

# Cousin's Lemma

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**Summary.** We formalize, in two different ways, that “the  $n$ -dimensional Euclidean metric space is a complete metric space” (version 1. with the results obtained in [13], [26], [25] and version 2., the results obtained in [13], [14], (*regi-strations*) [24]).

With the Cantor's theorem - in complete metric space (proof by Karol Pałk in [22]), we formalize “The Nested Intervals Theorem in 1-dimensional Euclidean metric space”.

Pierre Cousin's proof in 1892 [18] the lemma, published in 1895 [9] states that:

“Soit, sur le plan YOX, une aire connexe  $S$  limitée par un contour fermé simple ou complexe; on suppose qu'à chaque point de  $S$  ou de son périmètre correspond un cercle, de rayon non nul, ayant ce point pour centre : il est alors toujours possible de subdiviser  $S$  en régions, en nombre fini et assez petites pour que chacune d'elles soit complètement intérieure au cercle correspondant à un point convenablement choisi dans  $S$  ou sur son périmètre.”

(In the plane YOX let  $S$  be a connected area bounded by a closed contour, simple or complex; one supposes that at each point of  $S$  or its perimeter there is a circle, of non-zero radius, having this point as its centre; it is then always possible to subdivide  $S$  into regions, finite in number and sufficiently small for each one of them to be entirely inside a circle corresponding to a suitably chosen point in  $S$  or on its perimeter) [23].

Cousin's Lemma, used in Henstock and Kurzweil integral [29] (generalized Riemann integral), state that: “for any gauge  $\delta$ , there exists at least one  $\delta$ -fine tagged partition”. In the last section, we formalize this theorem. We use the suggestions given to the Cousin's Theorem p.11 in [5] and with notations: [4], [29], [19], [28] and [12].

MSC: 54D30 03B35

Keywords: Cousin's lemma; Cousin's theorem; nested intervals theorem

MML identifier: COUSIN, version: 8.1.04 5.36.1267

## 1. PRELIMINARIES

Now we state the proposition:

- (1) Let us consider non empty, increasing finite sequences  $p, q$  of elements of  $\mathbb{R}$ . Suppose  $p(\text{len } p) < q(1)$ . Then  $p \wedge q$  is a non empty, increasing finite sequence of elements of  $\mathbb{R}$ .

Let us consider real numbers  $a, b$ . Now we state the propositions:

- (2) If  $1 < a$  and  $0 < b < 1$ , then  $\log_a b < 0$ .  
 (3) If  $1 < a$  and  $1 < b$ , then  $0 < \log_a b$ .

Let us consider a finite sequence  $p$  and a natural number  $i$ .

Let us assume that  $i \in \text{dom } p$ . Now we state the propositions:

- (4) (i)  $i = 1$ , or  
 (ii)  $1 < i$ .  
 (5) (i)  $i = \text{len } p$ , or  
 (ii)  $i < \text{len } p$ .

Now we state the propositions:

- (6) Let us consider an object  $x$ . Then  $\prod\{\langle x \rangle\} = \{\langle x \rangle\}$ .  
 (7) Let us consider an element  $x$  of  $\mathcal{R}^1$ . Then there exists a real number  $r_3$  such that  $x = \langle r_3 \rangle$ .  
 (8) Let us consider a real number  $a$ . Then  $\langle a \rangle$  is a point of  $\mathcal{E}^1$ .  
 (9) Let us consider real numbers  $a, b$ . If  $a \leq b$ , then  $a \leq \frac{a+b}{2} \leq b$ .  
 (10) Let us consider real numbers  $a, b, c$ . If  $a \leq b < c$ , then  $a < \frac{b+c}{2}$ .

Let us consider real numbers  $a, b$ . Now we state the propositions:

- (11) If  $a < b$ , then  $\frac{a+b}{2} < b$ .  
 (12) If  $a \leq b$ , then  $[a, b]$  is a non empty, compact subset of  $\mathbb{R}$ .  
 (13) Let us consider a finite sequence  $f$ . Suppose  $2 \leq \text{len } f$ .  
 Then  $f_{|1}(\text{len } f_{|1}) = f(\text{len } f)$ .

2.  $\mathcal{E}^n$  IS COMPLETE - PROOF VERSION 1

From now on  $n$  denotes a natural number,  $s_1$  denotes a sequence of  $\mathcal{E}^n$ , and  $s_2$  denotes a sequence of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ .

Now we state the propositions:

- (14) Let us consider elements  $x, y$  of  $\mathcal{E}^n$ , and points  $g, h$  of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ . If  $x = g$  and  $y = h$ , then  $\rho(x, y) = \|g - h\|$ .
- (15) (i)  $s_1$  is a sequence of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$ , and  
 (ii)  $s_2$  is a sequence of  $\mathcal{E}^n$ .

