

Double Sequences and Iterated Limits in Regular Space

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Summary. First, we define in Mizar [5], the Cartesian product of two filters bases and the Cartesian product of two filters. After comparing the product of two Fréchet filters on \mathbb{N} (\mathcal{F}_1) with the Fréchet filter on $\mathbb{N} \times \mathbb{N}$ (\mathcal{F}_2), we compare $\lim_{\mathcal{F}_1}$ and $\lim_{\mathcal{F}_2}$ for all double sequences in a non empty topological space.

Endou, Okazaki and Shidama formalized in [14] the “convergence in Pringsheim’s sense” for double sequence of real numbers. We show some basic correspondences between the p -convergence and the filter convergence in a topological space. Then we formalize that the double sequence $(x_{m,n} = \frac{1}{m+1})_{(m,n)} \in \mathbb{N} \times \mathbb{N}$ converges in “Pringsheim’s sense” but not in Fréchet filter on $\mathbb{N} \times \mathbb{N}$ sense.

In the next section, we generalize some definitions: “is convergent in the first coordinate”, “is convergent in the second coordinate”, “the \lim in the first coordinate of”, “the \lim in the second coordinate of” according to [14], in Hausdorff space.

Finally, we generalize two theorems: (3) and (4) from [14] in the case of double sequences and we formalize the “iterated limit” theorem (“Double limit” [7], p. 81, par. 8.5 “*Double limite*” [6] (TG I,57)), all in regular space. We were inspired by the exercises (2.11.4), (2.17.5) [17] and the corrections B.10 [18].

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1. PRELIMINARIES

From now on x denotes an object, X, Y, Z denote sets, i, j, k, l, m, n denote natural numbers, r, s denote real numbers, n_1 denotes an element of the ordered \mathbb{N} , and A denotes a subset of $\mathbb{N} \times \mathbb{N}$.

Now we state the propositions:

- (1) Let us consider a finite subset W of X . If $X \setminus W \subseteq Z$, then $X \setminus Z$ is finite.
- (2) If $Z \subseteq X$ and $X \setminus Z$ is finite, then there exists a finite subset W of X such that $X \setminus W = Z$.
- (3) Let us consider sets X_1, X_2 , a family S_1 of subsets of X_1 , and a family S_2 of subsets of X_2 . Then $\{s, \text{ where } s \text{ is a subset of } X_1 \times X_2 : \text{ there exist sets } s_1, s_2 \text{ such that } s_1 \in S_1 \text{ and } s_2 \in S_2 \text{ and } s = s_1 \times s_2\}$ is a family of subsets of $X_1 \times X_2$.
- (4) If $x \in X \times Y$, then x is pair.
- (5) If $0 < r$, then there exists m such that m is not zero and $\frac{1}{m} < r$.
- (6) Let us consider points x, y of the metric space of real numbers. Then there exist real numbers x_1, y_1 such that
 - (i) $x = x_1$, and
 - (ii) $y = y_1$, and
 - (iii) $\rho(x, y) = \rho_{\mathbb{R}}(x, y)$, and
 - (iv) $\rho(x, y) = \rho^1(\langle x \rangle, \langle y \rangle)$, and
 - (v) $\rho(x, y) = |x_1 - y_1|$.
- (7) Let us consider points x, y of $(\mathcal{E}^1)_{\text{top}}$. Then there exist points x_2, y_2 of the metric space of real numbers and there exist real numbers x_1, y_1 such that $x_2 = x_1$ and $y_2 = y_1$ and $x = \langle x_1 \rangle$ and $y = \langle y_1 \rangle$ and $\rho(x_2, y_2) = \rho_{\mathbb{R}}(x_1, y_1)$ and $\rho(x_2, y_2) = \rho^1(\langle x_1 \rangle, \langle y_1 \rangle)$ and $\rho(x_2, y_2) = |x_1 - y_1|$.
- (8) Let us consider points x, y of \mathcal{E}^1 , and real numbers r, s . If $x = \langle r \rangle$ and $y = \langle s \rangle$, then $\rho(x, y) = |r - s|$. The theorem is a consequence of (7).

One can check that $\mathbb{N} \times \mathbb{N}$ is countable and $\mathbb{N} \times \mathbb{N}$ is denumerable.

