

On Multiset Ordering

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Summary. Formalization of a part of [11]. Unfortunately, not all is possible to be formalized. Namely, in the paper there is a mistake in the proof of Lemma 3. It states that there exists $x \in M_1$ such that $M_1(x) > N_1(x)$ and $(\forall y \in N_1)x \not\prec y$. It should be $M_1(x) \geq N_1(x)$. Nevertheless we do not know whether $x \in N_1$ or not and cannot prove the contradiction. In the article we referred to [8], [9] and [10].

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1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider natural numbers m, n . Then $n = m -' (m -' n) + (n -' m)$.
- (2) Let us consider natural numbers n, m . Then $m -' n \geq m - n$.

Let us consider natural numbers m, n, x, y . Now we state the propositions:

- (3) If $n = m -' x + y$, then $m -' n \leq x$ and $n -' m \leq y$. The theorem is a consequence of (2).
- (4) If $x \leq m$ and $n = m -' x + y$, then $x -' (m -' n) = y -' (n -' m)$. The theorem is a consequence of (3).

Now we state the propositions:

- (5) Let us consider natural numbers k, x_1, x_2, y_1, y_2 . Suppose $x_2 \leq k$ and $x_1 \leq k -' x_2 + y_2$. Then
 - (i) $x_2 + (x_1 -' y_2) \leq k$, and

$$(ii) \quad k -' x_2 + y_2 -' x_1 + y_1 = k -' (x_2 + (x_1 -' y_2)) + (y_2 -' x_1 + y_1).$$

PROOF: $x_2 + (x_1 -' y_2) \leq k$ by [12, (8)]. \square

(6) Let us consider natural numbers x, y . If $x + y > 0$, then $x > 0$ or $y > 0$.

From now on a, b denote objects and I, J denote sets.

Let us consider I . Let J be a non empty set. Let us note that every function from I into J is total and there exists a relational structure which is asymmetric, transitive, and non empty.

Let us consider I . One can verify that there exists a binary relation on I which is asymmetric and transitive.

Let R be a transitive relational structure. Observe that the internal relation of R is transitive.

Let R be an asymmetric relational structure. Let us observe that the internal relation of R is asymmetric.

Let us consider I . Let p, q be I -valued finite sequences. Let us observe that $p \wedge q$ is I -valued.

Now we state the proposition:

(7) Let us consider finite sequences p, q . Suppose $p \wedge q$ is I -valued. Then

(i) p is I -valued, and

(ii) q is I -valued.

Let us consider I . Let f be an I -valued finite sequence and n be a natural number. Let us note that $f \upharpoonright n$ is I -valued.

Now we state the propositions:

(8) Let us consider a finite sequence p . Suppose $a \in \text{rng } p$. Then there exist finite sequences q, r such that $p = (q \wedge \langle a \rangle) \wedge r$.

(9) Let us consider finite sequences p, q . Then $p \subset q$ if and only if $\text{len } p < \text{len } q$ and for every natural number i such that $i \in \text{dom } p$ holds $p(i) = q(i)$.

(10) Let us consider finite sequences p, q, r . Then $r \wedge p \subset r \wedge q$ if and only if $p \subset q$.

PROOF: If $r \wedge p \subset r \wedge q$, then $p \subset q$ by [4, (22)], (9), [15, (30)], [4, (28)]. \square

Let R be an asymmetric, non empty relational structure and x, y be elements of R . Let us observe that the predicate $x \leq y$ is asymmetric.

Now we state the proposition:

(11) Let us consider an asymmetric, non empty relational structure R , and elements x, y of R . Then $x \leq y$ if and only if $x < y$.

2. RELATIONAL EXTENSION

Let us consider I .

A multiset of I is an element of I^\otimes . Observe that every multiset of I is I -defined and natural-valued and every multiset of I is total.

Let m be a natural-valued function. Let us note that the functor support m is defined by the term

(Def. 1) $m^{-1}(\mathbb{N} \setminus \{0\})$.

Let us consider I . One can check that every multiset of I is finite-support.

Now we state the propositions:

(12) a is a multiset of I if and only if a is a bag of I .

(13) $1_{I^\otimes} = \text{EmptyBag } I$.

