sequence I of extended reals such that

(i) for every natural number n, $I(n) = M_2(E(n)) \cdot \chi_{D(n),X_1}(x)$, and

(ii) I is summable, and

(iii) $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \, \mathrm{d}M_2 = \sum I$.

Proof: For every natural number n, $\operatorname{dom}(F(n)) = X_2$. Reconsider $S_3 = X_2$ as an element of S_2. For every natural number n and for every set y such that $y \in E(n)$ holds $F(n)(y) = 0$ or $F(n)(y) = 1$ by [10, (3)], [18, (1)], [12, (39)]. For every natural number n and for every set y such that $y \notin E(n)$ holds $F(n)(y) = 0$. For every natural number n, $F(n)$ is non-negative and $F(n)$ is measurable on B by [8, (51)], [17, (37)], [18, (29)]. For every element y of X_2 such that $y \in B$ holds $F\#y$ is summable by [8, (51), (39)], [19, (16)], [29, (37)].

Consider I being a sequence of extended reals such that for every natural number n, $I(n) = \int F(n) \restriction B \, \mathrm{d}M_2$ and I is summable and $\int \lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa \in \mathbb{N}} \restriction B \, \mathrm{d}M_2 = \sum I$. For every natural number n, $I(n) =$

$M_2(E(n)) \cdot \chi_{D(n),X_1}(x)$ by [28, (61)], [10, (47), (49)], [18, (29)]. For every natural number n, $F(n)$ is measurable on S_3 by [18, (29)], [17, (37)]. For every natural number n, $F(n)$ is without $-\infty$. For every element y of X_2 such that $y \in S_3$ holds $(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}\#y$ is convergent by [19, (38)]. For every object y such that $y \in \operatorname{dom}\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}} \setminus B$ holds $(\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})(y) = 0$ by [19, (43)], [16, (52)]. For every object y such that $y \in \operatorname{dom}\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}}$ holds $(\lim(\sum_{\alpha=0}^{\kappa} F(\alpha))_{\kappa\in\mathbb{N}})(y) \geqslant 0$ by [19, (36)], [8, (51)], [19, (10), (38)]. □

(31) Let us consider a non empty set X, a σ-field S of subsets of X, an element A of S, and an extended real number p. Then $X \longmapsto p$ is measurable on A.
Proof: For every real number r, $A \cap \text{GTE-dom}(X \longmapsto p, r) \in S$ by [26, (7)], [7, (7)]. □

Let A, X be sets. The functor $\overline{\chi}_{A,X}$ yielding a function from X into $\overline{\mathbb{R}}$ is defined by

(Def. 7) for every object x such that $x \in X$ holds if $x \in A$, then $it(x) = +\infty$ and if $x \notin A$, then $it(x) = 0$.

Now we state the proposition:

(32) Let us consider a non empty set X, a σ-field S of subsets of X, and elements A, B of S. Then $\overline{\chi}_{A,X}$ is measurable on B.

Let X, A be sets. Let us observe that $\overline{\chi}_{A,X}$ is non-negative.

(33) Let us consider a non empty set X, a σ-field S of subsets of X, a σ-measure M on S, and an element A of S. Then

(i) if $M(A) \neq 0$, then $\int \overline{\chi}_{A,X}\, \mathrm{d}M = +\infty$, and

(ii) if $M(A) = 0$, then $\int \overline{\chi}_{A,X}\, \mathrm{d}M = 0$.

Proof: Reconsider $X_3 = X$ as an element of S. Reconsider $X_2 = X_3 \setminus A$ as an element of S. Reconsider $F = \overline{\chi}_{A,X}\restriction A$ as a partial function from X to $\overline{\mathbb{R}}$. Reconsider $O = \overline{\chi}_{A,X}\restriction X_2$ as a partial function from X to $\overline{\mathbb{R}}$. Reconsider $T = \overline{\chi}_{A,X}\restriction(X_2 \cup A)$ as a partial function from X to $\overline{\mathbb{R}}$. $\int F\, \mathrm{d}M = 0$. O is measurable on X_2. For every element x of X such that $x \in \operatorname{dom}(\overline{\chi}_{A,X}\restriction X_2)$ holds $(\overline{\chi}_{A,X}\restriction X_2)(x) = 0$ by [10, (47)]. $\int T\, \mathrm{d}M = \int O\, \mathrm{d}M + 0$. □

