

# Chebyshev Distance

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**Summary.** In [21], Marco Riccardi formalized that  $\mathbb{R}N$ -basis  $n$  is a basis (in the algebraic sense defined in [26]) of  $\mathcal{E}_T^n$  and in [20] he has formalized that  $\mathcal{E}_T^n$  is second-countable, we build (in the topological sense defined in [23]) a denumerable base of  $\mathcal{E}_T^n$ .

Then we introduce the  $n$ -dimensional intervals (interval in  $n$ -dimensional Euclidean space, *pavé (borné) de  $\mathbb{R}^n$*  [16], *semi-intervalle (borné) de  $\mathbb{R}^n$*  [22]).

We conclude with the definition of Chebyshev distance [11].

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## 1. PRELIMINARIES

From now on  $n$  denotes a natural number,  $r, s$  denote real numbers,  $x, y$  denote elements of  $\mathcal{R}^n$ ,  $p, q$  denote points of  $\mathcal{E}_T^n$ , and  $e$  denotes a point of  $\mathcal{E}^n$ .

Now we state the propositions:

- (1)  $|x - y| = |y - x|$ .
- (2) Let us consider a natural number  $i$ . If  $i \in \text{Seg } n$ , then  $|x|(i) \in \mathbb{R}$ .
- (3) Let us consider elements  $x, y$  of  $\mathbb{R}$ , and extended reals  $x_1, y_1$ . If  $x \leq x_1$  and  $y \leq y_1$ , then  $x + y \leq x_1 + y_1$ .
- (4) Let us consider real numbers  $a, c$ , and an extended real number  $b$ . Suppose  $a < b$  and  $[a, b] \subseteq [a, c]$ . Then  $b$  is a real number.
- (5) Let us consider a non empty set  $D$ , and a non empty subset  $D_1$  of  $D$ . Then  $D_1^n \subseteq D^n$ .

(6) Let us consider a non empty set  $X$ , and a function  $f$ . Suppose  $f = \text{Seg } n \mapsto X$ . Then  $f$  is a non-empty,  $n$ -element finite sequence.

Let  $n$  be a natural number. The functor  $\mathbb{R}(n)$  yielding a non-empty,  $n$ -element finite sequence is defined by the term

(Def. 1)  $\text{Seg } n \mapsto \mathbb{R}$ .

Now we state the propositions:

(7)  $\mathbb{R}(n) = \text{Seg } n \mapsto$  the carrier of  $\mathbb{R}^1$ .

(8)  $\prod(\text{Seg } n \mapsto \mathbb{R}) = \mathcal{R}^n$ .

(9)  $\prod \mathbb{R}(n) = \mathcal{R}^n$ .

(10) Let us consider a set  $X$ . Then  $\prod(\text{Seg } n \mapsto X) = X^n$ .

(11) Let us consider a non empty set  $D$ , and an  $n$ -tuple  $x$  of  $D$ . Then  $x \in D^{\text{Seg } n}$ .

(12) Let us consider a subset  $O_1$  of  $\mathcal{E}_T^n$ , and a subset  $O_2$  of  $(\mathcal{E}^n)_{\text{top}}$ . If  $O_1 = O_2$ , then  $O_1$  is open iff  $O_2$  is open.

(13) Suppose  $e = p$ . Then the set of all  $\text{OpenHypercube}(e, \frac{1}{m})$  where  $m$  is a non zero element of  $\mathbb{N}$  = the set of all  $\text{OpenHypercube}(p, \frac{1}{m})$  where  $m$  is a non zero element of  $\mathbb{N}$ .

(14) If  $q \in \text{OpenHypercube}(p, r)$ , then  $p \in \text{OpenHypercube}(q, r)$ .

(15) If  $q \in \text{OpenHypercube}(p, \frac{r}{2})$ , then  $\text{OpenHypercube}(q, \frac{r}{2}) \subseteq \text{OpenHypercube}(p, r)$ .

Let  $x$  be an element of  $\mathbb{R} \times \mathbb{R}$ . The functors:  $(x)_1$  and  $(x)_2$  yield elements of  $\mathbb{R}$ . Let  $n$  be a natural number and  $x$  be an element of  $\mathcal{R}^n \times \mathcal{R}^n$ . The functors:  $(x)_1$  and  $(x)_2$  yield elements of  $\mathcal{R}^n$ . Now we state the proposition:

(16) Let us consider an  $n$ -element finite sequence  $f$  of elements of  $\mathbb{R} \times \mathbb{R}$ . Then there exists an element  $x$  of  $\mathcal{R}^n \times \mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $(x)_1(i) = (f_i)_1$  and  $(x)_2(i) = (f_i)_2$ .

## 2. THE SET OF $n$ -TUPLES OF RATIONAL NUMBERS

Let us consider  $n$ . The functor  $\mathcal{Q}^n$  yielding a set of finite sequences of  $\mathbb{Q}$  is defined by the term

(Def. 2)  $\mathcal{Q}^n$ .

Now we state the proposition:

(17)  $\mathcal{Q}^0 = \{0\}$ .

One can check that  $\mathcal{Q}^0$  is trivial.

Let us consider  $n$ . One can check that  $\mathcal{Q}^n$  is non empty and every element of  $\mathcal{Q}^n$  is  $n$ -element and  $\mathcal{Q}^n$  is countable.

Let  $n$  be a positive natural number. Let us note that  $\mathcal{Q}^n$  is infinite and  $\mathcal{Q}^n$  is denumerable.

Now we state the proposition:

(18)  $\mathcal{Q}^n$  is a dense subset of  $\mathcal{E}_T^n$ .

PROOF:  $\mathcal{Q}^n$  is a subset of  $\mathcal{R}^n$ . Reconsider  $R = \mathcal{Q}^n$  as a subset of  $\mathcal{E}_T^n$ . For every subset  $Q$  of  $\mathcal{E}_T^n$  such that  $Q \neq \emptyset$  and  $Q$  is open holds  $R$  meets  $Q$  by [10, (67)], (12), [15, (23)], [13, (39)].  $\square$

Let us consider  $n$ . One can check that  $\mathcal{Q}^n$  is countable and dense as a subset of  $\mathcal{E}_T^n$ .

### 3. A COUNTABLE BASE OF AN $n$ -DIMENSIONAL EUCLIDEAN SPACE

(VERSION 1: [20]):

Let  $n$  be a natural number. Let us observe that there exists a basis of  $\mathcal{E}_T^n$  which is countable.

Let us consider  $n$  and  $e$ . Note that  $\text{OpenHypercubes } e$  is countable.

The functor  $\text{OpenHypercubes-}\mathbb{Q}(n)$  yielding a non empty set is defined by the term

(Def. 3)  $\{\text{OpenHypercubes } q, \text{ where } q \text{ is a point of } \mathcal{E}^n : q \in \mathcal{Q}^n\}$ .

Let  $q$  be an element of  $\mathcal{Q}^n$ . The functor  ${}^{\textcircled{q}}$  yielding a point of  $\mathcal{E}^n$  is defined by the term

(Def. 4)  $q$ .