Let R be a relational structure and x, y be elements of R . We say that $x \equiv y$ if and only if

(Def. 2) $x \not\prec y$ and $y \not\prec x$.

Observe that the predicate is symmetric.

We consider relational multiplicative magmas which extend multiplicative magmas and relational structures and are systems

$\langle \text{a carrier, a multiplication, an internal relation} \rangle$

where the carrier is a set, the multiplication is a binary operation on the carrier, the internal relation is a binary relation on the carrier.

We consider relational monoids which extend multiplicative loop structures and relational structures and are systems

$\langle \text{a carrier, a multiplication, a one, an internal relation} \rangle$

where the carrier is a set, the multiplication is a binary operation on the carrier, the one is an element of the carrier, the internal relation is a binary relation on the carrier.

Let M be a multiplicative loop structure.

A relational extension of M is a relational monoid and is defined by

(Def. 3) the multiplicative loop structure of $it =$ the multiplicative loop structure of M .

Let M be a non empty multiplicative loop structure. Let us observe that every relational extension of M is non empty.

Let M be a multiplicative loop structure. One can check that there exists a relational extension of M which is strict.

Let us consider a multiplicative loop structure N and a relational extension M of N . Now we state the propositions:

(14) a is an element of M if and only if a is an element of N .

(15) $1_N = 1_M$.

Let us consider I . Let M be a relational extension of I^\otimes . Let us observe that every element of M is function-like and relation-like and every element of M is I -defined, natural-valued, and finite-support and every element of M is total.

Now we state the proposition:

(16) Let us consider a relational extension M of I^\otimes . Then the carrier of $M = \text{Bags } I$. The theorem is a consequence of (12) and (14).

The scheme *RelEx* deals with a non empty multiplicative loop structure \mathcal{M} and a binary predicate \mathcal{R} and states that

(Sch. 1) There exists a strict relational extension N of \mathcal{M} such that for every elements x, y of N , $x \leq y$ iff $\mathcal{R}[x, y]$.

Now we state the proposition:

(17) Let us consider a multiplicative loop structure N , and strict relational extensions M_1, M_2 of N . Suppose for every elements m, n of M_1 for every elements x, y of M_2 such that $m = x$ and $n = y$ holds $m \leq n$ iff $x \leq y$. Then $M_1 = M_2$.

PROOF: The internal relation of $M_1 =$ the internal relation of M_2 by [7, (87)]. \square

3. DERSHOWITZ-MANNA ORDER

Let R be a non empty relational structure. The Dershowitz-Manna order R yielding a strict relational extension of $(\text{the carrier of } R)^\otimes$ is defined by

(Def. 4) for every elements m, n of it , $m \leq n$ iff there exist elements x, y of it such that $1_{it} \neq x \mid n$ and $m = n -' x + y$ and for every element b of R such that $y(b) > 0$ there exists an element a of R such that $x(a) > 0$ and $b \leq a$.

Now we state the proposition:

(18) Let us consider bags m, n of I . Then $n = m -' (m -' n) + (n -' m)$. The theorem is a consequence of (1).

Let us consider bags m, n, x, y of I . Now we state the propositions:

(19) If $n = m -' x + y$, then $m -' n \mid x$ and $n -' m \mid y$. The theorem is a consequence of (3).

(20) If $x \mid m$ and $n = m -' x + y$, then $x -' (m -' n) = y -' (n -' m)$. The theorem is a consequence of (4).

Now we state the propositions:

(21) Let us consider bags m, x, y of I . If $x \mid m$ and $x \neq y$, then $m \neq m -' x + y$.

(22) Let us consider a non empty set I , a binary relation R on I , and a reduction sequence r w.r.t. R . If $\text{len } r > 1$, then $r(\text{len } r) \in I$.

(23) Let us consider an asymmetric, transitive binary relation R on I . Then every reduction sequence w.r.t. R is one-to-one.

PROOF: For every natural numbers i, j such that $i > j$ and $i, j \in \text{dom } r$ holds $r(i) \neq r(j)$ by [1, (13)], [13, (22)], [1, (11)], [15, (25)]. \square

(24) Let us consider an asymmetric, transitive, non empty relational structure R , and a set X . Suppose X is finite and there exists an element x of R such that $x \in X$. Then there exists an element x of R such that x is maximal in X .