(34) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and a disjoint valued function K from $\mathbb{N}$ into $\text{MeasRect}(S_1, S_2)$. Suppose $\bigcup K \in \text{MeasRect}(S_1, S_2)$. Then $(\text{ProdpreMeas}(M_1, M_2))(\bigcup K) = \overline{\sum}(\text{ProdpreMeas}(M_1, M_2) \cdot K)$.
Proof: Consider A being an element of S_1, B being an element of S_2 such that $\bigcup K = A \times B$. Consider P being an element of S_1, Q being an element of S_2 such that $\bigcup K = P \times Q$ and $(\text{ProdpreMeas}(M_1, M_2))(\bigcup K) = M_1(P) \cdot$

$M_2(Q)$. Define $\mathcal{F}(\text{object}) = \chi_{K(\$_1),X_1 \times X_2}$. Consider X_6 being a sequence of partial functions from $X_1 \times X_2$ into $\overline{\mathbb{R}}$ such that for every natural number n, $X_6(n) = \mathcal{F}(n)$ from [24, Sch. 1]. Define $\mathcal{P}[\text{natural number}, \text{object}] \equiv \$_2 = \pi_1(K(\$_1))$. For every element i of $\mathbb{N}$, there exists an element A of S_1 such that $\mathcal{P}[i, A]$ by [2, (9)], [7, (7)]. Consider D being a function from $\mathbb{N}$ into S_1 such that for every element i of $\mathbb{N}$, $\mathcal{P}[i, D(i)]$ from [11, Sch. 3]. Define $\mathcal{Q}[\text{natural number}, \text{object}] \equiv \$_2 = \pi_2(K(\$_1))$. For every element i of $\mathbb{N}$, there exists an element B of S_2 such that $\mathcal{Q}[i, B]$ by [2, (9)], [7, (7)].

Consider E being a function from $\mathbb{N}$ into S_2 such that for every element i of $\mathbb{N}$, $\mathcal{Q}[i, E(i)]$ from [11, Sch. 3]. Define $\mathcal{O}(\text{object}) = \chi_{D(\$_1),X_1}$. Consider X_7 being a sequence of partial functions from X_1 into $\overline{\mathbb{R}}$ such that for every natural number n, $X_7(n) = \mathcal{O}(n)$ from [24, Sch. 1]. Define $\mathcal{T}(\text{object}) = \chi_{E(\$_1),X_2}$. Consider X_4 being a sequence of partial functions from X_2 into $\overline{\mathbb{R}}$ such that for every natural number n, $X_4(n) = \mathcal{T}(n)$ from [24, Sch. 1]. For every natural number n and for every objects x, y such that $x \in X_1$ and $y \in X_2$ holds $X_6(n)(x, y) = X_7(n)(x) \cdot X_4(n)(y)$ by [14, (87)], [2, (9)], (2). $(\text{ProdpreMeas}(M_1, M_2))(\bigcup K) = M_1(A) \cdot M_2(B)$ by [14, (110)]. Reconsider $C_1 = \chi_{A \times B, X_1 \times X_2}$ as a function from $X_1 \times X_2$ into $\overline{\mathbb{R}}$. For every element x of X_1, $M_2(B) \cdot \chi_{A,X_1}(x) = \int \text{curry}(C_1, x) \, \mathrm{d}M_2$ by (2), [13, (5)], [19, (14)], [23, (4)]. For every object n such that $n \in \mathbb{N}$ holds $X_7(n) \in \mathbb{R}^{X_1}$ by [12, (39)]. Reconsider $R_1 = X_7$ as a sequence of $\mathbb{R}^{X_1}$. For every natural number n, $D(n) \subseteq A$ and $E(n) \subseteq B$ by [2, (10)], [1, (1)]. For every element x of X_1, there exists a sequence X_5 of partial functions from X_2 into $\overline{\mathbb{R}}$ and there exists a sequence I of extended reals such that for every natural number n, $X_5(n) = R_1(n)(x) \cdot \chi_{E(n),X_2}$ and for every natural number n, $I(n) = M_2(E(n)) \cdot \chi_{D(n),X_1}(x)$ and I is summable and $\int \lim(\sum_{\alpha=0}^{\kappa} X_5(\alpha))_{\kappa \in \mathbb{N}} \, \mathrm{d}M_2 = \sum I$ by [13, (45)], (30).

