

Cousin's Lemma

Roland Coghetto
Rue de la Brasserie 5
7100 La Louvière, Belgium

Summary. We formalize, in two different ways, that “the n -dimensional Euclidean metric space is a complete metric space” (version 1. with the results obtained in [13], [26], [25] and version 2., the results obtained in [13], [14], (*regi-strations*) [24]).

With the Cantor's theorem - in complete metric space (proof by Karol Pałk in [22]), we formalize “The Nested Intervals Theorem in 1-dimensional Euclidean metric space”.

Pierre Cousin's proof in 1892 [18] the lemma, published in 1895 [9] states that:

“Soit, sur le plan YOX, une aire connexe S limitée par un contour fermé simple ou complexe; on suppose qu'à chaque point de S ou de son périmètre correspond un cercle, de rayon non nul, ayant ce point pour centre : il est alors toujours possible de subdiviser S en régions, en nombre fini et assez petites pour que chacune d'elles soit complètement intérieure au cercle correspondant à un point convenablement choisi dans S ou sur son périmètre.”

(In the plane YOX let S be a connected area bounded by a closed contour, simple or complex; one supposes that at each point of S or its perimeter there is a circle, of non-zero radius, having this point as its centre; it is then always possible to subdivide S into regions, finite in number and sufficiently small for each one of them to be entirely inside a circle corresponding to a suitably chosen point in S or on its perimeter) [23].

Cousin's Lemma, used in Henstock and Kurzweil integral [29] (generalized Riemann integral), state that: “for any gauge δ , there exists at least one δ -fine tagged partition”. In the last section, we formalize this theorem. We use the suggestions given to the Cousin's Theorem p.11 in [5] and with notations: [4], [29], [19], [28] and [12].

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1. PRELIMINARIES

Now we state the proposition:

- (1) Let us consider non empty, increasing finite sequences p, q of elements of \mathbb{R} . Suppose $p(\text{len } p) < q(1)$. Then $p \wedge q$ is a non empty, increasing finite sequence of elements of \mathbb{R} .

Let us consider real numbers a, b . Now we state the propositions:

- (2) If $1 < a$ and $0 < b < 1$, then $\log_a b < 0$.
 (3) If $1 < a$ and $1 < b$, then $0 < \log_a b$.

Let us consider a finite sequence p and a natural number i .

Let us assume that $i \in \text{dom } p$. Now we state the propositions:

- (4) (i) $i = 1$, or
 (ii) $1 < i$.
 (5) (i) $i = \text{len } p$, or
 (ii) $i < \text{len } p$.

Now we state the propositions:

- (6) Let us consider an object x . Then $\prod\{\langle x \rangle\} = \{\langle x \rangle\}$.
 (7) Let us consider an element x of \mathcal{R}^1 . Then there exists a real number r_3 such that $x = \langle r_3 \rangle$.
 (8) Let us consider a real number a . Then $\langle a \rangle$ is a point of \mathcal{E}^1 .
 (9) Let us consider real numbers a, b . If $a \leq b$, then $a \leq \frac{a+b}{2} \leq b$.
 (10) Let us consider real numbers a, b, c . If $a \leq b < c$, then $a < \frac{b+c}{2}$.

Let us consider real numbers a, b . Now we state the propositions:

- (11) If $a < b$, then $\frac{a+b}{2} < b$.
 (12) If $a \leq b$, then $[a, b]$ is a non empty, compact subset of \mathbb{R} .
 (13) Let us consider a finite sequence f . Suppose $2 \leq \text{len } f$.
 Then $f_{|1}(\text{len } f_{|1}) = f(\text{len } f)$.

2. \mathcal{E}^n IS COMPLETE - PROOF VERSION 1

From now on n denotes a natural number, s_1 denotes a sequence of \mathcal{E}^n , and s_2 denotes a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$.

Now we state the propositions:

(14) Let us consider elements x, y of \mathcal{E}^n , and points g, h of $\langle \mathcal{E}^n, \|\cdot\| \rangle$. If $x = g$ and $y = h$, then $\rho(x, y) = \|g - h\|$.

(15) (i) s_1 is a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$, and

(ii) s_2 is a sequence of \mathcal{E}^n .