PROOF: Reconsider $X_1 = X$ as a finite set. Set $Y = \{r, \text{ where } r \text{ is an element of } X_1^* : r \text{ is a reduction sequence w.r.t. the internal relation of } R\}$. Define $\mathcal{P}[\text{natural number}] \equiv \text{there exists a reduction sequence } r \text{ w.r.t. the internal relation of } R \text{ such that } r \in Y \text{ and } \text{len } r = \$_1$. For every natural number k such that $\mathcal{P}[k]$ holds $k \leq \overline{X_1}$ by (23), [1, (43)]. $\mathcal{P}[1]$ by [2, (6)], [4, (74), (39)]. Consider k being a natural number such that $\mathcal{P}[k]$ and for every natural number n such that $\mathcal{P}[n]$ holds $n \leq k$ from [1, Sch. 6]. Consider r being a reduction sequence w.r.t. the internal relation of R such that $r \in Y$ and $\text{len } r = k$. Consider q being an element of X_1^* such that $r = q$ and q is a reduction sequence w.r.t. the internal relation of R . \square

(25) Let us consider bags m, n of I . Then $m -' n \mid m$.

Let us consider I . Note that every element of Bags I is function-like and relation-like.

Now we state the proposition:

(26) Let us consider bags m, n of I . Then

(i) $m -' n \neq \text{EmptyBag } I$, or

(ii) $m = n$, or

(iii) $n -' m \neq \text{EmptyBag } I$.

Let R be an asymmetric, transitive, non empty relational structure. Let us observe that the Dershowitz-Manna order R is defined by

(Def. 5) for every elements m, n of it , $m \leq n$ iff $m \neq n$ and for every element a of R such that $m(a) > n(a)$ there exists an element b of R such that $a \leq b$ and $m(b) < n(b)$.

Now we state the proposition:

(27) Let us consider bags k, x_1, x_2, y_1, y_2 of I . Suppose $x_2 \mid k$ and $x_1 \mid k -' x_2 + y_2$. Then

- (i) $x_2 + (x_1 -' y_2) \mid k$, and
- (ii) $k -' x_2 + y_2 -' x_1 + y_1 = k -' (x_2 + (x_1 -' y_2)) + (y_2 -' x_1 + y_1)$.

The theorem is a consequence of (5).

Let R be an asymmetric, transitive, non empty relational structure. Let us observe that the Dershowitz-Manna order R is asymmetric and transitive.

Let us consider I . The functor $\text{DivOrder}(I)$ yielding a binary relation on $\text{Bags } I$ is defined by

(Def. 6) for every bags b_1, b_2 of I , $\langle b_1, b_2 \rangle \in it$ iff $b_1 \neq b_2$ and $b_1 \mid b_2$.

Now we state the proposition:

(28) Let us consider bags a, b, c of I . If $a \mid b \mid c$, then $a \mid c$.

Let us consider I . Note that $\text{DivOrder}(I)$ is asymmetric and transitive.

Let us consider an asymmetric, transitive, non empty relational structure R . Now we state the propositions:

(29) $\text{DivOrder}(\text{the carrier of } R) \subseteq$ the internal relation of the Dershowitz-Manna order R . The theorem is a consequence of (12) and (14).

(30) Suppose the internal relation of R is empty. Then the internal relation of the Dershowitz-Manna order $R = \text{DivOrder}(\text{the carrier of } R)$. The theorem is a consequence of (29).

Now we state the proposition:

(31) Let us consider asymmetric, transitive, non empty relational structures R_1, R_2 . Suppose the carrier of $R_1 =$ the carrier of R_2 and the internal relation of $R_1 \subseteq$ the internal relation of R_2 . Then the internal relation of the Dershowitz-Manna order $R_1 \subseteq$ the internal relation of the Dershowitz-Manna order R_2 . The theorem is a consequence of (12) and (14).

4. MONOIDAL ORDER

Let us consider I . Let f be a $(\text{Bags } I)$ -valued finite sequence. The functor $\sum f$ yielding a bag of I is defined by

(Def. 7) there exists a function F from \mathbb{N} into $\text{Bags } I$ such that $it = F(\text{len } f)$ and $F(0) = \text{EmptyBag } I$ and for every natural number i and for every bag b of I such that $i < \text{len } f$ and $b = f(i + 1)$ holds $F(i + 1) = F(i) + b$.