Let  $q$  be an object. Assume  $q \in \mathcal{Q}^n$ . The functor  $\text{object2}\mathbb{Q}(q, n)$  yielding an element of  $\mathcal{Q}^n$  is defined by the term

(Def. 5)  $q$ .

Let us note that  $\text{OpenHypercubes-}\mathbb{Q}(n)$  is countable and  $\bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$  is countable.

Now we state the propositions:

(19)  $\bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$  is an open family of subsets of  $\mathcal{E}_T^n$ . The theorem is a consequence of (12).

(20) Let us consider a non empty, open subset  $O$  of  $\mathcal{E}_T^n$ . Then there exists an element  $p$  of  $\mathcal{Q}^n$  such that  $p \in O$ . The theorem is a consequence of (18).

(21) Let us consider a family  $\mathcal{B}$  of subsets of  $\mathcal{E}_T^n$ .

Suppose  $\mathcal{B} = \bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$ . Then  $\mathcal{B}$  is quasi basis.

PROOF:  $F$  is quasi basis by (12), [15, (23)], [10, (67)], (20).  $\square$

Let us consider  $n$ . Observe that  $\bigcup \text{OpenHypercubes-}\mathbb{Q}(n)$  is non empty.

The functor  $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$  yielding a countable, open family of subsets of  $\mathcal{E}_T^n$  is defined by the term

(Def. 6)  $\bigcup \text{OpenHypercubes}\mathbb{Q}(n)$ .

Now we state the proposition:

(22)  $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n) = \{\text{OpenHypercube}(q, \frac{1}{m}),$   
 where  $q$  is a point of  $\mathcal{E}^n$ ,  $m$  is a positive natural number :  $q \in \mathcal{Q}^n\}$ .

(VERSION 2):

Let  $n$  be a natural number. Observe that there exists a basis of  $\mathcal{E}_T^n$  which is countable.

Now we state the propositions:

(23)  $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$  is a countable basis of  $\mathcal{E}_T^n$ .

(24) Let us consider an open subset  $O$  of  $\mathcal{E}_T^n$ . Then there exists a subset  $Y$  of  $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$  such that

- (i)  $Y$  is countable, and
- (ii)  $O = \bigcup Y$ .

The theorem is a consequence of (21).

Let us consider an open, non empty subset  $O$  of  $\mathcal{E}_T^n$ . Now we state the propositions:

(25) There exists a subset  $Y$  of  $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$  such that

- (i)  $Y$  is not empty, and
- (ii)  $O = \bigcup Y$ , and
- (iii) there exists a function  $g$  from  $\mathbb{N}$  into  $Y$  such that for every object  $x$ ,  $x \in O$  iff there exists an object  $y$  such that  $y \in \mathbb{N}$  and  $x \in g(y)$ .

The theorem is a consequence of (24).

(26) There exists a sequence  $s$  of  $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$  such that for every object  $x$ ,  $x \in O$  iff there exists an object  $y$  such that  $y \in \mathbb{N}$  and  $x \in s(y)$ . The theorem is a consequence of (25).

(27) There exists a sequence  $s$  of  $\text{OpenHypercubes}\mathbb{Q}\text{Union}(n)$  such that  $O = \bigcup s$ . The theorem is a consequence of (26).

#### 4. THE SET OF ALL LEFT OPEN REAL BOUNDED INTERVALS

The set of all left open real bounded intervals yielding a family of subsets of  $\mathbb{R}$  is defined by the term

(Def. 7) the set of all  $]a, b]$  where  $a, b$  are real numbers.

Let us note that the set of all left open real bounded intervals is non empty.

Now we state the propositions:

- (28) The set of all left open real bounded intervals  $\subseteq \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is left open interval}\}$ .
- (29) The set of all left open real bounded intervals is  $\cap$ -closed and  $\setminus_{fp}$ -closed and has the empty element and countable cover.
- (30) The set of all left open real bounded intervals is a semiring of  $\mathbb{R}$ .

### 5. THE SET OF ALL RIGHT OPEN REAL BOUNDED INTERVALS

The set of all right open real bounded intervals yielding a family of subsets of  $\mathbb{R}$  is defined by the term

(Def. 8) the set of all  $[a, b[$  where  $a, b$  are real numbers.

Observe that the set of all right open real bounded intervals is non empty.

Now we state the propositions:

- (31) The set of all right open real bounded intervals  $\subseteq \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is right open interval}\}$ .
- (32) The set of all right open real bounded intervals has the empty element.
- (33) (i) the set of all right open real bounded intervals is  $\cap$ -closed, and  
 (ii) the set of all right open real bounded intervals is  $\setminus_{fp}$ -closed and has the empty element.

The theorem is a consequence of (31), (32), and (4).

- (34) The set of all right open real bounded intervals has countable cover.

PROOF: Define  $\mathcal{F}[\text{object}, \text{object}] \equiv \mathbb{N}$  is an element of  $\mathbb{N}$  and  $\mathbb{N} \in$  the set of all right open real bounded intervals and there exists a real number  $x$  such that  $x = \mathbb{N}_1$  and  $\mathbb{N}_2 = [-x, x[$ . For every object  $x$  such that  $x \in \mathbb{N}$  there exists an object  $y$  such that  $y \in$  the set of all right open real bounded intervals and  $\mathcal{F}[x, y]$ . Consider  $f$  being a function such that  $\text{dom } f = \mathbb{N}$  and  $\text{rng } f \subseteq$  the set of all right open real bounded intervals and for every object  $x$  such that  $x \in \mathbb{N}$  holds  $\mathcal{F}[x, f(x)]$  from [7, Sch. 6].  $\text{rng } f$  is countable by [27, (4)], [14, (58)].  $\text{rng } f$  is a cover of  $\mathbb{R}$  by [2, (2)], [12, (8)], [3, (13)], [17, (45)].  $\square$

- (35) The set of all right open real bounded intervals is a semiring of  $\mathbb{R}$ .

6. FINITE PRODUCT OF LEFT OPEN INTERVALS

In the sequel  $n$  denotes a non zero natural number.

Let  $n$  be a non zero natural number. The functor  $\text{LeftOpenIntervals}(n)$  yielding a classical semiring family of  $\mathbb{R}(n)$  is defined by the term

(Def. 9)  $\text{Seg } n \longmapsto$  (the set of all left open real bounded intervals).

Now we state the propositions:

(36)  $\text{LeftOpenIntervals}(n) = \text{Seg } n \longmapsto$  the set of all  $]a, b]$  where  $a, b$  are real numbers.

(37)  $\text{MeasurableRectangleLeftOpenIntervals}(n)$  is a semiring of  $\mathcal{R}^n$ . The theorem is a consequence of (8).

Let us consider an object  $x$ .

Let us assume that  $x \in \text{MeasurableRectangleLeftOpenIntervals}(n)$ . Now we state the propositions:

(38) There exists a subset  $y$  of  $\mathcal{R}^n$  such that

(i)  $x = y$ , and

(ii) for every natural number  $i$  such that  $i \in \text{Seg } n$  there exist real numbers  $a, b$  such that for every element  $t$  of  $\mathcal{R}^n$  such that  $t \in y$  holds  $t(i) \in ]a, b]$ .