Reconsider $L_1 = \lim(\sum_{\alpha=0}^{\kappa} X_6(\alpha))_{\kappa \in \mathbb{N}}$ as a function from $X_1 \times X_2$ into $\overline{\mathbb{R}}$. For every element x of X_1 and for every element y of X_2, $(\text{curry}(C_1, x))(y) = (\text{curry}(L_1, x))(y)$. For every element x of X_1, $\text{curry}(C_1, x) = \text{curry}(L_1, x)$. For every element x of X_1, $M_2(B) \cdot \chi_{A,X_1}(x) = \int \text{curry}(L_1, x) \, \mathrm{d}M_2$. For every element x of X_1, there exists a sequence I of extended reals such that for every natural number n, $I(n) = M_2(E(n)) \cdot \chi_{D(n),X_1}(x)$ and $M_2(B) \cdot \chi_{A,X_1}(x) = \sum I$ by [8, (51)], [19, (38), (29), (30)]. Define $\mathcal{R}[\text{natural number}, \text{object}] \equiv$ if $M_2(E(\$_1)) = +\infty$, then $\$_2 = \overline{\chi}_{D(\$_1),X_1}$ and if $M_2(E(\$_1)) \neq +\infty$, then there exists a real number m_2 such that $m_2 = M_2(E(\$_1))$ and $\$_2 = m_2 \cdot \chi_{D(\$_1),X_1}$. For every element n of $\mathbb{N}$, there exists an element y of $X_1 \dot{\to} \overline{\mathbb{R}}$ such that $\mathcal{R}[n, y]$ by [13, (45)], [8, (51)]. Consider F_1 being a function from $\mathbb{N}$ into $X_1 \dot{\to} \overline{\mathbb{R}}$ such that for every element n of $\mathbb{N}$, $\mathcal{R}[n, F_1(n)]$ from [11, Sch. 3]. For every natural number

n, $\operatorname{dom}(F_1(n)) = X_1$. For every natural number n, $F_1(n)$ is non-negative by [8, (51)]. For every natural numbers n, m, $\operatorname{dom}(F_1(n)) = \operatorname{dom}(F_1(m))$.

Reconsider $X_3 = X_1$ as an element of S_1. For every natural number n, $F_1(n)$ is non-negative and $F_1(n)$ is measurable on A and $F_1(n)$ is measurable on X_3 by (32), [18, (29)], [17, (37)]. For every element x of X_1 such that $x \in A$ holds $F_1 \# x$ is summable by [8, (51), (39)], [20, (2)]. Consider J being a sequence of extended reals such that for every natural number n, $J(n) = \int F_1(n) \restriction A \, \mathrm{d}M_1$ and J is summable and $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \restriction A \, \mathrm{d}M_1 = \sum J$. For every natural number n, $J(n) = \int F_1(n) \, \mathrm{d}M_1$. Reconsider $X_3 = X_1$ as an element of S_1. For every element n of $\mathbb{N}$, $J(n) = (\mathrm{ProdpreMeas}(M_1, M_2) \cdot K)(n)$ by (33), [8, (51)], [18, (29)], [16, (86), (88)]. For every element x of X_1, $(\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}})(x) \geqslant 0$ by [19, (38)], [29, (37), (23)], [8, (51)]. For every natural number n, $F_1(n)$ is measurable on X_3 and $F_1(n)$ is without $-\infty$. For every object x such that $x \in \operatorname{dom} \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \setminus A$ holds $(\lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}})(x) = 0$ by [19, (30), (32)], [16, (52)]. $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \, \mathrm{d}M_1 =$
$\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \restriction A \, \mathrm{d}M_1$. $\int \lim(\sum_{\alpha=0}^{\kappa} F_1(\alpha))_{\kappa \in \mathbb{N}} \, \mathrm{d}M_1 = M_1(A) \cdot M_2(B)$ by [11, (63)], [19, (30), (32)], [8, (51)]. $\square$

(35) Let us consider a without $-\infty$ finite sequence f of elements of $\overline{\mathbb{R}}$, and a without $-\infty$ sequence s of extended reals. Suppose for every object n such that $n \in \operatorname{dom} f$ holds $f(n) = s(n)$.
Then $\sum f + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\operatorname{len} f)$.
Proof: Consider F being a sequence of $\overline{\mathbb{R}}$ such that $\sum f = F(\operatorname{len} f)$ and $F(0) = 0$ and for every natural number i such that $i < \operatorname{len} f$ holds $F(i+1) = F(i) + f(i+1)$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$_1 \leqslant \operatorname{len} f$, then $F(\$_1) + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\$_1)$ and $F(\$_1) \neq -\infty$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [3, (11)], [27, (25)], [16, (10)], [3, (13)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. $\square$