PROOF:  $s_1$  is a sequence of  $\langle \mathcal{E}^n, \|\cdot\| \rangle$  by [10, (67), (22)].  $s_2$  is a sequence of  $\mathcal{E}^n$  by [10, (22), (67)].  $\square$

Let us assume that  $s_1 = s_2$ . Now we state the propositions:

- (16)  $s_1$  is Cauchy if and only if  $s_2$  is Cauchy sequence by norm. The theorem is a consequence of (14).
- (17)  $s_1$  is convergent if and only if  $s_2$  is convergent. The theorem is a consequence of (14).
- (18) Let us consider a sequence  $S_1$  of  $\mathcal{E}^n$ . If  $S_1$  is Cauchy, then  $S_1$  is convergent. The theorem is a consequence of (15), (16), and (17).
- (19)  $\mathcal{E}^n$  is complete.

3.  $\mathcal{E}^n$  IS COMPLETE - PROOF VERSION 2

Now we state the propositions:

- (20) The distance by norm of  $\langle \mathcal{E}^n, \|\cdot\| \rangle = \rho^n$ . The theorem is a consequence of (14).
- (21)  $\text{MetricSpaceNorm}\langle \mathcal{E}^n, \|\cdot\| \rangle = \mathcal{E}^n$ . The theorem is a consequence of (20).
- (22)  $\mathcal{E}^n$  is complete. The theorem is a consequence of (21).

Let  $n$  be a natural number. Let us note that  $\mathcal{E}^n$  is complete.

4. THE NESTED INTERVALS THEOREM (1-DIMENSIONAL EUCLIDEAN SPACE)

Let  $a, b$  be sequences of real numbers. The functor  $\text{IntervalSeq}(a, b)$  yielding a sequence of subsets of  $\mathcal{R}^1$  is defined by

(Def. 1) for every natural number  $i$ ,  $it(i) = \prod \langle [a(i), b(i)] \rangle$ .

Now we state the propositions:

- (23) Let us consider sequences  $a, b$  of real numbers, and a natural number  $i$ . Then  $(\text{IntervalSeq}(a, b))(i) = \prod \langle [a(i), b(i)] \rangle$ .

(24) Let us consider sequences  $a, b$  of real numbers. Then  $\text{IntervalSeq}(a, b)$  is a sequence of subsets of  $\mathcal{E}^1$ .

(25)  $\prod\langle\mathbb{R}\rangle = \mathcal{R}^1$ .

(26) Let us consider real numbers  $a, b$ , and points  $x_1, x_2$  of  $\mathcal{E}^1$ . Suppose  $x_1 = \langle a \rangle$  and  $x_2 = \langle b \rangle$ . Then  $\rho(x_1, x_2) = |a - b|$ .

(27) Let us consider real numbers  $a, b$ , and a subset  $S$  of  $\mathcal{E}^1$ . Suppose  $a \leq b$  and  $S = \prod\langle[a, b]\rangle$ . Let us consider points  $x, y$  of  $\mathcal{E}^1$ . If  $x, y \in S$ , then  $\rho(x, y) \leq b - a$ .

PROOF: Set  $s = \prod\langle[a, b]\rangle$ . For every points  $x, y$  of  $\mathcal{E}^1$  such that  $x, y \in s$  holds  $\rho(x, y) \leq b - a$  by (6), [10, (67), (22)], (7).  $\square$

(28) Let us consider real numbers  $a, b$ , and a subset  $S$  of  $\mathcal{E}^1$ . If  $a \leq b$  and  $S = \prod\langle[a, b]\rangle$ , then  $S$  is bounded.

PROOF: Set  $s = \prod\langle[a, b]\rangle$ . There exists a real number  $r$  such that  $0 < r$  and for every points  $x, y$  of  $\mathcal{E}^1$  such that  $x, y \in s$  holds  $\rho(x, y) \leq r$  by (6), [10, (67), (22)], (7).  $\square$

Let us consider sequences  $a, b$  of real numbers.

Let us assume that for every natural number  $i$ ,  $a(i) \leq b(i)$  and  $a(i) \leq a(i+1)$  and  $b(i+1) \leq b(i)$ . Now we state the propositions:

(29)  $\text{IntervalSeq}(a, b)$  is a non-empty, pointwise bounded, closed sequence of subsets of  $\mathcal{E}^1$ .

PROOF: Reconsider  $s = \text{IntervalSeq}(a, b)$  as a sequence of subsets of  $\mathcal{E}^1$ .  $s$  is non-empty by (23), [1, (26)], [3, (2)].  $s$  is pointwise bounded by (23), (6), [10, (67), (22)].  $s$  is closed by (23), [10, (67), (22)], (25).  $\square$

(30)  $\text{IntervalSeq}(a, b)$  is non ascending. The theorem is a consequence of (23).

(31) Let us consider real numbers  $a, b, x$ . If  $a \leq x \leq b$ , then  $\langle x \rangle \in \prod\langle[a, b]\rangle$ .