PROOF: s_1 is a sequence of $\langle \mathcal{E}^n, \|\cdot\| \rangle$ by [10, (67), (22)]. s_2 is a sequence of \mathcal{E}^n by [10, (22), (67)]. \square

Let us assume that $s_1 = s_2$. Now we state the propositions:

(16) s_1 is Cauchy if and only if s_2 is Cauchy sequence by norm. The theorem is a consequence of (14).

(17) s_1 is convergent if and only if s_2 is convergent. The theorem is a consequence of (14).

(18) Let us consider a sequence S_1 of \mathcal{E}^n . If S_1 is Cauchy, then S_1 is convergent. The theorem is a consequence of (15), (16), and (17).

(19) \mathcal{E}^n is complete.

3. \mathcal{E}^n IS COMPLETE - PROOF VERSION 2

Now we state the propositions:

(20) The distance by norm of $\langle \mathcal{E}^n, \|\cdot\| \rangle = \rho^n$. The theorem is a consequence of (14).

(21) $\text{MetricSpaceNorm}\langle \mathcal{E}^n, \|\cdot\| \rangle = \mathcal{E}^n$. The theorem is a consequence of (20).

(22) \mathcal{E}^n is complete. The theorem is a consequence of (21).

Let n be a natural number. Let us note that \mathcal{E}^n is complete.

4. THE NESTED INTERVALS THEOREM (1-DIMENSIONAL EUCLIDEAN SPACE)

Let a, b be sequences of real numbers. The functor $\text{IntervalSeq}(a, b)$ yielding a sequence of subsets of \mathcal{R}^1 is defined by

(Def. 1) for every natural number i , $it(i) = \prod \langle [a(i), b(i)] \rangle$.

Now we state the propositions:

(23) Let us consider sequences a, b of real numbers, and a natural number i . Then $(\text{IntervalSeq}(a, b))(i) = \prod \langle [a(i), b(i)] \rangle$.

(24) Let us consider sequences a, b of real numbers. Then $\text{IntervalSeq}(a, b)$ is a sequence of subsets of \mathcal{E}^1 .

(25) $\prod\langle\mathbb{R}\rangle = \mathcal{R}^1$.

(26) Let us consider real numbers a, b , and points x_1, x_2 of \mathcal{E}^1 . Suppose $x_1 = \langle a \rangle$ and $x_2 = \langle b \rangle$. Then $\rho(x_1, x_2) = |a - b|$.

(27) Let us consider real numbers a, b , and a subset S of \mathcal{E}^1 . Suppose $a \leq b$ and $S = \prod\langle[a, b]\rangle$. Let us consider points x, y of \mathcal{E}^1 . If $x, y \in S$, then $\rho(x, y) \leq b - a$.

PROOF: Set $s = \prod\langle[a, b]\rangle$. For every points x, y of \mathcal{E}^1 such that $x, y \in s$ holds $\rho(x, y) \leq b - a$ by (6), [10, (67), (22)], (7). \square

(28) Let us consider real numbers a, b , and a subset S of \mathcal{E}^1 . If $a \leq b$ and $S = \prod\langle[a, b]\rangle$, then S is bounded.

PROOF: Set $s = \prod\langle[a, b]\rangle$. There exists a real number r such that $0 < r$ and for every points x, y of \mathcal{E}^1 such that $x, y \in s$ holds $\rho(x, y) \leq r$ by (6), [10, (67), (22)], (7). \square

Let us consider sequences a, b of real numbers.

Let us assume that for every natural number i , $a(i) \leq b(i)$ and $a(i) \leq a(i+1)$ and $b(i+1) \leq b(i)$. Now we state the propositions:

(29) $\text{IntervalSeq}(a, b)$ is a non-empty, pointwise bounded, closed sequence of subsets of \mathcal{E}^1 .

PROOF: Reconsider $s = \text{IntervalSeq}(a, b)$ as a sequence of subsets of \mathcal{E}^1 . s is non-empty by (23), [1, (26)], [3, (2)]. s is pointwise bounded by (23), (6), [10, (67), (22)]. s is closed by (23), [10, (67), (22)], (25). \square

(30) $\text{IntervalSeq}(a, b)$ is non ascending. The theorem is a consequence of (23).