The theorem is a consequence of (37).

(39) There exists a subset  $y$  of  $\mathcal{R}^n$  and there exists an  $n$ -element finite sequence  $f$  of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $x = y$  and for every element  $t$  of  $\mathcal{R}^n$ ,  $t \in y$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ](f_i)_1, (f_i)_2]$ .

PROOF:  $\text{MeasurableRectangleLeftOpenIntervals}(n)$  is a family of subsets of  $\mathcal{R}^n$ . Reconsider  $y = x$  as a subset of  $\mathcal{R}^n$ . Consider  $g$  being a function such that  $x = \prod g$  and  $g \in \prod \text{LeftOpenIntervals}(n)$ . Define  $\mathcal{P}[\text{natural number, set}] \equiv$  there exists an element  $x$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\$2 = x$  and  $g(\$1) = ](x)_1, (x)_2]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$ . There exists a finite sequence  $f_1$  of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f_1 = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_{1i}]$  from [25, Sch. 1]. Consider  $f_1$  being a finite sequence of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f_1 = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $x$  of  $\mathbb{R} \times \mathbb{R}$  such that  $f_{1i} = x$  and  $g(i) = ](x)_1, (x)_2]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $g(i) = ](f_{1i})_1, (f_{1i})_2]$ . For every element  $t$  of  $\mathcal{R}^n$  such that  $t \in y$  for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ](f_{1i})_1, (f_{1i})_2]$ . For every element  $t$  of  $\mathcal{R}^n$  such that for every natural

number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ](f_{1i})_1, (f_{1i})_2]$  holds  $t \in y$  by [6, (93)].  $\square$

- (40) There exists a subset  $y$  of  $\mathcal{R}^n$  and there exist elements  $a, b$  of  $\mathcal{R}^n$  such that  $x = y$  and for every object  $s, s \in y$  iff there exists an element  $t$  of  $\mathcal{R}^n$  such that  $s = t$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ]a(i), b(i)]$ . The theorem is a consequence of (39) and (16).

Now we state the proposition:

- (41) Let us consider a set  $x$ . Suppose  $x \in \text{MeasurableRectangleLeftOpenIntervals}(n)$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ]a(i), b(i)]$ . The theorem is a consequence of (39) and (16).

### 7. FINITE PRODUCT OF RIGHT OPEN INTERVALS

Let  $n$  be a non zero natural number. The functor  $\text{RightOpenIntervals}(n)$  yielding a classical semiring family of  $\mathbb{R}(n)$  is defined by the term

(Def. 10)  $\text{Seg } n \longmapsto$  (the set of all right open real bounded intervals).

From now on  $n$  denotes a non zero natural number.

Now we state the propositions:

- (42)  $\text{RightOpenIntervals}(n) = \text{Seg } n \longmapsto$  the set of all  $[a, b[$  where  $a, b$  are real numbers.
- (43)  $\text{MeasurableRectangleRightOpenIntervals}(n)$  is a semiring of  $\mathcal{R}^n$ . The theorem is a consequence of (8).

Let us consider an object  $x$ .

Let us assume that  $x \in \text{MeasurableRectangleRightOpenIntervals}(n)$ . Now we state the propositions:

- (44) There exists a subset  $y$  of  $\mathcal{R}^n$  such that
- (i)  $x = y$ , and
  - (ii) for every natural number  $i$  such that  $i \in \text{Seg } n$  there exist real numbers  $a, b$  such that for every element  $t$  of  $\mathcal{R}^n$  such that  $t \in y$  holds  $t(i) \in [a, b[$ .

The theorem is a consequence of (43).

- (45) There exists a subset  $y$  of  $\mathcal{R}^n$  and there exists an  $n$ -element finite sequence  $f$  of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $x = y$  and for every element  $t$  of  $\mathcal{R}^n, t \in y$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [(f_i)_1, (f_i)_2[$ .

PROOF:  $\text{MeasurableRectangleRightOpenIntervals}(n)$  is a family of subsets of  $\mathcal{R}^n$ . Reconsider  $y = x$  as a subset of  $\mathcal{R}^n$ . Consider  $g$  being a function

such that  $x = \prod g$  and  $g \in \prod \text{RightOpenIntervals}(n)$ . Define  $\mathcal{P}[\text{natural number, set}] \equiv$  there exists an element  $x$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\$2 = x$  and  $g(\$1) = [(x)_1, (x)_2[$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$ . There exists a finite sequence  $f_1$  of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f_1 = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_{1,i}]$  from [25, Sch. 1]. Consider  $f_1$  being a finite sequence of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f_1 = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $x$  of  $\mathbb{R} \times \mathbb{R}$  such that  $f_{1,i} = x$  and  $g(i) = [(x)_1, (x)_2[$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $g(i) = [(f_{1,i})_1, (f_{1,i})_2[$ . For every element  $t$  of  $\mathcal{R}^n$  such that  $t \in y$  for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [(f_{1,i})_1, (f_{1,i})_2[$ . For every element  $t$  of  $\mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [(f_{1,i})_1, (f_{1,i})_2[$  holds  $t \in y$  by [6, (93)].  $\square$

- (46) There exists a subset  $y$  of  $\mathcal{R}^n$  and there exist elements  $a, b$  of  $\mathcal{R}^n$  such that  $x = y$  and for every object  $s, s \in y$  iff there exists an element  $t$  of  $\mathcal{R}^n$  such that  $s = t$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [a(i), b(i)[$ . The theorem is a consequence of (45) and (16).

Now we state the proposition:

- (47) Let us consider a set  $x$ . Suppose  $x \in \text{MeasurableRectangle RightOpenIntervals}(n)$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [a(i), b(i)[$ . The theorem is a consequence of (45) and (16).

### 8. $n$ -DIMENSIONAL PRODUCT OF SUBSET FAMILY

In the sequel  $n$  denotes a natural number,  $X$  denotes a set, and  $S$  denotes a family of subsets of  $X$ .

Let us consider  $n$  and  $X$ . The functor  $\text{Product}(n, X)$  yielding a set is defined by

- (Def. 11) for every object  $f, f \in it$  iff there exists a function  $g$  such that  $f = \prod g$  and  $g \in \prod(\text{Seg } n \mapsto X)$ .

Now we state the propositions:

- (48)  $\text{Product}(n, X) \subseteq 2^{(\bigcup(\text{Seg } n \mapsto X))^{\text{dom}(\text{Seg } n \mapsto X)}}$ .
- (49)  $\text{Product}(n, S)$  is a family of subsets of  $\prod(\text{Seg } n \mapsto X)$ .

PROOF: Reconsider  $S_1 = \text{Product}(n, S)$  as a subset of  $2^{(\bigcup(\text{Seg } n \mapsto S))^{\text{dom}(\text{Seg } n \mapsto S)}}$ .  $S_1 \subseteq 2^{\prod(\text{Seg } n \mapsto X)}$  by [1, (9)], [24, (13), (7)], [9, (77), (81)].  $\square$



(50) Let us consider a non empty family  $S$  of subsets of  $X$ . Then  $\text{Product}(n, S) =$  the set of all  $\prod f$  where  $f$  is an  $n$ -tuple of  $S$ .