Now we state the proposition:

(32) $\sum \varepsilon_{\text{Bags } I} = \text{EmptyBag } I$.

Let us consider I . Let b be a bag of I . One can verify that $\langle b \rangle$ is $(\text{Bags } I)$ -valued as a finite sequence.

Now we state the proposition:

(33) Let us consider a (Bags I)-valued finite sequence p , and a bag b of I . Then $\sum(p \wedge \langle b \rangle) = \sum p + b$.

PROOF: Set $f = p \wedge \langle b \rangle$. Consider F being a function from \mathbb{N} into Bags I such that $\sum f = F(\text{len } f)$ and $F(0) = \text{EmptyBag } I$ and for every natural number i and for every bag b of I such that $i < \text{len } f$ and $b = f(i + 1)$ holds $F(i + 1) = F(i) + b$. Consider F_1 being a function from \mathbb{N} into Bags I such that $\sum p = F_1(\text{len } p)$ and $F_1(0) = \text{EmptyBag } I$ and for every natural number i and for every bag b of I such that $i < \text{len } p$ and $b = p(i + 1)$ holds $F_1(i + 1) = F_1(i) + b$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq \text{len } p$, then $F(\$1) = F_1(\$1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [5, (16)], [1, (13), (11)], [15, (25)]. For every natural number i , $\mathcal{P}[i]$ from [1, Sch. 2]. \square

From now on b denotes a bag of I .

Now we state the propositions:

(34) $\sum \langle b \rangle = b$. The theorem is a consequence of (33) and (32).

(35) Let us consider (Bags I)-valued finite sequences p, q . Then $\sum(p \wedge q) = \sum p + \sum q$.

PROOF: Set $f = p \wedge q$. Consider F being a function from \mathbb{N} into Bags I such that $\sum f = F(\text{len } f)$ and $F(0) = \text{EmptyBag } I$ and for every natural number i and for every bag b of I such that $i < \text{len } f$ and $b = f(i + 1)$ holds $F(i + 1) = F(i) + b$. Consider F_1 being a function from \mathbb{N} into Bags I such that $\sum p = F_1(\text{len } p)$ and $F_1(0) = \text{EmptyBag } I$ and for every natural number i and for every bag b of I such that $i < \text{len } p$ and $b = p(i + 1)$ holds $F_1(i + 1) = F_1(i) + b$. Consider F_2 being a function from \mathbb{N} into Bags I such that $\sum q = F_2(\text{len } q)$ and $F_2(0) = \text{EmptyBag } I$ and for every natural number i and for every bag b of I such that $i < \text{len } q$ and $b = q(i + 1)$ holds $F_2(i + 1) = F_2(i) + b$. Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \leq \text{len } p$, then $F(\$1) = F_1(\$1)$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [4, (22)], [1, (11), (13)], [15, (25)]. For every natural number i , $\mathcal{P}[i]$ from [1, Sch. 2]. Define $\mathcal{Q}[\text{natural number}] \equiv$ if $\$1 \leq \text{len } q$, then $F(\text{len } p + \$1) = \sum p + F_2(\$1)$. For every natural number i such that $\mathcal{Q}[i]$ holds $\mathcal{Q}[i + 1]$ by [4, (22)], [1, (13), (11)], [15, (25)]. For every natural number i , $\mathcal{Q}[i]$ from [1, Sch. 2]. \square

Let us consider a (Bags I)-valued finite sequence p . Now we state the propositions:

(36) $\sum(\langle b \rangle \wedge p) = b + \sum p$. The theorem is a consequence of (35) and (34).

(37) If $b \in \text{rng } p$, then $b \mid \sum p$. The theorem is a consequence of (8), (7), (33), and (35).