(31) Let us consider real numbers a, b, x . If $a \leq x \leq b$, then $\langle x \rangle \in \prod\langle[a, b]\rangle$.

PROOF: Reconsider $P = \langle x \rangle$ as a point of \mathcal{E}^1 . There exists a function g such that $g = P$ and $\text{dom } g = \text{dom}\langle[a, b]\rangle$ and for every object y such that $y \in \text{dom}\langle[a, b]\rangle$ holds $g(y) \in \langle[a, b]\rangle(y)$ by [3, (2)]. \square

(32) Let us consider real numbers a, b , and a subset S of \mathcal{E}^1 . If $a \leq b$ and $S = \prod\langle[a, b]\rangle$, then $\emptyset S = b - a$. The theorem is a consequence of (28), (31), (27), (8), and (26).

(33) Let us consider sequences a, b of real numbers. Suppose for every natural number i , $a(i) \leq b(i)$ and a is non-decreasing and b is non-increasing. Then

(i) a is convergent, and

(ii) b is convergent.

(34) Let us consider sequences a, b of real numbers. Suppose $a(0) \leq b(0)$ and for every natural number i , $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or

$a(i + 1) = \frac{a(i)+b(i)}{2}$ and $b(i + 1) = b(i)$. Let us consider a natural number i . Then $a(i) \leq b(i)$.

PROOF: Define $\mathcal{P}[\text{object}] \equiv$ there exists a natural number i such that $\$1 = i$ and $a(i) \leq b(i)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number k , $\mathcal{P}[k]$ from [2, Sch. 2]. \square

Let us consider sequences a, b of real numbers, a sequence S of subsets of \mathcal{E}^1 , and a natural number i . Now we state the propositions:

- (35) Suppose $a(0) \leq b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every natural number i , $a(i + 1) = a(i)$ and $b(i + 1) = \frac{a(i)+b(i)}{2}$ or $a(i + 1) = \frac{a(i)+b(i)}{2}$ and $b(i + 1) = b(i)$. Then
 - (i) $a(i) \leq b(i)$, and
 - (ii) $a(i) \leq a(i + 1)$, and
 - (iii) $b(i + 1) \leq b(i)$, and
 - (iv) $(\emptyset S)(i) = b(i) - a(i)$.

The theorem is a consequence of (34), (9), (24), (23), and (32).

- (36) Suppose $a(0) = b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every natural number i , $a(i + 1) = a(i)$ and $b(i + 1) = \frac{a(i)+b(i)}{2}$ or $a(i + 1) = \frac{a(i)+b(i)}{2}$ and $b(i + 1) = b(i)$. Then
 - (i) $a(i) = a(0)$, and
 - (ii) $b(i) = b(0)$, and
 - (iii) $(\emptyset S)(i) = 0$.

PROOF: Define $\mathcal{P}[\text{natural number}] \equiv a(\$1) = a(0)$ and $b(\$1) = b(0)$. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$. For every natural number k , $\mathcal{P}[k]$ from [2, Sch. 2]. \square

- (37) Let us consider sequences a, b of real numbers. Suppose for every natural number i , $a(i + 1) = a(i)$ and $b(i + 1) = \frac{a(i)+b(i)}{2}$ or $a(i + 1) = \frac{a(i)+b(i)}{2}$ and $b(i + 1) = b(i)$. Let us consider a natural number i , and a real number r . If $r = 2^i$ and $r \neq 0$, then $b(i) - a(i) \leq \frac{b(0)-a(0)}{r}$.

PROOF: Define $\mathcal{P}[\text{object}] \equiv$ there exists a natural number i and there exists a real number r such that $\$1 = i$ and $r = 2^i$ and $r \neq 0$ and $b(i) - a(i) \leq \frac{b(0)-a(0)}{r}$. $\mathcal{P}[0]$ by [17, (4)]. For every natural number k such that $\mathcal{P}[k]$ holds $\mathcal{P}[k + 1]$ by [17, (87), (6)]. For every natural number k , $\mathcal{P}[k]$ from [2, Sch. 2]. Consider i_1 being a natural number, r_1 being a real number such that $i = i_1$ and $r_1 = 2^{i_1}$ and $r_1 \neq 0$ and $b(i_1) - a(i_1) \leq \frac{b(0)-a(0)}{r_1}$. \square

- (38) Let us consider sequences a, b of real numbers, and a sequence S of subsets of \mathcal{E}^1 . Suppose $a(0) \leq b(0)$ and $S = \text{IntervalSeq}(a, b)$ and for every

natural number i , $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then

- (i) $\emptyset S$ is convergent, and
- (ii) $\lim \emptyset S = 0$.