PROOF: Reconsider  $P = \langle x \rangle$  as a point of  $\mathcal{E}^1$ . There exists a function  $g$  such that  $g = P$  and  $\text{dom } g = \text{dom}\langle[a, b]\rangle$  and for every object  $y$  such that  $y \in \text{dom}\langle[a, b]\rangle$  holds  $g(y) \in \langle[a, b]\rangle(y)$  by [3, (2)].  $\square$

(32) Let us consider real numbers  $a, b$ , and a subset  $S$  of  $\mathcal{E}^1$ . If  $a \leq b$  and  $S = \prod\langle[a, b]\rangle$ , then  $\emptyset S = b - a$ . The theorem is a consequence of (28), (31), (27), (8), and (26).

(33) Let us consider sequences  $a, b$  of real numbers. Suppose for every natural number  $i$ ,  $a(i) \leq b(i)$  and  $a$  is non-decreasing and  $b$  is non-increasing. Then

(i)  $a$  is convergent, and

(ii)  $b$  is convergent.

(34) Let us consider sequences  $a, b$  of real numbers. Suppose  $a(0) \leq b(0)$  and for every natural number  $i$ ,  $a(i+1) = a(i)$  and  $b(i+1) = \frac{a(i)+b(i)}{2}$  or

$a(i + 1) = \frac{a(i)+b(i)}{2}$  and  $b(i + 1) = b(i)$ . Let us consider a natural number  $i$ . Then  $a(i) \leq b(i)$ .

PROOF: Define  $\mathcal{P}[\text{object}] \equiv$  there exists a natural number  $i$  such that  $\$1 = i$  and  $a(i) \leq b(i)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [2, Sch. 2].  $\square$

Let us consider sequences  $a, b$  of real numbers, a sequence  $S$  of subsets of  $\mathcal{E}^1$ , and a natural number  $i$ . Now we state the propositions:

- (35) Suppose  $a(0) \leq b(0)$  and  $S = \text{IntervalSeq}(a, b)$  and for every natural number  $i$ ,  $a(i + 1) = a(i)$  and  $b(i + 1) = \frac{a(i)+b(i)}{2}$  or  $a(i + 1) = \frac{a(i)+b(i)}{2}$  and  $b(i + 1) = b(i)$ . Then
  - (i)  $a(i) \leq b(i)$ , and
  - (ii)  $a(i) \leq a(i + 1)$ , and
  - (iii)  $b(i + 1) \leq b(i)$ , and
  - (iv)  $(\emptyset S)(i) = b(i) - a(i)$ .

The theorem is a consequence of (34), (9), (24), (23), and (32).

- (36) Suppose  $a(0) = b(0)$  and  $S = \text{IntervalSeq}(a, b)$  and for every natural number  $i$ ,  $a(i + 1) = a(i)$  and  $b(i + 1) = \frac{a(i)+b(i)}{2}$  or  $a(i + 1) = \frac{a(i)+b(i)}{2}$  and  $b(i + 1) = b(i)$ . Then
  - (i)  $a(i) = a(0)$ , and
  - (ii)  $b(i) = b(0)$ , and
  - (iii)  $(\emptyset S)(i) = 0$ .

PROOF: Define  $\mathcal{P}[\text{natural number}] \equiv a(\$1) = a(0)$  and  $b(\$1) = b(0)$ . For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [2, Sch. 2].  $\square$

- (37) Let us consider sequences  $a, b$  of real numbers. Suppose for every natural number  $i$ ,  $a(i + 1) = a(i)$  and  $b(i + 1) = \frac{a(i)+b(i)}{2}$  or  $a(i + 1) = \frac{a(i)+b(i)}{2}$  and  $b(i + 1) = b(i)$ . Let us consider a natural number  $i$ , and a real number  $r$ . If  $r = 2^i$  and  $r \neq 0$ , then  $b(i) - a(i) \leq \frac{b(0)-a(0)}{r}$ .

PROOF: Define  $\mathcal{P}[\text{object}] \equiv$  there exists a natural number  $i$  and there exists a real number  $r$  such that  $\$1 = i$  and  $r = 2^i$  and  $r \neq 0$  and  $b(i) - a(i) \leq \frac{b(0)-a(0)}{r}$ .  $\mathcal{P}[0]$  by [17, (4)]. For every natural number  $k$  such that  $\mathcal{P}[k]$  holds  $\mathcal{P}[k + 1]$  by [17, (87), (6)]. For every natural number  $k$ ,  $\mathcal{P}[k]$  from [2, Sch. 2]. Consider  $i_1$  being a natural number,  $r_1$  being a real number such that  $i = i_1$  and  $r_1 = 2^{i_1}$  and  $r_1 \neq 0$  and  $b(i_1) - a(i_1) \leq \frac{b(0)-a(0)}{r_1}$ .  $\square$

- (38) Let us consider sequences  $a, b$  of real numbers, and a sequence  $S$  of subsets of  $\mathcal{E}^1$ . Suppose  $a(0) \leq b(0)$  and  $S = \text{IntervalSeq}(a, b)$  and for every

natural number  $i$ ,  $a(i+1) = a(i)$  and  $b(i+1) = \frac{a(i)+b(i)}{2}$  or  $a(i+1) = \frac{a(i)+b(i)}{2}$  and  $b(i+1) = b(i)$ . Then

- (i)  $\emptyset S$  is convergent, and
- (ii)  $\lim \emptyset S = 0$ .