Now we state the propositions:

- (9) the set of all $\langle 0, n \rangle$ where n is a natural number is infinite.
 PROOF: Define $\mathcal{F}(\text{object}) = \langle 0, \$1 \rangle$. Consider f being a function such that $\text{dom } f = \mathbb{N}$ and for every object x such that $x \in \mathbb{N}$ holds $f(x) = \mathcal{F}(x)$ from [9, Sch. 3]. f is one-to-one. $\text{rng } f =$ the set of all $\langle 0, n \rangle$ where n is a natural number by [9, (3)]. \square
- (10) If $i \leq k$ and $j \leq l$, then $\mathbb{Z}_i \times \mathbb{Z}_j \subseteq \mathbb{Z}_k \times \mathbb{Z}_l$.
- (11) $(\mathbb{N} \setminus \mathbb{Z}_m) \times (\mathbb{N} \setminus \mathbb{Z}_n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_m \times \mathbb{Z}_n$.
- (12) If $n = n_1$ and $n \leq m$, then $m \in \uparrow n_1$.
- (13) If $n = n_1$ and $m \in \uparrow n_1$, then $n \leq m$.
- (14) If $n = n_1$, then $\uparrow n_1 = \mathbb{N} \setminus \mathbb{Z}_n$.

PROOF: $\uparrow n_1 \subseteq \mathbb{N} \setminus \mathbb{Z}_n$ by [12, (50)], (13), [1, (44)]. $\mathbb{N} \setminus \mathbb{Z}_n \subseteq \uparrow n_1$ by [1, (44)], [12, (50)]. \square

- (15) $\pi_1(A) = \{x, \text{ where } x \text{ is an element of } \mathbb{N} : \text{ there exists an element } y \text{ of } \mathbb{N} \text{ such that } \langle x, y \rangle \in A\}$.
- (16) $\pi_2(A) = \{y, \text{ where } y \text{ is an element of } \mathbb{N} : \text{ there exists an element } x \text{ of } \mathbb{N} \text{ such that } \langle x, y \rangle \in A\}$.
- (17) Let us consider a finite subset A of $\mathbb{N} \times \mathbb{N}$. Then there exists m and there exists n such that $A \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$. The theorem is a consequence of (15) and (16).
- (18) Let us consider a non empty set X . Then every filter of X is a proper filter of 2_{\subseteq}^X .
- (19) Let us consider a non empty set X , and a filter \mathcal{F} of X . Then there exists a filter base \mathcal{B} of X such that
 - (i) $\mathcal{B} = \mathcal{F}$, and
 - (ii) $[\mathcal{B}] = \mathcal{F}$.
- (20) Let us consider a non empty topological space T , and a filter \mathcal{F} of the carrier of T . If $x \in \text{LimFilter}(\mathcal{F})$, then x is a cluster point of \mathcal{F}, T .
- (21) Let us consider an element B of the base of Frechet filter. Then there exists n such that $B = \mathbb{N} \setminus \mathbb{Z}_n$. The theorem is a consequence of (14).
- (22) Let us consider a subset B of \mathbb{N} . Suppose $B = \mathbb{N} \setminus \mathbb{Z}_n$. Then B is an element of the base of Frechet filter. The theorem is a consequence of (14).

2. CARTESIAN PRODUCT OF TWO FILTERS

From now on X, Y, X_1, X_2 denote non empty sets, $\mathcal{A}_1, \mathcal{B}_1$ denote filter bases of X_1 , $\mathcal{A}_2, \mathcal{B}_2$ denote filter bases of X_2 , \mathcal{F}_1 denotes a filter of X_1 , \mathcal{F}_2 denotes a filter of X_2 , \mathcal{B}_3 denotes a generalized basis of \mathcal{F}_1 .

Let X_1, X_2 be non empty sets, \mathcal{B}_1 be a filter base of X_1 , and \mathcal{B}_2 be a filter base of X_2 . The functor $\mathcal{B}_1 \times \mathcal{B}_2$ yielding a filter base of $X_1 \times X_2$ is defined by the term

(Def. 1) the set of all $B_1 \times B_2$ where B_1 is an element of \mathcal{B}_1 , B_2 is an element of \mathcal{B}_2 .