Now we state the proposition:

(38) Let us consider a (Bags I)-valued finite sequence p , and an object i . Suppose $i \in \text{support } \sum p$. Then there exists b such that

- (i) $b \in \text{rng } p$, and
- (ii) $i \in \text{support } b$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every (Bags I)-valued finite sequence p such that $\text{len } p = \$1$ for every object i such that $i \in \text{support } \sum p$ there exists b such that $b \in \text{rng } p$ and $i \in \text{support } b$. $\mathcal{P}[0]$. For every natural number j such that $\mathcal{P}[j]$ holds $\mathcal{P}[j + 1]$ by [3, (3)], (7), [4, (40)], [15, (25)]. For every natural number j , $\mathcal{P}[j]$ from [1, Sch. 2]. \square

Let us consider I and b .

A partition of b is a (Bags I)-valued finite sequence and is defined by

(Def. 8) $b = \sum it$.

Observe that the functor $\langle b \rangle$ yields a partition of b . Let R be a relational structure, M be a relational extension of (the carrier of R) $^\otimes$, b be an element of M , and p be a partition of b . We say that p is co-ordered if and only if

(Def. 9) for every natural number i such that $i, i + 1 \in \text{dom } p$ for every elements b_1, b_2 of M such that $b_1 = p(i)$ and $b_2 = p(i + 1)$ holds $b_2 \leq b_1$.

Let R be a non empty relational structure and b be a bag of the carrier of R . We say that p is ordered if and only if

(Def. 10) for every bag m of the carrier of R such that $m \in \text{rng } p$ for every element x of R such that $m(x) > 0$ holds $m(x) = b(x)$ and for every bag m of the carrier of R such that $m \in \text{rng } p$ for every elements x, y of R such that $m(x) > 0$ and $m(y) > 0$ and $x \neq y$ holds $x \equiv y$ and for every bag m of the carrier of R such that $m \in \text{rng } p$ holds $m \neq \text{EmptyBag}(\text{the carrier of } R)$ and for every natural number i such that $i, i + 1 \in \text{dom } p$ for every element x of R such that $p_{i+1}(x) > 0$ there exists an element y of R such that $p_i(y) > 0$ and $x \leq y$.

In the sequel R denotes an asymmetric, transitive, non empty relational structure, a, b, c denote bags of the carrier of R , and x, y, z denote elements of R .

Now we state the propositions:

(39) $\langle a \rangle$ is ordered if and only if $a \neq \text{EmptyBag}(\text{the carrier of } R)$ and for every x and y such that $a(x) > 0$ and $a(y) > 0$ and $x \neq y$ holds $x \equiv y$.

(40) Let us consider a (Bags I)-valued finite sequence p , and bags a, b of I . Then $\langle a \rangle \wedge p$ is a partition of b if and only if $a \mid b$ and p is a partition of $b -' a$. The theorem is a consequence of (36).

From now on p denotes a partition of $b -' a$ and q denotes a partition of b .

Now we state the proposition:

(41) If $q = \langle a \rangle \wedge p$ and q is ordered, then p is ordered. The theorem is a consequence of (37) and (25).

Let us consider I . Let m be a bag of I and J be a set. The functor $m \downarrow J$ yielding a bag of I is defined by

(Def. 11) for every object i such that $i \in I$ holds if $i \in J$, then $it(i) = m(i)$ and if $i \notin J$, then $it(i) = 0$.

From now on J denotes a set and m denotes a bag of I .

Now we state the propositions:

(42) $\text{support}(m \downarrow J) = J \cap \text{support } m$.

(43) $m \downarrow J + m \downarrow (I \setminus J) = m$.

(44) $m \downarrow J \mid m$.

(45) If $\text{support } m \subseteq J$, then $m \downarrow J = m$.

(46) $\text{support}(m -' m \downarrow J) = \text{support } m \setminus J$.

(47) If q is ordered and $q = \langle a \rangle \wedge p$ and $a(x) > 0$, then $a(x) = b(x)$.

(48) If q is ordered and $q = \langle a \rangle \wedge p$ and $a(x) > 0$ and $a(y) > 0$ and $x \neq y$, then $x \equiv y$.

(49) If q is ordered and $q = \langle a \rangle \wedge p$, then $a \neq \text{EmptyBag}(\text{the carrier of } R)$.

(50) Let us consider a bag c of the carrier of R , and a (Bags(the carrier of R))-valued finite sequence r . Suppose q is ordered and $q = \langle a, c \rangle \wedge r$ and $c(y) > 0$. Then there exists x such that

(i) $a(x) > 0$, and

(ii) $y \leq x$.