The theorem is a consequence of (36), (35), (34), (33), (3), and (37).

(39) Let us consider sequences  $a, b$  of real numbers. Suppose  $a(0) \leq b(0)$  and for every natural number  $i$ ,  $a(i+1) = a(i)$  and  $b(i+1) = \frac{a(i)+b(i)}{2}$  or  $a(i+1) = \frac{a(i)+b(i)}{2}$  and  $b(i+1) = b(i)$ . Then  $\cap \text{IntervalSeq}(a, b)$  is not empty. The theorem is a consequence of (24), (35), (29), (30), and (38).

(40) Let us consider a real number  $r$ , and sequences  $a, b$  of real numbers. Suppose  $0 < r$  and  $a(0) \leq b(0)$  and for every natural number  $i$ ,  $a(i+1) = a(i)$  and  $b(i+1) = \frac{a(i)+b(i)}{2}$  or  $a(i+1) = \frac{a(i)+b(i)}{2}$  and  $b(i+1) = b(i)$ . Then there exists a real number  $c$  such that

- (i) for every natural number  $j$ ,  $a(j) \leq c \leq b(j)$ , and
- (ii) there exists a natural number  $k$  such that  $c-r < a(k)$  and  $b(k) < c+r$ .

The theorem is a consequence of (39), (23), (24), (35), (29), and (38).

## 5. TAGGED PARTITION

Now we state the propositions:

(41) Let us consider a non empty, closed interval subset  $I$  of  $\mathbb{R}$ . Then there exist real numbers  $a, b$  such that

- (i)  $a \leq b$ , and
- (ii)  $I = [a, b]$ .

(42) Let us consider non empty, closed interval subsets  $I_1, I_2$  of  $\mathbb{R}$ . Suppose  $\sup I_1 = \inf I_2$ . Then there exist real numbers  $a, b, c$  such that

- (i)  $a \leq c \leq b$ , and
- (ii)  $I_1 = [a, c]$ , and
- (iii)  $I_2 = [c, b]$ .

The theorem is a consequence of (41).

Let  $A$  be a non empty, closed interval subset of  $\mathbb{R}$  and  $D$  be a partition of  $A$ . The set of tagged partitions of  $D$  yielding a subset of  $\mathbb{R}^*$  is defined by

(Def. 2) for every object  $x$ ,  $x \in \text{it}$  iff there exists a non empty, non-decreasing finite sequence  $s$  of elements of  $\mathbb{R}$  such that  $x = s$  and  $\text{dom } s = \text{dom } D$  and for every natural number  $i$  such that  $i \in \text{dom } s$  holds  $s(i) \in \text{divset}(D, i)$ .

Now we state the propositions:

- (43) Let us consider a non empty, closed interval subset  $A$  of  $\mathbb{R}$ , and a partition  $D$  of  $A$ . Then  $D \in$  the set of tagged partitions of  $D$ .

PROOF: For every natural number  $i$  such that  $i \in \text{dom } D$  holds  $D(i) \in \text{divset}(D, i)$  by [15, (19)], (4).  $\square$

- (44) Let us consider real numbers  $a, b$ , and a non empty, closed interval subset  $I_4$  of  $\mathbb{R}$ . If  $I_4 = [a, b]$ , then  $\langle b \rangle$  is a partition of  $I_4$ .

PROOF:  $\langle b \rangle$  is a partition of  $I_4$  by [3, (39)], [15, (19)].  $\square$

Let  $I$  be a non empty, closed interval subset of  $\mathbb{R}$  and  $\varphi$  be a positive yielding function from  $I$  into  $\mathbb{R}$ .

A tagged partition of  $I$  and  $\varphi$  is defined by

- (Def. 3) there exists a partition  $D$  of  $I$  and there exists an element  $T$  of the set of tagged partitions of  $D$  such that  $it = \langle D, T \rangle$ .

Let  $T_1$  be a tagged partition of  $I$  and  $\varphi$ . We say that  $T_1$  is  $\delta$ -fine if and only if

- (Def. 4) there exists a partition  $D$  of  $I$  and there exists an element  $T$  of the set of tagged partitions of  $D$  such that  $T_1 = \langle D, T \rangle$  and for every natural number  $i$  such that  $i \in \text{dom } D$  holds  $\text{vol}(\text{divset}(D, i)) \leq \varphi(T(i))$ .

## 6. PARTITION COMPOSITION

Let us consider a real number  $r$ . Now we state the propositions:

- (45) (i)  $\sup\{r\} = r$ , and

(ii)  $\inf\{r\} = r$ .

- (46)  $\text{vol}(\{r\}) = 0$ . The theorem is a consequence of (45).