Now we state the propositions:

- (23) Suppose $\mathcal{F}_1 = [\mathcal{B}_1)$ and $\mathcal{F}_1 = [\mathcal{A}_1)$ and $\mathcal{F}_2 = [\mathcal{B}_2)$ and $\mathcal{F}_2 = [\mathcal{A}_2)$. Then $[\mathcal{B}_1 \times \mathcal{B}_2) = [\mathcal{A}_1 \times \mathcal{A}_2)$.
- (24) If $\mathcal{B}_3 = \mathcal{B}_1$, then $[\mathcal{B}_1] = \mathcal{F}_1$.

(25) There exists \mathcal{B}_1 such that $[\mathcal{B}_1] = \mathcal{F}_1$. The theorem is a consequence of (24).

Let X_1, X_2 be non empty sets, \mathcal{F}_1 be a filter of X_1 , and \mathcal{F}_2 be a filter of X_2 . The functor $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ yielding a filter of $X_1 \times X_2$ is defined by

(Def. 2) there exists a filter base \mathcal{B}_1 of X_1 and there exists a filter base \mathcal{B}_2 of X_2 such that $[\mathcal{B}_1] = \mathcal{F}_1$ and $[\mathcal{B}_2] = \mathcal{F}_2$ and $it = [\mathcal{B}_1 \times \mathcal{B}_2]$.

Let \mathcal{B}_1 be a generalized basis of \mathcal{F}_1 and \mathcal{B}_2 be a generalized basis of \mathcal{F}_2 . The functor $\mathcal{B}_1 \times \mathcal{B}_2$ yielding a generalized basis of $\langle \mathcal{F}_1, \mathcal{F}_2 \rangle$ is defined by

(Def. 3) there exists a filter base \mathcal{B}_3 of X_1 and there exists a filter base \mathcal{B}_4 of X_2 such that $\mathcal{B}_1 = \mathcal{B}_3$ and $\mathcal{B}_2 = \mathcal{B}_4$ and $it = \mathcal{B}_3 \times \mathcal{B}_4$.

Let n be a natural number. The functor $\uparrow^2(n)$ yielding a subset of $\mathbb{N} \times \mathbb{N}$ is defined by

(Def. 4) for every element x of $\mathbb{N} \times \mathbb{N}$, $x \in it$ iff there exist natural numbers n_1, n_2 such that $n_1 = (x)_1$ and $n_2 = (x)_2$ and $n \leq n_1$ and $n \leq n_2$.

Now we state the proposition:

(26) $\langle n, n \rangle \in \uparrow^2(n)$.

Let us consider n . One can check that $\uparrow^2(n)$ is non empty.

Now we state the propositions:

(27) If $\langle i, j \rangle \in \uparrow^2(n)$, then $\langle i + k, j \rangle, \langle i, j + l \rangle \in \uparrow^2(n)$.

(28) $\uparrow^2(n)$ is an infinite subset of $\mathbb{N} \times \mathbb{N}$. The theorem is a consequence of (17).

(29) If $n_1 = n$, then $\uparrow^2(n) = \uparrow n_1 \times \uparrow n_1$. The theorem is a consequence of (12) and (13).

(30) If $m = n - 1$, then $\uparrow^2(n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \text{Seg } m \times \text{Seg } m$.

PROOF: Reconsider $y = x$ as an element of $\mathbb{N} \times \mathbb{N}$. Consider n_1, n_2 being natural numbers such that $n_1 = (y)_1$ and $n_2 = (y)_2$ and $n \leq n_1$ and $n \leq n_2$. $x \notin \text{Seg } m \times \text{Seg } m$ by [3, (1)]. \square

(31) $\uparrow^2(n) \subseteq \mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_n \times \mathbb{Z}_n$.

PROOF: Reconsider $y = x$ as an element of $\mathbb{N} \times \mathbb{N}$. Consider n_1, n_2 being natural numbers such that $n_1 = (y)_1$ and $n_2 = (y)_2$ and $n \leq n_1$ and $n \leq n_2$. $x \notin \mathbb{Z}_n \times \mathbb{Z}_n$ by [16, (10)]. \square

(32) $\uparrow^2(n) = (\mathbb{N} \setminus \mathbb{Z}_n) \times (\mathbb{N} \setminus \mathbb{Z}_n)$. The theorem is a consequence of (14) and (29).