PROOF:  $\text{Product}(n, S) \subseteq$  the set of all  $\prod f$  where  $f$  is an  $n$ -tuple of  $S$  by (10), [6, (131)]. the set of all  $\prod f$  where  $f$  is an  $n$ -tuple of  $S \subseteq \text{Product}(n, S)$  by [6, (131)], (10).  $\square$

(51) Let us consider a non zero natural number  $n$ . Then  $\text{Product}(n, X) \subseteq 2^{(\cup X)^{\text{Seg } n}}$ .

Let us consider a non zero natural number  $n$ , a non empty set  $X$ , and a non empty family  $S$  of subsets of  $X$ .

Let us assume that  $S \neq \{\emptyset\}$ . Now we state the propositions:

(52)  $\text{Product}(n, S) \subseteq 2^{X^{\text{Seg } n}}$ . The theorem is a consequence of (51) and (5).

(53)  $\cup \text{Product}(n, S) \subseteq X^{\text{Seg } n}$ . The theorem is a consequence of (52).

Let  $n$  be a natural number and  $X$  be a non empty set. Let us note that  $\text{Product}(n, X)$  is non empty.

Now we state the proposition:

(54) Let us consider a non empty set  $X$ , a non empty family  $S$  of subsets of  $X$ , and a subset  $x$  of  $X^{\text{Seg } n}$ . Then  $x$  is an element of  $\text{Product}(n, S)$  if and only if there exists an  $n$ -tuple  $s$  of  $S$  such that for every element  $t$  of  $X^{\text{Seg } n}$ , for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in s(i)$  iff  $t \in x$ .

## 9. THE SET OF ALL CLOSED REAL BOUNDED INTERVALS

The set of all closed real bounded intervals yielding a family of subsets of  $\mathbb{R}$  is defined by the term

(Def. 12) the set of all  $[a, b]$  where  $a, b$  are real numbers.

Now we state the proposition:

(55) The set of all closed real bounded intervals =  $\{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is closed interval}\}$ .

Let us note that the set of all closed real bounded intervals is non empty.

Now we state the propositions:

(56) The set of all closed real bounded intervals is  $\cap$ -closed and has the empty element and countable cover.

PROOF: The set of all closed real bounded intervals is  $\cap$ -closed. There exists a countable subset  $X$  of the set of all closed real bounded intervals such that  $X$  is a cover of  $\mathbb{R}$  by [27, (4)], [14, (58)], [2, (2)], [12, (8)].  $\square$

(57) Let us consider a natural number  $n$ . Then  $\text{Seg } n \mapsto$  (the set of all closed real bounded intervals) is an  $n$ -element finite sequence.

## 10. THE SET OF ALL OPEN REAL BOUNDED INTERVALS

The set of all open real bounded intervals yielding a family of subsets of  $\mathbb{R}$  is defined by the term

(Def. 13) the set of all  $]a, b[$  where  $a, b$  are real numbers.

Now we state the proposition:

(58) The set of all open real bounded intervals  $\subseteq \{I, \text{ where } I \text{ is a subset of } \mathbb{R} : I \text{ is open interval}\}$ .

Let us observe that the set of all open real bounded intervals is non empty.

Now we state the propositions:

(59) The set of all open real bounded intervals is  $\cap$ -closed and has the empty element and countable cover.

PROOF: The set of all open real bounded intervals is  $\cap$ -closed. There exists a countable subset  $X$  of the set of all open real bounded intervals such that  $X$  is a cover of  $\mathbb{R}$  by [27, (4)], [14, (58)], [2, (2)], [12, (8)].  $\square$

(60) Let us consider a natural number  $n$ . Then  $\text{Seg } n \mapsto$  (the set of all open real bounded intervals) is an  $n$ -element finite sequence.

11.  $n$ -DIMENSIONAL SUBSET FAMILY OF  $\mathbb{R}$ 

From now on  $n$  denotes a natural number and  $S$  denotes a family of subsets of  $\mathbb{R}$ .

Now we state the proposition:

(61)  $\text{Product}(n, S)$  is a family of subsets of  $\mathcal{R}^n$ . The theorem is a consequence of (49) and (8).

Let us consider  $n$  and  $S$ . One can check that the functor  $\text{Product}(n, S)$  yields a family of subsets of  $\mathcal{R}^n$ . Now we state the propositions:

(62) Let us consider a non empty family  $S$  of subsets of  $\mathbb{R}$ , and a subset  $x$  of  $\mathcal{R}^n$ . Then  $x$  is an element of  $\text{Product}(n, S)$  if and only if there exists an  $n$ -tuple  $s$  of  $S$  such that for every element  $t$  of  $\mathcal{R}^n$ , for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in s(i)$  iff  $t \in x$ .

PROOF: If  $x$  is an element of  $\text{Product}(n, S)$ , then there exists an  $n$ -tuple  $s$  of  $S$  such that for every element  $t$  of  $\mathcal{R}^n$ , for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in s(i)$  iff  $t \in x$  by [6, (93)]. If there exists an  $n$ -tuple  $s$  of  $S$  such that for every element  $t$  of  $\mathcal{R}^n$ , for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in s(i)$  iff  $t \in x$ , then  $x$  is an element of  $\text{Product}(n, S)$  by [6, (93)].  $\square$

(63) Let us consider a non zero natural number  $n$ , and an  $n$ -tuple  $s$  of the set of all closed real bounded intervals. Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = [a(i), b(i)]$ .

PROOF:  $s \in$  (the set of all closed real bounded intervals) $^{\text{Seg } n}$ . Consider  $f$  being a function such that  $s = f$  and  $\text{dom } f = \text{Seg } n$  and  $\text{rng } f \subseteq$  the set of all closed real bounded intervals. Define  $\mathcal{P}[\text{object}, \text{object}] \equiv$  there exists an element  $f$  of  $\mathbb{R} \times \mathbb{R}$  such that  $f = \$_2$  and  $s(\$_1) = [(f)_1, (f)_2]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$  by [7, (3)]. Consider  $f$  being a finite sequence of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_i]$  from [25, Sch. 1]. Consider  $z$  being an element of  $\mathcal{R}^n \times \mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $(z)_1(i) = (f_i)_1$  and  $(z)_2(i) = (f_i)_2$ . Reconsider  $a = (z)_1, b = (z)_2$  as an element of  $\mathcal{R}^n$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = [a(i), b(i)]$ .  $\square$

(64) Let us consider a non zero natural number  $n$ , and an element  $x$  of  $\text{Product}(n, \text{the set of all closed real bounded intervals})$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [a(i), b(i)]$ . The theorem is a consequence of (62) and (63).

Let us consider a non zero natural number  $n$ , a subset  $x$  of  $\mathcal{R}^n$ , and elements  $a, b$  of  $\mathcal{R}^n$ . Now we state the propositions:

(65) Suppose for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [a(i), b(i)]$ . Then  $x$  is an element of  $\text{Product}(n, \text{the set of all closed real bounded intervals})$ .