The theorem is a consequence of (36), (35), (34), (33), (3), and (37).

(39) Let us consider sequences a, b of real numbers. Suppose $a(0) \leq b(0)$ and for every natural number i , $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then $\cap \text{IntervalSeq}(a, b)$ is not empty. The theorem is a consequence of (24), (35), (29), (30), and (38).

(40) Let us consider a real number r , and sequences a, b of real numbers. Suppose $0 < r$ and $a(0) \leq b(0)$ and for every natural number i , $a(i+1) = a(i)$ and $b(i+1) = \frac{a(i)+b(i)}{2}$ or $a(i+1) = \frac{a(i)+b(i)}{2}$ and $b(i+1) = b(i)$. Then there exists a real number c such that

- (i) for every natural number j , $a(j) \leq c \leq b(j)$, and
- (ii) there exists a natural number k such that $c-r < a(k)$ and $b(k) < c+r$.

The theorem is a consequence of (39), (23), (24), (35), (29), and (38).

5. TAGGED PARTITION

Now we state the propositions:

(41) Let us consider a non empty, closed interval subset I of \mathbb{R} . Then there exist real numbers a, b such that

- (i) $a \leq b$, and
- (ii) $I = [a, b]$.

(42) Let us consider non empty, closed interval subsets I_1, I_2 of \mathbb{R} . Suppose $\sup I_1 = \inf I_2$. Then there exist real numbers a, b, c such that

- (i) $a \leq c \leq b$, and
- (ii) $I_1 = [a, c]$, and
- (iii) $I_2 = [c, b]$.

The theorem is a consequence of (41).

Let A be a non empty, closed interval subset of \mathbb{R} and D be a partition of A . The set of tagged partitions of D yielding a subset of \mathbb{R}^* is defined by

(Def. 2) for every object x , $x \in \text{it}$ iff there exists a non empty, non-decreasing finite sequence s of elements of \mathbb{R} such that $x = s$ and $\text{dom } s = \text{dom } D$ and for every natural number i such that $i \in \text{dom } s$ holds $s(i) \in \text{divset}(D, i)$.

Now we state the propositions:

- (43) Let us consider a non empty, closed interval subset A of \mathbb{R} , and a partition D of A . Then $D \in$ the set of tagged partitions of D .

PROOF: For every natural number i such that $i \in \text{dom } D$ holds $D(i) \in \text{divset}(D, i)$ by [15, (19)], (4). \square

- (44) Let us consider real numbers a, b , and a non empty, closed interval subset I_4 of \mathbb{R} . If $I_4 = [a, b]$, then $\langle b \rangle$ is a partition of I_4 .

PROOF: $\langle b \rangle$ is a partition of I_4 by [3, (39)], [15, (19)]. \square

Let I be a non empty, closed interval subset of \mathbb{R} and φ be a positive yielding function from I into \mathbb{R} .

A tagged partition of I and φ is defined by

- (Def. 3) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $it = \langle D, T \rangle$.

Let T_1 be a tagged partition of I and φ . We say that T_1 is δ -fine if and only if

- (Def. 4) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $T_1 = \langle D, T \rangle$ and for every natural number i such that $i \in \text{dom } D$ holds $\text{vol}(\text{divset}(D, i)) \leq \varphi(T(i))$.

6. PARTITION COMPOSITION

Let us consider a real number r . Now we state the propositions:

- (45) (i) $\sup\{r\} = r$, and

(ii) $\inf\{r\} = r$.

- (46) $\text{vol}(\{r\}) = 0$. The theorem is a consequence of (45).

- (47) Let us consider non empty, closed interval subsets I_1, I_2 of \mathbb{R} , and a positive yielding function φ from I_1 into \mathbb{R} . Suppose $I_2 \subseteq I_1$. Then $\varphi|_{I_2}$ is a positive yielding function from I_2 into \mathbb{R} .