(51) If $x \in I$ and for every y such that $y \in I$ and $y \neq x$ holds $x \equiv y$, then x is maximal in I .

(52) If q is ordered and $q = \langle a \rangle \wedge p$ and $c \in \text{rng } p$ and $c(x) > 0$, then there exists y such that $a(y) > 0$ and $x \leq y$.

PROOF: Consider i being an object such that $i \in \text{dom } p$ and $c = p(i)$. Define $\mathcal{P}[\text{natural number}] \equiv \text{if } \$1 \in \text{dom } p$, then for every x such that $p_{\$1}(x) > 0$ there exists y such that $a(y) > 0$ and $x \leq y$. $\mathcal{P}[1]$ by [4, (28)], [15, (25)], [4, (40)]. For every natural number i such that $i \geq 1$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i + 1]$ by [1, (13)], [15, (25)], [4, (28)], [16, (3)]. For every natural number i such that $i \geq 1$ holds $\mathcal{P}[i]$ from [1, Sch. 8]. \square

Let us assume that q is ordered and $q = \langle a \rangle \wedge p$. Now we state the propositions:

(53) x is maximal in support b if and only if $a(x) > 0$.

PROOF: $a \mid \sum q = b$. There exists no y such that $y \in \text{support } b$ and $x < y$ by (48), (38), [4, (31), (39)]. \square

(54) $a = b \upharpoonright \{x : x \text{ is maximal in support } b\}$. The theorem is a consequence of (53) and (47).

Now we state the propositions:

(55) Let us consider a $(\text{Bags } I)$ -valued finite sequence p . Suppose $\sum p = \text{EmptyBag } I$ and for every bag a of I such that $a \in \text{rng } p$ holds $a \neq \text{EmptyBag } I$. Then $p = \emptyset$. The theorem is a consequence of (37).

(56) Let us consider bags a, b of I . If $a \neq \text{EmptyBag } I$, then $a + b \neq \text{EmptyBag } I$.

(57) Let us consider partitions p, q of b . If p is ordered and q is ordered, then $p = q$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ for every b and q such that $\text{len } q = \$_1$ and q is ordered for every partition p of b such that p is ordered holds $q = p$. $\mathcal{P}[0]$. For every natural number i such that $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [5, (130)], (40), (49), (36). For every natural number i , $\mathcal{P}[i]$ from [1, Sch. 2]. \square

Let us consider I . Let a, b be bags of I . One can verify that the functor $\langle a, b \rangle$ yields an element of $\text{Bags } I \times \text{Bags } I$. Now we state the proposition:

(58) Suppose $a \neq \text{EmptyBag}(\text{the carrier of } R)$. Then $\{x : x \text{ is maximal in support } a\} \neq \emptyset$. The theorem is a consequence of (24).

Let us consider R and b . The ordered partition of b yielding a $(\text{Bags}(\text{the carrier of } R))$ -valued finite sequence is defined by

(Def. 12) there exist functions F, G from \mathbb{N} into $\text{Bags}(\text{the carrier of } R)$ such that $F(0) = b$ and $G(0) = \text{EmptyBag}(\text{the carrier of } R)$ and for every natural number i , $G(i+1) = F(i) \upharpoonright \{x : x \text{ is maximal in support}(F(i))\}$ and $F(i+1) = F(i) -' G(i+1)$ and there exists a natural number i such that $F(i) = \text{EmptyBag}(\text{the carrier of } R)$ and $it = G \upharpoonright \text{Seg } i$ and for every natural number j such that $j < i$ holds $F(j) \neq \text{EmptyBag}(\text{the carrier of } R)$.

One can verify that the ordered partition of b yields a partition of b . Let us note that the ordered partition of b is ordered as a partition of b .

Now we state the proposition:

(59) $b = \text{EmptyBag}(\text{the carrier of } R)$ if and only if the ordered partition of $b = \emptyset$. The theorem is a consequence of (32).