PROOF: Define  $\mathcal{P}[\text{object}, \text{object}] \equiv$  there exists a natural number  $n$  such that  $\$1 = n$  and  $\$2 = [a(n), b(n)]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of the set of all closed real bounded intervals such that  $\mathcal{P}[i, d]$ . There exists a finite sequence  $g$  of elements of the set of all closed real bounded intervals such that  $\text{len } g = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, g_i]$  from [25, Sch. 1]. Consider  $g$  being a finite sequence of elements of the set of all closed real bounded intervals such that  $\text{len } g = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, g_i]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $g(i) = [a(i), b(i)]$ . There exists a function  $g$  such that  $x = \prod g$  and  $g \in \prod(\text{Seg } n \mapsto \text{(the set of all closed real bounded intervals)})$  by [4, (89)], [24, (13), (7)], [1, (9)].  $\square$

(66) Suppose for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number

$i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ]a(i), b(i)[$ . Then  $x$  is an element of  $\text{Product}(n, \text{the set of all left open real bounded intervals})$ .

PROOF: Define  $\mathcal{P}[\text{object}, \text{object}] \equiv$  there exists a natural number  $n$  such that  $\$1 = n$  and  $\$2 = ]a(n), b(n)[$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of the set of all left open real bounded intervals such that  $\mathcal{P}[i, d]$ . There exists a finite sequence  $g$  of elements of the set of all left open real bounded intervals such that  $\text{len } g = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, g_i]$  from [25, Sch. 1]. Consider  $g$  being a finite sequence of elements of the set of all left open real bounded intervals such that  $\text{len } g = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, g_i]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $g(i) = ]a(i), b(i)[$ . There exists a function  $g$  such that  $x = \prod g$  and  $g \in \prod(\text{Seg } n \mapsto (\text{the set of all left open real bounded intervals}))$  by [4, (89)], [24, (13), (7)], [1, (9)].  $\square$

- (67) Suppose for every element  $t$  of  $\mathcal{R}^n$ ,  $t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [a(i), b(i)[$ . Then  $x$  is an element of  $\text{Product}(n, \text{the set of all right open real bounded intervals})$ .

PROOF: Define  $\mathcal{P}[\text{object}, \text{object}] \equiv$  there exists a natural number  $n$  such that  $\$1 = n$  and  $\$2 = [a(n), b(n)[$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of the set of all right open real bounded intervals such that  $\mathcal{P}[i, d]$ . There exists a finite sequence  $g$  of elements of the set of all right open real bounded intervals such that  $\text{len } g = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, g_i]$  from [25, Sch. 1]. Consider  $g$  being a finite sequence of elements of the set of all right open real bounded intervals such that  $\text{len } g = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, g_i]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $g(i) = [a(i), b(i)[$ . There exists a function  $g$  such that  $x = \prod g$  and  $g \in \prod(\text{Seg } n \mapsto (\text{the set of all right open real bounded intervals}))$  by [4, (89)], [24, (13), (7)], [1, (9)].  $\square$

Now we state the propositions:

- (68) Let us consider a non zero natural number  $n$ , and an  $n$ -tuple  $s$  of the set of all left open real bounded intervals. Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = ]a(i), b(i)[$ .

PROOF:  $s \in (\text{the set of all left open real bounded intervals})^{\text{Seg } n}$ . Consider  $f$  being a function such that  $s = f$  and  $\text{dom } f = \text{Seg } n$  and  $\text{rng } f \subseteq \text{the set of all left open real bounded intervals}$ . Define  $\mathcal{P}[\text{object}, \text{object}] \equiv$  there exists an element  $f$  of  $\mathbb{R} \times \mathbb{R}$  such that  $f = \$2$  and  $s(\$1) = ](f)_1, (f)_2[$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$  by [7, (3)]. Consider  $f$  being a finite sequence of

elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_i]$  from [25, Sch. 1]. Consider  $z$  being an element of  $\mathcal{R}^n \times \mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $(z)_1(i) = (f_i)_1$  and  $(z)_2(i) = (f_i)_2$ . Reconsider  $a = (z)_1, b = (z)_2$  as an element of  $\mathcal{R}^n$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = ]a(i), b(i)]. \square$

(69) Let us consider a non zero natural number  $n$ , and an element  $x$  of  $\text{Product}(n, \text{the set of all left open real bounded intervals})$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ]a(i), b(i)]$ . The theorem is a consequence of (62) and (68).

(70) Let us consider a non zero natural number  $n$ , and an  $n$ -tuple  $s$  of the set of all right open real bounded intervals. Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = [a(i), b(i)[$ .

PROOF:  $s \in (\text{the set of all right open real bounded intervals})^{\text{Seg } n}$ . Consider  $f$  being a function such that  $s = f$  and  $\text{dom } f = \text{Seg } n$  and  $\text{rng } f \subseteq \text{the set of all right open real bounded intervals}$ . Define  $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$2 \text{ and } s(\$1) = [(f)_1, (f)_2[$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$  by [7, (3)]. Consider  $f$  being a finite sequence of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_i]$  from [25, Sch. 1]. Consider  $z$  being an element of  $\mathcal{R}^n \times \mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $(z)_1(i) = (f_i)_1$  and  $(z)_2(i) = (f_i)_2$ . Reconsider  $a = (z)_1, b = (z)_2$  as an element of  $\mathcal{R}^n$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = [a(i), b(i)[. \square$

(71) Let us consider a non zero natural number  $n$ , and an element  $x$  of  $\text{Product}(n, \text{the set of all right open real bounded intervals})$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [a(i), b(i)[$ . The theorem is a consequence of (62) and (70).

## 12. CLOSED/OPEN/LEFT-OPEN/RIGHT-OPEN – HYPER INTERVAL

From now on  $n$  denotes a natural number and  $a, b, c, d$  denote elements of  $\mathcal{R}^n$ .

Let us consider  $n, a,$  and  $b$ . The functor  $\text{ClosedHyperInterval}(a, b)$  yielding a subset of  $\mathcal{R}^n$  is defined by

(Def. 14) for every object  $x$ ,  $x \in it$  iff there exists an element  $y$  of  $\mathcal{R}^n$  such that  $x = y$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $y(i) \in [a(i), b(i)]$ .

The functor  $\text{OpenHyperInterval}(a, b)$  yielding a subset of  $\mathcal{R}^n$  is defined by

(Def. 15) for every object  $x$ ,  $x \in it$  iff there exists an element  $y$  of  $\mathcal{R}^n$  such that  $x = y$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $y(i) \in ]a(i), b(i)[$ .

The functor  $\text{LeftOpenHyperInterval}(a, b)$  yielding a subset of  $\mathcal{R}^n$  is defined by

(Def. 16) for every object  $x$ ,  $x \in it$  iff there exists an element  $y$  of  $\mathcal{R}^n$  such that  $x = y$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $y(i) \in ]a(i), b(i)[$ .