- (48) Let us consider a non empty, closed interval subset I of \mathbb{R} , and a real number c . Suppose $c \in I$. Then

(i) $[\inf I, c]$ is a non empty, closed interval subset of \mathbb{R} , and

(ii) $[c, \sup I]$ is a non empty, closed interval subset of \mathbb{R} , and

(iii) $\sup[\inf I, c] = \inf[c, \sup I]$.

The theorem is a consequence of (41).

Let I_5, I_6 be non empty, closed interval subsets of \mathbb{R} , D_4 be a partition of I_5 , and D_6 be a partition of I_6 . Assume $\sup I_5 \leq \inf I_6$. The functor $D_4 \cdot D_6$

yielding a non empty, increasing finite sequence of elements of \mathbb{R} is defined by the term

$$(\text{Def. 5}) \quad \begin{cases} D_4 \wedge D_6, & \text{if } D_6(1) \neq \sup I_5, \\ D_4 \wedge D_{6|1}, & \text{otherwise.} \end{cases}$$

Now we state the propositions:

- (49) Let us consider non empty, closed interval subsets I_5, I_6 of \mathbb{R} , a partition D_4 of I_5 , and a partition D_6 of I_6 . Suppose $\sup I_5 = \inf I_6$ and $\text{len } D_6 = 1$ and $D_6(1) = \inf I_6$. Then $D_4 \cdot D_6 = D_4$.
- (50) Let us consider non empty, closed interval subsets I_1, I_2, I of \mathbb{R} . Suppose $\sup I_1 \leq \inf I_2$ and $\inf I \leq \inf I_1$ and $\sup I_2 \leq \sup I$. Then $I_1 \cup I_2 \subseteq I$.
- (51) Let us consider non empty, closed interval subsets I_1, I_2, I of \mathbb{R} , a partition D_1 of I_1 , and a partition D_2 of I_2 . Suppose $\sup I_1 \leq \inf I_2$ and $I = [\inf I_1, \sup I_2]$. Then $D_1 \cdot D_2$ is a partition of I . The theorem is a consequence of (50).
- (52) Let us consider a non empty, closed interval subset I of \mathbb{R} , and a partition D of I . Then the set of tagged partitions of D is not empty.
- (53) Let us consider a non empty, increasing finite sequence s of elements of \mathbb{R} , and a real number r . Suppose $s(\text{len } s) < r$. Then $s \wedge \langle r \rangle$ is a non empty, increasing finite sequence of elements of \mathbb{R} . The theorem is a consequence of (1).
- (54) Let us consider non empty, increasing finite sequences s_1, s_2 of elements of \mathbb{R} , and a real number r . Suppose $s_1(\text{len } s_1) < r < s_2(1)$. Then $(s_1 \wedge \langle r \rangle) \wedge s_2$ is a non empty, increasing finite sequence of elements of \mathbb{R} . The theorem is a consequence of (53) and (1).
- (55) Let us consider non empty, closed interval subsets I_1, I_2, I of \mathbb{R} . Suppose $\sup I_1 = \inf I_2$ and $I = I_1 \cup I_2$. Then
 - (i) $\inf I = \inf I_1$, and
 - (ii) $\sup I = \sup I_2$.
- (56) Let us consider a non empty, closed interval subset I of \mathbb{R} , and a partition D of I . Then
 - (i) $\text{divset}(D, 1) = [\inf I, D(1)]$, and
 - (ii) for every natural number j such that $j \in \text{dom } D$ and $j \neq 1$ holds $\text{divset}(D, j) = [D(j-1), D(j)]$.

PROOF: For every natural number j such that $j \in \text{dom } D$ and $j \neq 1$ holds $\text{divset}(D, j) = [D(j-1), D(j)]$ by [12, (4)]. \square
- (57) Let us consider a real number r , and finite sequences p, q of elements of \mathbb{R} . Then $\text{len}((p \wedge \langle r \rangle) \wedge q) = \text{len } p + \text{len } q + 1$.