- (47) Let us consider non empty, closed interval subsets  $I_1, I_2$  of  $\mathbb{R}$ , and a positive yielding function  $\varphi$  from  $I_1$  into  $\mathbb{R}$ . Suppose  $I_2 \subseteq I_1$ . Then  $\varphi|_{I_2}$  is a positive yielding function from  $I_2$  into  $\mathbb{R}$ .

- (48) Let us consider a non empty, closed interval subset  $I$  of  $\mathbb{R}$ , and a real number  $c$ . Suppose  $c \in I$ . Then

(i)  $[\inf I, c]$  is a non empty, closed interval subset of  $\mathbb{R}$ , and

(ii)  $[c, \sup I]$  is a non empty, closed interval subset of  $\mathbb{R}$ , and

(iii)  $\sup[\inf I, c] = \inf[c, \sup I]$ .

The theorem is a consequence of (41).

Let  $I_5, I_6$  be non empty, closed interval subsets of  $\mathbb{R}$ ,  $D_4$  be a partition of  $I_5$ , and  $D_6$  be a partition of  $I_6$ . Assume  $\sup I_5 \leq \inf I_6$ . The functor  $D_4 \cdot D_6$

yielding a non empty, increasing finite sequence of elements of  $\mathbb{R}$  is defined by the term

$$(\text{Def. 5}) \quad \begin{cases} D_4 \wedge D_6, & \text{if } D_6(1) \neq \sup I_5, \\ D_4 \wedge D_{6|1}, & \text{otherwise.} \end{cases}$$

Now we state the propositions:

- (49) Let us consider non empty, closed interval subsets  $I_5, I_6$  of  $\mathbb{R}$ , a partition  $D_4$  of  $I_5$ , and a partition  $D_6$  of  $I_6$ . Suppose  $\sup I_5 = \inf I_6$  and  $\text{len } D_6 = 1$  and  $D_6(1) = \inf I_6$ . Then  $D_4 \cdot D_6 = D_4$ .
- (50) Let us consider non empty, closed interval subsets  $I_1, I_2, I$  of  $\mathbb{R}$ . Suppose  $\sup I_1 \leq \inf I_2$  and  $\inf I \leq \inf I_1$  and  $\sup I_2 \leq \sup I$ . Then  $I_1 \cup I_2 \subseteq I$ .
- (51) Let us consider non empty, closed interval subsets  $I_1, I_2, I$  of  $\mathbb{R}$ , a partition  $D_1$  of  $I_1$ , and a partition  $D_2$  of  $I_2$ . Suppose  $\sup I_1 \leq \inf I_2$  and  $I = [\inf I_1, \sup I_2]$ . Then  $D_1 \cdot D_2$  is a partition of  $I$ . The theorem is a consequence of (50).
- (52) Let us consider a non empty, closed interval subset  $I$  of  $\mathbb{R}$ , and a partition  $D$  of  $I$ . Then the set of tagged partitions of  $D$  is not empty.
- (53) Let us consider a non empty, increasing finite sequence  $s$  of elements of  $\mathbb{R}$ , and a real number  $r$ . Suppose  $s(\text{len } s) < r$ . Then  $s \wedge \langle r \rangle$  is a non empty, increasing finite sequence of elements of  $\mathbb{R}$ . The theorem is a consequence of (1).
- (54) Let us consider non empty, increasing finite sequences  $s_1, s_2$  of elements of  $\mathbb{R}$ , and a real number  $r$ . Suppose  $s_1(\text{len } s_1) < r < s_2(1)$ . Then  $(s_1 \wedge \langle r \rangle) \wedge s_2$  is a non empty, increasing finite sequence of elements of  $\mathbb{R}$ . The theorem is a consequence of (53) and (1).
- (55) Let us consider non empty, closed interval subsets  $I_1, I_2, I$  of  $\mathbb{R}$ . Suppose  $\sup I_1 = \inf I_2$  and  $I = I_1 \cup I_2$ . Then
- (i)  $\inf I = \inf I_1$ , and
  - (ii)  $\sup I = \sup I_2$ .
- (56) Let us consider a non empty, closed interval subset  $I$  of  $\mathbb{R}$ , and a partition  $D$  of  $I$ . Then
- (i)  $\text{divset}(D, 1) = [\inf I, D(1)]$ , and
  - (ii) for every natural number  $j$  such that  $j \in \text{dom } D$  and  $j \neq 1$  holds  $\text{divset}(D, j) = [D(j-1), D(j)]$ .
- PROOF: For every natural number  $j$  such that  $j \in \text{dom } D$  and  $j \neq 1$  holds  $\text{divset}(D, j) = [D(j-1), D(j)]$  by [12, (4)].  $\square$
- (57) Let us consider a real number  $r$ , and finite sequences  $p, q$  of elements of  $\mathbb{R}$ . Then  $\text{len}((p \wedge \langle r \rangle) \wedge q) = \text{len } p + \text{len } q + 1$ .