The functor  $\text{RightOpenHyperInterval}(a, b)$  yielding a subset of  $\mathcal{R}^n$  is defined by

(Def. 17) for every object  $x$ ,  $x \in it$  iff there exists an element  $y$  of  $\mathcal{R}^n$  such that  $x = y$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $y(i) \in [a(i), b(i)[$ .

Now we state the proposition:

$$(72) \quad \text{ClosedHyperInterval}(a, a) = \{a\}.$$

PROOF:  $\text{ClosedHyperInterval}(a, a) \subseteq \{a\}$  by [6, (124)].

$$\{a\} \subseteq \text{ClosedHyperInterval}(a, a). \quad \square$$

Let us consider  $n$  and  $a$ . Let us observe that  $\text{ClosedHyperInterval}(a, a)$  is trivial.

Now we state the proposition:

- (73) (i)  $\text{OpenHyperInterval}(a, b) \subseteq \text{LeftOpenHyperInterval}(a, b)$ , and  
 (ii)  $\text{OpenHyperInterval}(a, b) \subseteq \text{RightOpenHyperInterval}(a, b)$ , and  
 (iii)  $\text{LeftOpenHyperInterval}(a, b) \subseteq \text{ClosedHyperInterval}(a, b)$ , and  
 (iv)  $\text{RightOpenHyperInterval}(a, b) \subseteq \text{ClosedHyperInterval}(a, b)$ .

Let us consider  $n$ ,  $a$ , and  $b$ . We say that  $a \leq b$  if and only if

(Def. 18) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $a(i) \leq b(i)$ .

One can verify that the predicate is reflexive.

Now we state the propositions:

$$(74) \quad \text{If } a \leq b \leq c, \text{ then } a \leq c.$$

$$(75) \quad \text{If } a \leq c \text{ and } d \leq b,$$

$$\text{then } \text{ClosedHyperInterval}(c, d) \subseteq \text{ClosedHyperInterval}(a, b).$$

$$(76) \quad \text{If } a \leq b, \text{ then } \text{ClosedHyperInterval}(a, b) \text{ is not empty. The theorem is a consequence of (75) and (72).}$$

Let us consider  $n$ ,  $a$ , and  $b$ . We say that  $a < b$  if and only if

(Def. 19) for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $a(i) < b(i)$ .

Now we state the propositions:

(77) If  $a < b < c$ , then  $a < c$ .

(78) If  $b < a$  and  $n$  is not zero, then  $\text{ClosedHyperInterval}(a, b)$  is empty.

(79)  $n \mapsto r$  is an element of  $\mathcal{R}^n$ .

PROOF: Set  $f = n \mapsto r$ .  $f \in \mathbb{R}^{\text{Seg } n}$  by [6, (112), (93)].  $\square$

Let us consider  $n$  and  $r$ . Note that the functor  $n \mapsto r$  yields an element of  $\mathcal{R}^n$ . One can check that the functor  $\langle r \rangle$  yields an element of  $\mathcal{R}^1$ . Now we state the propositions:

(80) Let us consider a non zero natural number  $n$ , and a point  $e$  of  $\mathcal{E}^n$ . Then there exists an element  $a$  of  $\mathcal{R}^n$  such that

(i)  $a = e$ , and

(ii)  $\text{OpenHypercube}(e, r) = \text{OpenHyperInterval}(a - n \mapsto r, a + n \mapsto r)$ .

PROOF: Reconsider  $a = e$  as an element of  $\mathcal{R}^n$ . Reconsider  $p = e$  as a point of  $\mathcal{E}_T^n$ . Consider  $e_0$  being a point of  $\mathcal{E}^n$  such that  $p = e_0$  and  $\text{OpenHypercube}(e_0, r) = \text{OpenHypercube}(p, r)$ .  $\text{OpenHypercube}(e, r) \subseteq \text{OpenHyperInterval}(a - n \mapsto r, a + n \mapsto r)$  by [8, (27)], [6, (57)], [8, (11)], [18, (4)].  $\text{OpenHyperInterval}(a - n \mapsto r, a + n \mapsto r) \subseteq \text{OpenHypercube}(e, r)$  by [10, (22)], [8, (27)], [6, (57)], [8, (11)].  $\square$

(81) Let us consider a point  $p$  of  $\mathcal{E}_T^n$ . Then there exists an element  $a$  of  $\mathcal{R}^n$  such that

(i)  $a = p$ , and

(ii)  $\text{ClosedHypercube}(p, b) = \text{ClosedHyperInterval}(a - b, a + b)$ .

PROOF: Reconsider  $a = p$  as an element of  $\mathcal{R}^n$ .  $\text{ClosedHypercube}(p, b) \subseteq \text{ClosedHyperInterval}(a - b, a + b)$  by [10, (22)], [8, (11), (27)].  $\text{ClosedHyperInterval}(a - b, a + b) \subseteq \text{ClosedHypercube}(p, b)$  by [10, (22)], [8, (11), (27)].  $\square$

### 13. CORRESPONDANCE BETWEEN INTERVAL AND 1-DIMENSIONAL HYPER INTERVAL

Let us consider a real number  $x$ . Now we state the propositions:

(82)  $x \in [r, s]$  if and only if  $1 \mapsto x \in \text{ClosedHyperInterval}(\langle r \rangle, \langle s \rangle)$ .

PROOF: Set  $a_1 = \langle r \rangle$ . Set  $b_1 = \langle s \rangle$ . For every real number  $x$  such that  $x \in [r, s]$  holds  $1 \mapsto x \in \text{ClosedHyperInterval}(a_1, b_1)$  by [4, (2)], [24, (7)]. For every real number  $x$  such that  $1 \mapsto x \in \text{ClosedHyperInterval}(a_1, b_1)$  holds  $x \in [r, s]$  by [24, (7)].  $\square$

(83)  $x \in ]r, s[$  if and only if  $1 \mapsto x \in \text{OpenHyperInterval}(\langle r \rangle, \langle s \rangle)$ .

PROOF: Set  $a_1 = \langle r \rangle$ . Set  $b_1 = \langle s \rangle$ . For every real number  $x$  such that  $x \in ]r, s[$  holds  $1 \mapsto x \in \text{OpenHyperInterval}(a_1, b_1)$  by [4, (2)], [24, (7)]. For every real number  $x$  such that  $1 \mapsto x \in \text{OpenHyperInterval}(a_1, b_1)$  holds  $x \in ]r, s[$  by [24, (7)].  $\square$

(84)  $x \in ]r, s]$  if and only if  $1 \mapsto x \in \text{LeftOpenHyperInterval}(\langle r \rangle, \langle s \rangle)$ .

PROOF: Set  $a_1 = \langle r \rangle$ . Set  $b_1 = \langle s \rangle$ . For every real number  $x$  such that  $x \in ]r, s]$  holds  $1 \mapsto x \in \text{LeftOpenHyperInterval}(a_1, b_1)$  by [4, (2)], [24, (7)]. For every real number  $x$  such that  $1 \mapsto x \in \text{LeftOpenHyperInterval}(a_1, b_1)$  holds  $x \in ]r, s]$  by [24, (7)].  $\square$

(85)  $x \in [r, s[$  if and only if  $1 \mapsto x \in \text{RightOpenHyperInterval}(\langle r \rangle, \langle s \rangle)$ .