- (58) Let us consider a non empty, closed interval subset I of \mathbb{R} , and a partition D of I . Then every element of the set of tagged partitions of D is not empty. The theorem is a consequence of (43).
- (59) Let us consider a non empty, closed interval subset I of \mathbb{R} , a partition D of I , and an element T of the set of tagged partitions of D . Then $\text{rng } T \subseteq \mathbb{R}$. The theorem is a consequence of (43).

Let I be a non empty, closed interval subset of \mathbb{R} , φ be a positive yielding function from I into \mathbb{R} , and T_1 be a tagged partition of I and φ . The functor T_1 -partition yielding a partition of I is defined by

(Def. 6) there exists a partition D of I and there exists an element T of the set of tagged partitions of D such that $it = D$ and $T_1 = \langle D, T \rangle$.

7. EXAMPLES OF PARTITIONS

In the sequel r, s denote real numbers.

Now we state the proposition:

- (60) Let us consider a function φ from $[r, s]$ into $]0, +\infty[$. Suppose $r \leq s$. Then the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$ is a family of subsets of $[r, s]_{\mathbb{T}}$.

Let us consider a function φ from $[r, s]$ into $]0, +\infty[$ and a family S of subsets of $[r, s]_{\mathbb{T}}$.

Let us assume that $r \leq s$ and $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Now we state the propositions:

- (61) S is a cover of $[r, s]_{\mathbb{T}}$.
 PROOF: $[r, s] \subseteq \bigcup S$ by [8, (3)]. \square
- (62) S is open.
 PROOF: For every subset P of $[r, s]_{\mathbb{T}}$ such that $P \in S$ holds P is open by [11, (17)], [20, (35)], [11, (15), (9), (10)]. \square
- (63) Suppose $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Then S is connected.
 PROOF: For every subset X of $[r, s]_{\mathbb{T}}$ such that $X \in S$ holds X is connected by [16, (43)]. \square
- (64) Let us consider a function φ from $[r, s]$ into $]0, +\infty[$, and a family S of subsets of $[r, s]_{\mathbb{T}}$. Suppose $r \leq s$ and $S =$ the set of all $]x - \varphi(x), x + \varphi(x)[\cap [r, s]$ where x is an element of $[r, s]$. Let us consider an interval cover I of S . Then

- (i) I is a finite sequence of elements of $2^{\mathbb{R}}$, and
- (ii) $\text{rng } I \subseteq S$, and

- (iii) $\bigcup \text{rng } I = [r, s]$, and
- (iv) for every natural number n such that $1 \leq n$ holds if $n \leq \text{len } I$, then I_n is not empty and if $n + 1 \leq \text{len } I$, then $\inf I_n \leq \inf I_{n+1}$ and $\sup I_n \leq \sup I_{n+1}$ and $\inf I_{n+1} < \sup I_n$ and if $n + 2 \leq \text{len } I$, then $\sup I_n \leq \inf I_{n+2}$, and
- (v) if $[r, s] \in S$, then $I = \langle [r, s] \rangle$, and
- (vi) if $[r, s] \notin S$, then there exists a real number p such that $r < p \leq s$ and $I(1) = [r, p[$ and there exists a real number q such that $r \leq p < s$ and $I(\text{len } I) =]p, s]$ and for every natural number n such that $1 < n < \text{len } I$ there exist real numbers p, q such that $r \leq p < q \leq s$ and $I(n) =]p, q[$.

The theorem is a consequence of (61), (62), and (63).

(65) Let us consider real numbers r, s, t, x . Then

- (i) if $r \leq x - t$ and $x + t \leq s$, then $]x - t, x + t[\cap [r, s] =]x - t, x + t[$, and
- (ii) if $r \leq x - t$ and $s < x + t$, then $]x - t, x + t[\cap [r, s] =]x - t, s]$, and
- (iii) if $x - t < r$ and $x + t \leq s$, then $]x - t, x + t[\cap [r, s] = [r, x + t[$, and
- (iv) if $x - t < r$ and $s < x + t$, then $]x - t, x + t[\cap [r, s] = [r, s]$.