- (58) Let us consider a non empty, closed interval subset  $I$  of  $\mathbb{R}$ , and a partition  $D$  of  $I$ . Then every element of the set of tagged partitions of  $D$  is not empty. The theorem is a consequence of (43).
- (59) Let us consider a non empty, closed interval subset  $I$  of  $\mathbb{R}$ , a partition  $D$  of  $I$ , and an element  $T$  of the set of tagged partitions of  $D$ . Then  $\text{rng } T \subseteq \mathbb{R}$ . The theorem is a consequence of (43).

Let  $I$  be a non empty, closed interval subset of  $\mathbb{R}$ ,  $\varphi$  be a positive yielding function from  $I$  into  $\mathbb{R}$ , and  $T_1$  be a tagged partition of  $I$  and  $\varphi$ . The functor  $T_1$ -partition yielding a partition of  $I$  is defined by

(Def. 6) there exists a partition  $D$  of  $I$  and there exists an element  $T$  of the set of tagged partitions of  $D$  such that  $it = D$  and  $T_1 = \langle D, T \rangle$ .

### 7. EXAMPLES OF PARTITIONS

In the sequel  $r, s$  denote real numbers.

Now we state the proposition:

- (60) Let us consider a function  $\varphi$  from  $[r, s]$  into  $]0, +\infty[$ . Suppose  $r \leq s$ . Then the set of all  $]x - \varphi(x), x + \varphi(x)[ \cap [r, s]$  where  $x$  is an element of  $[r, s]$  is a family of subsets of  $[r, s]_{\mathbb{T}}$ .

Let us consider a function  $\varphi$  from  $[r, s]$  into  $]0, +\infty[$  and a family  $S$  of subsets of  $[r, s]_{\mathbb{T}}$ .

Let us assume that  $r \leq s$  and  $S =$  the set of all  $]x - \varphi(x), x + \varphi(x)[ \cap [r, s]$  where  $x$  is an element of  $[r, s]$ . Now we state the propositions:

- (61)  $S$  is a cover of  $[r, s]_{\mathbb{T}}$ .  
 PROOF:  $[r, s] \subseteq \bigcup S$  by [8, (3)].  $\square$
- (62)  $S$  is open.  
 PROOF: For every subset  $P$  of  $[r, s]_{\mathbb{T}}$  such that  $P \in S$  holds  $P$  is open by [11, (17)], [20, (35)], [11, (15), (9), (10)].  $\square$
- (63) Suppose  $S =$  the set of all  $]x - \varphi(x), x + \varphi(x)[ \cap [r, s]$  where  $x$  is an element of  $[r, s]$ . Then  $S$  is connected.  
 PROOF: For every subset  $X$  of  $[r, s]_{\mathbb{T}}$  such that  $X \in S$  holds  $X$  is connected by [16, (43)].  $\square$
- (64) Let us consider a function  $\varphi$  from  $[r, s]$  into  $]0, +\infty[$ , and a family  $S$  of subsets of  $[r, s]_{\mathbb{T}}$ . Suppose  $r \leq s$  and  $S =$  the set of all  $]x - \varphi(x), x + \varphi(x)[ \cap [r, s]$  where  $x$  is an element of  $[r, s]$ . Let us consider an interval cover  $I$  of  $S$ . Then

- (i)  $I$  is a finite sequence of elements of  $2^{\mathbb{R}}$ , and
- (ii)  $\text{rng } I \subseteq S$ , and

- (iii)  $\bigcup \text{rng } I = [r, s]$ , and
- (iv) for every natural number  $n$  such that  $1 \leq n$  holds if  $n \leq \text{len } I$ , then  $I_n$  is not empty and if  $n + 1 \leq \text{len } I$ , then  $\inf I_n \leq \inf I_{n+1}$  and  $\sup I_n \leq \sup I_{n+1}$  and  $\inf I_{n+1} < \sup I_n$  and if  $n + 2 \leq \text{len } I$ , then  $\sup I_n \leq \inf I_{n+2}$ , and
- (v) if  $[r, s] \in S$ , then  $I = \langle [r, s] \rangle$ , and
- (vi) if  $[r, s] \notin S$ , then there exists a real number  $p$  such that  $r < p \leq s$  and  $I(1) = [r, p[$  and there exists a real number  $q$  such that  $r \leq p < s$  and  $I(\text{len } I) = ]p, s]$  and for every natural number  $n$  such that  $1 < n < \text{len } I$  there exist real numbers  $p, q$  such that  $r \leq p < q \leq s$  and  $I(n) = ]p, q[$ .

The theorem is a consequence of (61), (62), and (63).

(65) Let us consider real numbers  $r, s, t, x$ . Then

- (i) if  $r \leq x - t$  and  $x + t \leq s$ , then  $]x - t, x + t[ \cap [r, s] = ]x - t, x + t[$ , and
- (ii) if  $r \leq x - t$  and  $s < x + t$ , then  $]x - t, x + t[ \cap [r, s] = ]x - t, s]$ , and
- (iii) if  $x - t < r$  and  $x + t \leq s$ , then  $]x - t, x + t[ \cap [r, s] = [r, x + t[$ , and
- (iv) if  $x - t < r$  and  $s < x + t$ , then  $]x - t, x + t[ \cap [r, s] = [r, s]$ .