PROOF: Set  $a_1 = \langle r \rangle$ . Set  $b_1 = \langle s \rangle$ . For every real number  $x$  such that  $x \in [r, s[$  holds  $1 \mapsto x \in \text{RightOpenHyperInterval}(a_1, b_1)$  by [4, (2)], [24, (7)]. For every real number  $x$  such that  $1 \mapsto x \in \text{RightOpenHyperInterval}(a_1, b_1)$  holds  $x \in [r, s[$  by [24, (7)].  $\square$

#### 14. CORRESPONDANCE BETWEEN MEASURABLE RECTANGLE AND PRODUCT

From now on  $n$  denotes a non zero natural number.

Now we state the propositions:

(86) Let us consider an  $n$ -tuple  $s$  of the set of all open real bounded intervals. Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = ]a(i), b(i)[$ .

PROOF:  $s \in (\text{the set of all open real bounded intervals})^{\text{Seg } n}$ . Consider  $f$  being a function such that  $s = f$  and  $\text{dom } f = \text{Seg } n$  and  $\text{rng } f \subseteq \text{the set of all open real bounded intervals}$ . Define  $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$_2 \text{ and } s(\$_1) = ](f)_1, (f)_2[$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$  by [7, (3)]. Consider  $f$  being a finite sequence of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_i]$  from [25, Sch. 1]. Consider  $z$  being an element of  $\mathcal{R}^n \times \mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $(z)_1(i) = (f_i)_1$  and  $(z)_2(i) = (f_i)_2$ . Reconsider  $a = (z)_1, b = (z)_2$  as an element of  $\mathcal{R}^n$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = ]a(i), b(i)[$ .  $\square$

(87) Let us consider an element  $x$  of  $\text{Product}(n, \text{the set of all open real bounded intervals})$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that



$i \in \text{Seg } n$  holds  $t(i) \in ]a(i), b(i)[$ . The theorem is a consequence of (62) and (86).

- (88) Let us consider an  $n$ -tuple  $s$  of the set of all left open real bounded intervals. Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = ]a(i), b(i)[$ .

PROOF:  $s \in (\text{the set of all left open real bounded intervals})^{\text{Seg } n}$ . Consider  $f$  being a function such that  $s = f$  and  $\text{dom } f = \text{Seg } n$  and  $\text{rng } f \subseteq \text{the set of all left open real bounded intervals}$ . Define  $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$_2 \text{ and } s(\$_1) = [(f)_1, (f)_2]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$  by [7, (3)]. Consider  $f$  being a finite sequence of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_i]$  from [25, Sch. 1]. Consider  $z$  being an element of  $\mathcal{R}^n \times \mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $(z)_1(i) = (f_i)_1$  and  $(z)_2(i) = (f_i)_2$ . Reconsider  $a = (z)_1, b = (z)_2$  as an element of  $\mathcal{R}^n$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = ]a(i), b(i)[$ .  $\square$

- (89) Let us consider an element  $x$  of  $\text{Product}(n, \text{the set of all left open real bounded intervals})$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every element  $t$  of  $\mathcal{R}^n, t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in ]a(i), b(i)[$ . The theorem is a consequence of (62) and (88).

- (90) Let us consider an  $n$ -tuple  $s$  of the set of all right open real bounded intervals. Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = [a(i), b(i)[$ .

PROOF:  $s \in (\text{the set of all right open real bounded intervals})^{\text{Seg } n}$ . Consider  $f$  being a function such that  $s = f$  and  $\text{dom } f = \text{Seg } n$  and  $\text{rng } f \subseteq \text{the set of all right open real bounded intervals}$ . Define  $\mathcal{P}[\text{object}, \text{object}] \equiv \text{there exists an element } f \text{ of } \mathbb{R} \times \mathbb{R} \text{ such that } f = \$_2 \text{ and } s(\$_1) = [(f)_1, (f)_2]$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  there exists an element  $d$  of  $\mathbb{R} \times \mathbb{R}$  such that  $\mathcal{P}[i, d]$  by [7, (3)]. Consider  $f$  being a finite sequence of elements of  $\mathbb{R} \times \mathbb{R}$  such that  $\text{len } f = n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $\mathcal{P}[i, f_i]$  from [25, Sch. 1]. Consider  $z$  being an element of  $\mathcal{R}^n \times \mathcal{R}^n$  such that for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $(z)_1(i) = (f_i)_1$  and  $(z)_2(i) = (f_i)_2$ . Reconsider  $a = (z)_1, b = (z)_2$  as an element of  $\mathcal{R}^n$ . For every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $s(i) = [a(i), b(i)[$ .  $\square$

- (91) Let us consider an element  $x$  of  $\text{Product}(n, \text{the set of all right open real bounded intervals})$ . Then there exist elements  $a, b$  of  $\mathcal{R}^n$  such that

for every element  $t$  of  $\mathcal{R}^n$ ,  $t \in x$  iff for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $t(i) \in [a(i), b(i)[$ . The theorem is a consequence of (62) and (90).

- (92)  $\text{MeasurableRectangleLeftOpenIntervals}(n) = \text{Product}(n, \text{the set of all left open real bounded intervals})$ . The theorem is a consequence of (40) and (66).
- (93)  $\text{MeasurableRectangleRightOpenIntervals}(n) = \text{Product}(n, \text{the set of all right open real bounded intervals})$ . The theorem is a consequence of (46) and (67).

### 15. CHEBYSHEV DISTANCE

In the sequel  $n$  denotes a non zero natural number and  $x, y, z$  denote elements of  $\mathcal{R}^n$ .

Let us consider  $n$ . The functor  $D_{\text{Chebyshev}}^n$  yielding a function from  $\mathcal{R}^n \times \mathcal{R}^n$  into  $\mathbb{R}$  is defined by

(Def. 20) for every elements  $x, y$  of  $\mathcal{R}^n$ ,  $it(x, y) = \sup \text{rng}|x - y|$ .

Now we state the propositions:

- (94) (i) the set of all  $|x(i) - y(i)|$  where  $i$  is an element of  $\text{Seg } n$  is real-membered, and

(ii) the set of all  $|x(i) - y(i)|$  where  $i$  is an element of  $\text{Seg } n = \text{rng}|x - y|$ .

PROOF: Set  $S$  = the set of all  $|x(i) - y(i)|$  where  $i$  is an element of  $\text{Seg } n$ .  $S \subseteq \text{rng}|x - y|$  by [8, (27)], [6, (124)]. For every object  $t$  such that  $t \in \text{rng}|x - y|$  holds  $t \in S$  by [6, (124)], [8, (27)].  $\square$

- (95) There exists an extended real-membered set  $S$  such that

(i)  $S$  = the set of all  $|x(i) - y(i)|$  where  $i$  is an element of  $\text{Seg } n$ , and

(ii)  $(D_{\text{Chebyshev}}^n)(x, y) = \sup S$ .

The theorem is a consequence of (94).