(66) Let us consider real numbers r, s, t, x , and a subset X_1 of \mathbb{R} . Suppose $0 < t$ and $r \leq x \leq s$ and $X_1 =]x - t, x + t[\cap [r, s]$. Then

- (i) if $r \leq x - t$ and $x + t \leq s$, then $\inf X_1 = x - t$ and $\sup X_1 = x + t$, and
- (ii) if $r \leq x - t$ and $s < x + t$, then $\inf X_1 = x - t$ and $\sup X_1 = s$, and
- (iii) if $x - t < r$ and $x + t \leq s$, then $\inf X_1 = r$ and $\sup X_1 = x + t$, and
- (iv) if $x - t < r$ and $s < x + t$, then $\inf X_1 = r$ and $\sup X_1 = s$.

The theorem is a consequence of (65).

Let us consider real numbers a, b, c , non empty, compact subsets I_5, I_6 of \mathbb{R} , a partition D_4 of I_5 , a partition D_6 of I_6 , and natural numbers i, j .

Let us assume that $a \leq c \leq b$ and $I_5 = [a, c]$ and $I_6 = [c, b]$. Now we state the propositions:

(67) Suppose $i \in \text{dom } D_4$ and $j \in \text{dom } D_6$. Then

- (i) if $i < \text{len } D_4$, then $D_4(i) < D_6(j)$, and
- (ii) if $i = \text{len } D_4$ and $c < D_6(1)$, then $D_4(i) < D_6(j)$, and
- (iii) if $D_6(1) = c$, then $D_4(\text{len } D_4) = D_6(1)$.

PROOF: If $i < \text{len } D_4$, then $D_4(i) < D_6(j)$ by [3, (3)]. If $i = \text{len } D_4$ and $c < D_6(1)$, then $D_4(i) < D_6(j)$ by [7, (6)], [3, (91)]. \square

(68) If $i \in \text{dom } D_4$ and $j \in \text{dom } D_6$, then if $c < D_6(1)$, then $D_4(i) < D_6(j)$. The theorem is a consequence of (67).

(69) Let us consider real numbers a, b, c , and non empty, compact subsets I_4, I_5, I_6 of \mathbb{R} . Suppose $a \leq c \leq b$ and $I_4 = [a, b]$ and $I_5 = [a, c]$ and $I_6 = [c, b]$. Let us consider a partition D_4 of I_5 , and a partition D_6 of I_6 . Suppose $c < D_6(1)$. Then $D_4 \wedge D_6$ is a partition of I_4 .

PROOF: Set $D_5 = D_4 \wedge D_6$. For every extended reals e_1, e_2 such that $e_1, e_2 \in \text{dom } D_5$ and $e_1 < e_2$ holds $D_5(e_1) < D_5(e_2)$ by [3, (25)], (68), [2, (11)], [3, (1)]. $\text{rng } D_5 \subseteq I_4$ by [3, (31)]. $D_5(\text{len } D_5) = \sup I_4$ by [3, (3), (22)], [15, (19)]. \square

(70) Let us consider real numbers a, b , and a non empty, closed interval subset I_4 of \mathbb{R} . Suppose $a \leq b$ and $I_4 = [a, b]$. Let us consider a partition D_3 of I_4 . If $\text{len } D_3 = 1$, then $D_3 = \langle b \rangle$.

(71) Let us consider real numbers a, b , a non empty, compact subset I_4 of \mathbb{R} , and a partition D_3 of I_4 . Suppose $2 \leq \text{len } D_3$. Then $D_{3|1}$ is a partition of I_4 .

PROOF: Set $D = D_{3|1}$. D is a non empty, increasing finite sequence of elements of \mathbb{R} by [3, (60)]. $\text{rng } D \subseteq I_4$ by [7, (33)]. $D(\text{len } D) = \sup I_4$ by [3, (3)]. \square

(72) Let us consider real numbers a, b . Suppose $a < b$. Then $\langle a, b \rangle$ is a non empty, increasing finite sequence of elements of \mathbb{R} .

PROOF: Set $s = \langle a, b \rangle$. s is increasing by [3, (44), (2)]. \square

(73) Let us consider real numbers a, b , and a non empty, closed interval subset I_4 of \mathbb{R} . Suppose $a < b$ and $I_4 = [a, b]$. Then $\langle a, b \rangle$ is a partition of I_4 .