(66) Let us consider real numbers  $r, s, t, x$ , and a subset  $X_1$  of  $\mathbb{R}$ . Suppose  $0 < t$  and  $r \leq x \leq s$  and  $X_1 = ]x - t, x + t[ \cap [r, s]$ . Then

- (i) if  $r \leq x - t$  and  $x + t \leq s$ , then  $\inf X_1 = x - t$  and  $\sup X_1 = x + t$ , and
- (ii) if  $r \leq x - t$  and  $s < x + t$ , then  $\inf X_1 = x - t$  and  $\sup X_1 = s$ , and
- (iii) if  $x - t < r$  and  $x + t \leq s$ , then  $\inf X_1 = r$  and  $\sup X_1 = x + t$ , and
- (iv) if  $x - t < r$  and  $s < x + t$ , then  $\inf X_1 = r$  and  $\sup X_1 = s$ .

The theorem is a consequence of (65).

Let us consider real numbers  $a, b, c$ , non empty, compact subsets  $I_5, I_6$  of  $\mathbb{R}$ , a partition  $D_4$  of  $I_5$ , a partition  $D_6$  of  $I_6$ , and natural numbers  $i, j$ .

Let us assume that  $a \leq c \leq b$  and  $I_5 = [a, c]$  and  $I_6 = [c, b]$ . Now we state the propositions:

(67) Suppose  $i \in \text{dom } D_4$  and  $j \in \text{dom } D_6$ . Then

- (i) if  $i < \text{len } D_4$ , then  $D_4(i) < D_6(j)$ , and
- (ii) if  $i = \text{len } D_4$  and  $c < D_6(1)$ , then  $D_4(i) < D_6(j)$ , and
- (iii) if  $D_6(1) = c$ , then  $D_4(\text{len } D_4) = D_6(1)$ .

PROOF: If  $i < \text{len } D_4$ , then  $D_4(i) < D_6(j)$  by [3, (3)]. If  $i = \text{len } D_4$  and  $c < D_6(1)$ , then  $D_4(i) < D_6(j)$  by [7, (6)], [3, (91)].  $\square$

(68) If  $i \in \text{dom } D_4$  and  $j \in \text{dom } D_6$ , then if  $c < D_6(1)$ , then  $D_4(i) < D_6(j)$ . The theorem is a consequence of (67).

(69) Let us consider real numbers  $a, b, c$ , and non empty, compact subsets  $I_4, I_5, I_6$  of  $\mathbb{R}$ . Suppose  $a \leq c \leq b$  and  $I_4 = [a, b]$  and  $I_5 = [a, c]$  and  $I_6 = [c, b]$ . Let us consider a partition  $D_4$  of  $I_5$ , and a partition  $D_6$  of  $I_6$ . Suppose  $c < D_6(1)$ . Then  $D_4 \cap D_6$  is a partition of  $I_4$ .

PROOF: Set  $D_5 = D_4 \cap D_6$ . For every extended reals  $e_1, e_2$  such that  $e_1, e_2 \in \text{dom } D_5$  and  $e_1 < e_2$  holds  $D_5(e_1) < D_5(e_2)$  by [3, (25)], (68), [2, (11)], [3, (1)].  $\text{rng } D_5 \subseteq I_4$  by [3, (31)].  $D_5(\text{len } D_5) = \sup I_4$  by [3, (3), (22)], [15, (19)].  $\square$

(70) Let us consider real numbers  $a, b$ , and a non empty, closed interval subset  $I_4$  of  $\mathbb{R}$ . Suppose  $a \leq b$  and  $I_4 = [a, b]$ . Let us consider a partition  $D_3$  of  $I_4$ . If  $\text{len } D_3 = 1$ , then  $D_3 = \langle b \rangle$ .

(71) Let us consider real numbers  $a, b$ , a non empty, compact subset  $I_4$  of  $\mathbb{R}$ , and a partition  $D_3$  of  $I_4$ . Suppose  $2 \leq \text{len } D_3$ . Then  $D_{3|1}$  is a partition of  $I_4$ .

PROOF: Set  $D = D_{3|1}$ .  $D$  is a non empty, increasing finite sequence of elements of  $\mathbb{R}$  by [3, (60)].  $\text{rng } D \subseteq I_4$  by [7, (33)].  $D(\text{len } D) = \sup I_4$  by [3, (3)].  $\square$

(72) Let us consider real numbers  $a, b$ . Suppose  $a < b$ . Then  $\langle a, b \rangle$  is a non empty, increasing finite sequence of elements of  $\mathbb{R}$ .

PROOF: Set  $s = \langle a, b \rangle$ .  $s$  is increasing by [3, (44), (2)].  $\square$

(73) Let us consider real numbers  $a, b$ , and a non empty, closed interval subset  $I_4$  of  $\mathbb{R}$ . Suppose  $a < b$  and  $I_4 = [a, b]$ . Then  $\langle a, b \rangle$  is a partition of  $I_4$ .