Let us consider R . The functor $\prec_{\mathcal{M}} R$ yielding a strict relational extension of $(\text{the carrier of } R)^{\otimes}$ is defined by

(Def. 13) for every elements m, n of it , $m \leq n$ iff $m \neq n$ and for every x such that $m(x) > 0$ holds $m(x) < n(x)$ or there exists y such that $n(y) > 0$ and $x \leq y$.

Let us note that $\prec_{\mathcal{M}} R$ is asymmetric and transitive.

Let us consider I . Let R be a relation between I and I .

The functor $\text{LexOrder}(I, R)$ yielding a binary relation on I^* is defined by

(Def. 14) for every I -valued finite sequences p, q , $\langle p, q \rangle \in \text{it}$ iff $p \subset q$ or there exists a natural number k such that $k \in \text{dom } p$ and $k \in \text{dom } q$ and $\langle p(k), q(k) \rangle \in R$ and for every natural number n such that $1 \leq n < k$ holds $p(n) = q(n)$.

Let R be a transitive binary relation on I . One can verify that $\text{LexOrder}(I, R)$ is transitive.

Let R be an asymmetric binary relation on I . Note that $\text{LexOrder}(I, R)$ is asymmetric.

Now we state the proposition:

(60) Let us consider an asymmetric binary relation R on I , and I -valued finite sequences p, q, r . Then $\langle p, q \rangle \in \text{LexOrder}(I, R)$ if and only if $\langle r \hat{\ } p, r \hat{\ } q \rangle \in \text{LexOrder}(I, R)$. The theorem is a consequence of (10).

Let us consider R . The functor $\prec\prec_{\mathcal{M}} R$ yielding a strict relational extension of (the carrier of R)[⊗] is defined by

(Def. 15) for every elements m, n of it , $m \leq n$ iff $\langle \text{the ordered partition of } m, \text{the ordered partition of } n \rangle \in \text{LexOrder}(\text{(the carrier of } \prec\prec_{\mathcal{M}} R), \text{(the internal relation of } \prec\prec_{\mathcal{M}} R))$.

Observe that $\prec\prec_{\mathcal{M}} R$ is asymmetric and transitive.

Now we state the propositions:

(61) Let us consider elements a, b of the Dershowitz-Manna order R . Suppose $a \leq b$. Then $b \neq \text{EmptyBag}(\text{the carrier of } R)$. The theorem is a consequence of (29).

(62) Let us consider elements a, b, c, d of the Dershowitz-Manna order R , and a bag e of the carrier of R . Suppose $a \leq b$ and $e \mid a$ and $e \mid b$. If $c = a -' e$ and $d = b -' e$, then $c \leq d$.

(63) Let us consider a (Bags I)-valued finite sequence p , and an object x . Suppose $x \in I$ and $(\sum p)(x) > 0$. Then there exists a natural number i such that

- (i) $i \in \text{dom } p$, and
- (ii) $p_i(x) > 0$.

PROOF: Define $\mathcal{P}[\text{object}] \equiv$ for every (Bags I)-valued finite sequence p such that $p = \$_1$ and $(\sum p)(x) > 0$ there exists a natural number i such that $i \in \text{dom } p$ and $p_i(x) > 0$. $\mathcal{P}[\emptyset]$ by (32), [14, (7)]. For every finite sequence p and for every object a such that $\mathcal{P}[p]$ holds $\mathcal{P}[p \hat{\ } \langle a \rangle]$ by (7), [4, (40)], [15, (25)], [6, (102)]. For every finite sequence p , $\mathcal{P}[p]$ from [4, Sch. 3]. \square

(64) If q is ordered and $q_1(x) = 0$ and $b(x) > 0$, then there exists y such that $q_1(y) > 0$ and $x \leq y$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv$ if $\$1 \in \text{dom } q$, then for every x such that $q_{\$1}(x) > 0$ there exists y such that $q_1(y) > 0$ and $x \leq y$. $\mathcal{P}[2]$ by [15, (25)]. For every natural number i such that $2 \leq i$ and $\mathcal{P}[i]$ holds $\mathcal{P}[i+1]$ by [1, (11)], [15, (25)], [16, (3)]. For every natural number i such that $i \geq 2$ holds $\mathcal{P}[i]$ from [1, Sch. 8]. Consider i being a natural number such that $i \in \text{dom } q$ and $q_i(x) > 0$. \square

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