PROOF: $\langle a, b \rangle$ is a partition of I_4 by (72), [6, (127)], [3, (44)], [15, (19)]. \square

8. COUSIN'S LEMMA

Now we state the proposition:

(74) Let us consider real numbers a, b , and a positive yielding function φ from $[a, b]$ into \mathbb{R} . Suppose $a \leq b$. Then there exists a non empty, increasing finite sequence x of elements of \mathbb{R} and there exists a non empty finite sequence t of elements of \mathbb{R} such that $x(1) = a$ and $x(\text{len } x) = b$ and $t(1) = a$ and $\text{dom } x = \text{dom } t$ and for every natural number i such that $i - 1, i \in \text{dom } t$ holds $t(i) - \varphi(t(i)) \leq x(i - 1) \leq t(i)$ and for every natural number i such that $i \in \text{dom } t$ holds $t(i) \leq x(i) \leq t(i) + \varphi(t(i))$.

PROOF: Define $\mathcal{P}[\text{object}] \equiv$ there exists a non empty, increasing finite sequence x of elements of \mathbb{R} and there exists a non empty finite sequence t of elements of \mathbb{R} such that $x(1) = a$ and $x(\text{len } x) = \$_1$ and $t(1) = a$ and $\text{dom } x = \text{dom } t$ and for every natural number i such that $i - 1, i \in \text{dom } t$ holds $t(i) - \varphi(t(i)) \leq x(i - 1) \leq t(i)$ and for every natural number i such that $i \in \text{dom } t$ holds $t(i) \leq x(i) \leq t(i) + \varphi(t(i))$. Consider C being a set such that for every object x , $x \in C$ iff $x \in [a, b]$ and $\mathcal{P}[x]$. For every object x such that $x \in C$ holds x is real. Reconsider $c = \sup C$ as a real number. $c \in [a, b]$. Consider d being an element of $\overline{\mathbb{R}}$ such that $d \in C$ and $c - \varphi(c) < d$. Consider D_0 being a non empty, increasing finite sequence of elements of \mathbb{R} , T_0 being a non empty finite sequence of elements of \mathbb{R} such that $D_0(1) = a$ and $D_0(\text{len } D_0) = d$ and $T_0(1) = a$ and $\text{dom } D_0 = \text{dom } T_0$ and for every natural number i such that $i - 1, i \in \text{dom } T_0$ holds $T_0(i) - \varphi(T_0(i)) \leq D_0(i - 1) \leq T_0(i)$ and for every natural number i such that $i \in \text{dom } T_0$ holds $T_0(i) \leq D_0(i) \leq T_0(i) + \varphi(T_0(i))$. $c \in C$ and $\mathcal{P}[c]$ by (1), [27, (32)], [3, (22), (39), (1)]. $c = b$ by (1), [27, (32)], [3, (22), (39), (1)]. \square

(75) COUSIN'S LEMMA:

Let us consider a non empty, closed interval subset I of \mathbb{R} , and a positive yielding function φ from I into \mathbb{R} . Then there exists a tagged partition T_1 of I and φ such that T_1 is δ -fine.

PROOF: Consider a, b being real numbers such that $a \leq b$ and $I = [a, b]$. Reconsider $r = \frac{1}{2}$ as a positive real number. Reconsider $\phi = r \cdot \varphi$ as a positive yielding function from I into \mathbb{R} . Consider x being a non empty, increasing finite sequence of elements of \mathbb{R} , t being a non empty finite sequence of elements of \mathbb{R} such that $x(1) = a$ and $x(\text{len } x) = b$ and $t(1) = a$ and $\text{dom } x = \text{dom } t$ and for every natural number i such that $i - 1, i \in \text{dom } t$ holds $t(i) - \phi(t(i)) \leq x(i - 1) \leq t(i)$ and for every natural number i such that $i \in \text{dom } t$ holds $t(i) \leq x(i) \leq t(i) + \phi(t(i))$. Reconsider $D = x$ as a partition of I . Reconsider $T = t$ as an element of the set of tagged partitions of D . Reconsider $T_1 = \langle D, T \rangle$ as a tagged partition of I and φ . T_1 is δ -fine by [15, (19)], (4), [8, (3)], [21, (20)]. \square

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