PROOF:  $\langle a, b \rangle$  is a partition of  $I_4$  by (72), [6, (127)], [3, (44)], [15, (19)].  $\square$

### 8. COUSIN'S LEMMA

Now we state the proposition:

(74) Let us consider real numbers  $a, b$ , and a positive yielding function  $\varphi$  from  $[a, b]$  into  $\mathbb{R}$ . Suppose  $a \leq b$ . Then there exists a non empty, increasing finite sequence  $x$  of elements of  $\mathbb{R}$  and there exists a non empty finite sequence  $t$  of elements of  $\mathbb{R}$  such that  $x(1) = a$  and  $x(\text{len } x) = b$  and  $t(1) = a$  and  $\text{dom } x = \text{dom } t$  and for every natural number  $i$  such that  $i - 1, i \in \text{dom } t$  holds  $t(i) - \varphi(t(i)) \leq x(i - 1) \leq t(i)$  and for every natural number  $i$  such that  $i \in \text{dom } t$  holds  $t(i) \leq x(i) \leq t(i) + \varphi(t(i))$ .

PROOF: Define  $\mathcal{P}[\text{object}] \equiv$  there exists a non empty, increasing finite sequence  $x$  of elements of  $\mathbb{R}$  and there exists a non empty finite sequence  $t$  of elements of  $\mathbb{R}$  such that  $x(1) = a$  and  $x(\text{len } x) = \$_1$  and  $t(1) = a$  and  $\text{dom } x = \text{dom } t$  and for every natural number  $i$  such that  $i - 1, i \in \text{dom } t$  holds  $t(i) - \varphi(t(i)) \leq x(i - 1) \leq t(i)$  and for every natural number  $i$  such that  $i \in \text{dom } t$  holds  $t(i) \leq x(i) \leq t(i) + \varphi(t(i))$ . Consider  $C$  being a set such that for every object  $x$ ,  $x \in C$  iff  $x \in [a, b]$  and  $\mathcal{P}[x]$ . For every object  $x$  such that  $x \in C$  holds  $x$  is real. Reconsider  $c = \sup C$  as a real number.  $c \in [a, b]$ . Consider  $d$  being an element of  $\overline{\mathbb{R}}$  such that  $d \in C$  and  $c - \varphi(c) < d$ . Consider  $D_0$  being a non empty, increasing finite sequence of elements of  $\mathbb{R}$ ,  $T_0$  being a non empty finite sequence of elements of  $\mathbb{R}$  such that  $D_0(1) = a$  and  $D_0(\text{len } D_0) = d$  and  $T_0(1) = a$  and  $\text{dom } D_0 = \text{dom } T_0$  and for every natural number  $i$  such that  $i - 1, i \in \text{dom } T_0$  holds  $T_0(i) - \varphi(T_0(i)) \leq D_0(i - 1) \leq T_0(i)$  and for every natural number  $i$  such that  $i \in \text{dom } T_0$  holds  $T_0(i) \leq D_0(i) \leq T_0(i) + \varphi(T_0(i))$ .  $c \in C$  and  $\mathcal{P}[c]$  by (1), [27, (32)], [3, (22), (39), (1)].  $c = b$  by (1), [27, (32)], [3, (22), (39), (1)].  $\square$

(75) COUSIN'S LEMMA:

Let us consider a non empty, closed interval subset  $I$  of  $\mathbb{R}$ , and a positive yielding function  $\varphi$  from  $I$  into  $\mathbb{R}$ . Then there exists a tagged partition  $T_1$  of  $I$  and  $\varphi$  such that  $T_1$  is  $\delta$ -fine.

PROOF: Consider  $a, b$  being real numbers such that  $a \leq b$  and  $I = [a, b]$ . Reconsider  $r = \frac{1}{2}$  as a positive real number. Reconsider  $\phi = r \cdot \varphi$  as a positive yielding function from  $I$  into  $\mathbb{R}$ . Consider  $x$  being a non empty, increasing finite sequence of elements of  $\mathbb{R}$ ,  $t$  being a non empty finite sequence of elements of  $\mathbb{R}$  such that  $x(1) = a$  and  $x(\text{len } x) = b$  and  $t(1) = a$  and  $\text{dom } x = \text{dom } t$  and for every natural number  $i$  such that  $i - 1, i \in \text{dom } t$  holds  $t(i) - \phi(t(i)) \leq x(i - 1) \leq t(i)$  and for every natural number  $i$  such that  $i \in \text{dom } t$  holds  $t(i) \leq x(i) \leq t(i) + \phi(t(i))$ . Reconsider  $D = x$  as a partition of  $I$ . Reconsider  $T = t$  as an element of the set of tagged partitions of  $D$ . Reconsider  $T_1 = \langle D, T \rangle$  as a tagged partition of  $I$  and  $\varphi$ .  $T_1$  is  $\delta$ -fine by [15, (19)], (4), [8, (3)], [21, (20)].  $\square$

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Received December 31, 2015