(36) Let us consider a non-negative finite sequence f of elements of $\overline{\mathbb{R}}$, and a sequence s of extended reals. Suppose for every object n such that $n \in \operatorname{dom} f$ holds $f(n) = s(n)$ and for every element n of $\mathbb{N}$ such that $n \notin \operatorname{dom} f$ holds $s(n) = 0$. Then

(i) $\sum f = \sum s$, and

(ii) $\sum f = \overline{\sum} s$.

Proof: For every object n such that $n \in \operatorname{dom} s$ holds $0 \leqslant s(n)$ by [8, (51)]. $\sum f + s(0) = (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\operatorname{len} f)$. Define $\mathcal{P}[\text{natural number}] \equiv (\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}}(\operatorname{len} f) = ((\sum_{\alpha=0}^{\kappa} s(\alpha))_{\kappa \in \mathbb{N}} \uparrow \operatorname{len} f)(\$_1)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k+1]$ by [27, (25)]. For every natural number k, $\mathcal{P}[k]$ from [3, Sch. 2]. $\square$

(37) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, a σ-measure M_2 on S_2, and a disjoint valued finite sequence F of elements of $\mathrm{MeasRect}(S_1, S_2)$. Suppose $\bigcup F \in \mathrm{MeasRect}(S_1, S_2)$. Then $(\mathrm{ProdpreMeas}(M_1, M_2))(\bigcup F) = \sum(\mathrm{ProdpreMeas}(M_1, M_2) \cdot F)$.

PROOF: Set $S = \mathrm{MeasRect}(S_1, S_2)$. Define $\mathcal{P}[\text{object}, \text{object}] \equiv$ if $\$_1 \in \mathrm{dom}\, F$, then $\$_2 = F(\$_1)$ and if $\$_1 \notin \mathrm{dom}\, F$, then $\$_2 = \emptyset$. For every element n of $\mathbb{N}$, there exists an element y of S such that $\mathcal{P}[n, y]$ by [10, (3)]. Consider G being a function from $\mathbb{N}$ into S such that for every element n of $\mathbb{N}$, $\mathcal{P}[n, G(n)]$ from [11, Sch. 3]. For every object x such that $x \notin \mathrm{dom}\, F$ holds $G(x) = \emptyset$. For every objects x, y such that $x \neq y$ holds $G(x)$ misses $G(y)$. $(\mathrm{ProdpreMeas}(M_1, M_2))(\bigcup F) = \overline{\sum}(\mathrm{ProdpreMeas}(M_1, M_2) \cdot G)$. For every object n such that $n \in \mathrm{dom}(\mathrm{ProdpreMeas}(M_1, M_2) \cdot F)$ holds $(\mathrm{ProdpreMeas}(M_1, M_2) \cdot F)(n) = (\mathrm{ProdpreMeas}(M_1, M_2) \cdot G)(n)$ by [10, (11), (12), (13)]. For every element n of $\mathbb{N}$ such that $n \notin \mathrm{dom}(\mathrm{ProdpreMeas}(M_1, M_2) \cdot F)$ holds $(\mathrm{ProdpreMeas}(M_1, M_2) \cdot G)(n) = 0$ by [10, (3), (11), (13)]. $\square$

(38) Let us consider non empty sets X_1, X_2, a σ-field S_1 of subsets of X_1, a σ-field S_2 of subsets of X_2, a σ-measure M_1 on S_1, and a σ-measure M_2 on S_2. Then $\mathrm{ProdpreMeas}(M_1, M_2)$ is a pre-measure of $\mathrm{MeasRect}(S_1, S_2)$. The theorem is a consequence of (37) and (34).

Let X_1, X_2 be non empty sets, S_1 be a σ-field of subsets of X_1, S_2 be a σ-field of subsets of X_2, M_1 be a σ-measure on S_1, and M_2 be a σ-measure on S_2. Let us observe that the functor $\mathrm{ProdpreMeas}(M_1, M_2)$ yields a pre-measure of $\mathrm{MeasRect}(S_1, S_2)$.

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