- (96)  $(D_{\text{Chebyshev}}^n)(x, y) = |x - y|(\text{max-diff-index}(x, y))$ .

PROOF:  $(D_{\text{Chebyshev}}^n)(x, y) \leq |x - y|(\text{max-diff-index}(x, y))$  by [15, (5)].  $\square$

- (97)  $(D_{\text{Chebyshev}}^n)(x, y) = 0$  if and only if  $x = y$ .

PROOF: Consider  $S$  being an extended real-membered set such that  $S =$  the set of all  $|x(i) - y(i)|$  where  $i$  is an element of  $\text{Seg } n$  and

$(D_{\text{Chebyshev}}^n)(x, y) = \sup S$ .  $S = \{0\}$  by [19, (2)], [3, (53)], [4, (1)].  $\square$

- (98)  $(D_{\text{Chebyshev}}^n)(x, y) = (D_{\text{Chebyshev}}^n)(y, x)$ . The theorem is a consequence of (1).

(99)  $(D_{\text{Chebyshev}}^n)(x, y) \leq (D_{\text{Chebyshev}}^n)(x, z) + (D_{\text{Chebyshev}}^n)(z, y).$

PROOF: Reconsider  $s_1 = \sup \text{rng}|x - y|$ ,  $s_2 = \sup \text{rng}|x - z|$ ,  $s_3 = \sup \text{rng}|z - y|$  as a real number.  $s_1 \leq s_2 + s_3$  by [8, (27)], [5, (56)], [6, (124)], (2).  $\square$

(100)  $D_{\text{Chebyshev}}^n$  is a metric of  $\mathcal{R}^n$ . The theorem is a consequence of (97), (98), and (99).

(101)  $\rho^2([0, 0], [1, 1]) = \sqrt{2}.$

(102)  $(D_{\text{Chebyshev}}^2)([0, 0], [1, 1]) = 1.$

PROOF: Consider  $S$  being an extended real-membered set such that  $S =$  the set of all  $|[0, 0](i) - [1, 1](i)|$  where  $i$  is an element of Seg 2 and  $(D_{\text{Chebyshev}}^2)([0, 0], [1, 1]) = \sup S$ .  $S = \{0 - 1\}$  by [4, (2), (44)].  $\square$

Let us consider elements  $x, y$  of  $\mathcal{R}^1$ . Now we state the propositions:

(103)  $(D_{\text{Chebyshev}}^1)(x, y) = |x(1) - y(1)|.$

PROOF: Consider  $S$  being an extended real-membered set such that  $S =$  the set of all  $|x(i) - y(i)|$  where  $i$  is an element of Seg 1 and  $(D_{\text{Chebyshev}}^1)(x, y) = \sup S$ .  $S = \{|x(1) - y(1)|\}$  by [4, (2)].  $\square$

(104)  $\rho^1(x, y) = |x(1) - y(1)|.$

Now we state the propositions:

(105)  $\rho^1 = D_{\text{Chebyshev}}^1$ . The theorem is a consequence of (104) and (103).

(106)  $\rho^2 \neq D_{\text{Chebyshev}}^2$ . The theorem is a consequence of (101) and (102).

Let  $n$  be a non zero natural number. The functor  $L_\infty(n)$  yielding a strict metric space is defined by the term

(Def. 21)  $\langle \mathcal{R}^n, D_{\text{Chebyshev}}^n \rangle.$

Let us observe that  $L_\infty(n)$  is non empty.

The functor  $\mathcal{E}_\infty^n(n)$  yielding a strict real linear topological structure is defined by

(Def. 22) the topological structure of  $it = (L_\infty(n))_{\text{top}}$  and the RLS structure of  $it = \mathbb{R}_{\mathbb{R}}^{\text{Seg } n}.$

Now we state the proposition:

(107) The RLS structure of  $\mathcal{E}_{\mathbb{T}}^n =$  the RLS structure of  $\mathcal{E}_\infty^n(n).$

Let  $n$  be a non zero natural number. Let us note that  $\mathcal{E}_\infty^n(n)$  is non empty.

Now we state the propositions:

(108) Let us consider an element  $x$  of  $\mathcal{R}^0$ . Then

(i)  $\text{Intervals}(x, r)$  is empty, and

(ii)  $\prod \text{Intervals}(x, r) = \{\emptyset\}.$

(109) If  $r \leq 0$ , then  $\prod \text{Intervals}(x, r)$  is empty.

In the sequel  $p$  denotes an element of  $L_\infty(n)$ .

Let  $n$  be a non zero natural number and  $p$  be an element of  $L_\infty(n)$ . The functor  ${}^{\textcircled{a}}p$  yielding an element of  $\mathcal{R}^n$  is defined by the term

(Def. 23)  $p$ .

Now we state the propositions:

(110)  $\text{Ball}(p, r) = \prod \text{Intervals}({}^{\textcircled{a}}p, r)$ . The theorem is a consequence of (109), (95), and (96).

(111) Let us consider a point  $e$  of  $\mathcal{E}^n$ . If  $e = p$ , then  $\text{Ball}(p, r) = \text{OpenHypercube}(e, r)$ . The theorem is a consequence of (110).

Let  $n$  be a non zero natural number,  $p$  be an element of  $L_\infty(n)$ , and  $r$  be a negative real number. Let us note that  $\overline{\text{Ball}}(p, r)$  is empty.

Now we state the propositions:

(112) Let us consider an object  $t$ . Then  $t \in \overline{\text{Ball}}(p, r)$  if and only if there exists a function  $f$  such that  $t = f$  and  $\text{dom } f = \text{Seg } n$  and for every natural number  $i$  such that  $i \in \text{Seg } n$  holds  $f(i) \in [({}^{\textcircled{a}}p)(i) - r, ({}^{\textcircled{a}}p)(i) + r]$ . The theorem is a consequence of (95).

(113) Let us consider a point  $p$  of  $\mathcal{E}_T^n$ , and an element  $q$  of  $L_\infty(n)$ . Suppose  $q = p$ . Then  $\overline{\text{Ball}}(q, r) = \text{ClosedHypercube}(p, n \mapsto r)$ .

PROOF: For every object  $x$  such that  $x \in \overline{\text{Ball}}(q, r)$  holds  $x \in \text{ClosedHypercube}(p, n \mapsto r)$  by (112), [6, (57), (93)], [10, (22)]. For every object  $x$  such that  $x \in \text{ClosedHypercube}(p, n \mapsto r)$  holds  $x \in \overline{\text{Ball}}(q, r)$  by [10, (22)], [6, (131), (124), (57)].  $\square$

(114)  $\text{Ball}(p, r) = \text{OpenHyperInterval}({}^{\textcircled{a}}p - n \mapsto r, {}^{\textcircled{a}}p + n \mapsto r)$ . The theorem is a consequence of (80) and (110).

(115)  $\overline{\text{Ball}}(p, r) = \text{ClosedHyperInterval}({}^{\textcircled{a}}p - n \mapsto r, {}^{\textcircled{a}}p + n \mapsto r)$ . The theorem is a consequence of (81) and (113).

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