(33) There exists n such that $\uparrow^2(n) \subseteq (\mathbb{N} \setminus \mathbb{Z}_i) \times (\mathbb{N} \setminus \mathbb{Z}_j)$. The theorem is a consequence of (4).

(34) If $n = \max(i, j)$, then $\uparrow^2(n) \subseteq (\uparrow^2(i)) \cap (\uparrow^2(j))$.

Let n be a natural number. The functor $\downarrow^2(n)$ yielding a subset of $\mathbb{N} \times \mathbb{N}$ is defined by

(Def. 5) for every element x of $\mathbb{N} \times \mathbb{N}$, $x \in it$ iff there exist natural numbers n_1, n_2 such that $n_1 = (x)_1$ and $n_2 = (x)_2$ and $n_1 < n$ and $n_2 < n$.

Now we state the propositions:

(35) $\downarrow^2(n) = \mathbb{Z}_n \times \mathbb{Z}_n$.

PROOF: $\downarrow^2(n) \subseteq \mathbb{Z}_n \times \mathbb{Z}_n$ by [1, (44)]. Consider y_2, y_1 being objects such that $y_2 \in \mathbb{Z}_n$ and $y_1 \in \mathbb{Z}_n$ and $x = \langle y_2, y_1 \rangle$. \square

(36) Let us consider a finite subset A of $\mathbb{N} \times \mathbb{N}$. Then there exists n such that $A \subseteq \downarrow^2(n)$.

PROOF: Consider m, n such that $A \subseteq \mathbb{Z}_m \times \mathbb{Z}_n$. Reconsider $m_1 = \max(m, n)$ as a natural number. $A \subseteq \downarrow^2(m_1)$ by [1, (39)], [11, (96)], (35). \square

(37) $\downarrow^2(n)$ is a finite subset of $\mathbb{N} \times \mathbb{N}$. The theorem is a consequence of (35).

3. COMPARISON BETWEEN CARTESIAN PRODUCT OF FRECHET FILTER ON \mathbb{N} AND THE FRECHET FILTER OF $\mathbb{N} \times \mathbb{N}$

Let us consider an element x of (the base of Frechet filter) \times (the base of Frechet filter). Now we state the propositions:

(38) There exists i and there exists j such that $x = (\mathbb{N} \setminus \mathbb{Z}_i) \times (\mathbb{N} \setminus \mathbb{Z}_j)$. The theorem is a consequence of (21).

(39) There exists n such that $\uparrow^2(n) \subseteq x$. The theorem is a consequence of (38) and (33).

(40) (The base of Frechet filter) \times (the base of Frechet filter) is a filter base of $\mathbb{N} \times \mathbb{N}$.

(41) There exists a generalized basis \mathcal{B} of $\text{FrechetFilter}(\mathbb{N})$ such that

(i) $\mathcal{B} =$ the base of Frechet filter, and

(ii) $\mathcal{B} \times \mathcal{B}$ is a generalized basis of $\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$.

The functor $\uparrow_{\mathbb{N}}^2$ yielding a filter base of $\mathbb{N} \times \mathbb{N}$ is defined by the term

(Def. 6) the set of all $\uparrow^2(n)$ where n is a natural number.

Now we state the propositions:

(42) $\uparrow_{\mathbb{N}}^2$ and (the base of Frechet filter) \times (the base of Frechet filter) are equivalent generators. The theorem is a consequence of (22), (32), and (39).

(43) $[(\text{the base of Frechet filter}) \times (\text{the base of Frechet filter})] = \langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$. The theorem is a consequence of (41).

(44) $[\uparrow_{\mathbb{N}}^2] = \langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$.

(45) $\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$ is finer than $\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})$.
 The theorem is a consequence of (17), (11), (22), and (43).

(46) (i) $\mathbb{N} \times \mathbb{N} \setminus$ the set of all $\langle 0, n \rangle$ where n is a natural number $\in \langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$, and

(ii) $\mathbb{N} \times \mathbb{N} \setminus$ the set of all $\langle 0, n \rangle$ where n is a natural number $\notin \text{FrechetFilter}(\mathbb{N} \times \mathbb{N})$.

PROOF: Set $X = \mathbb{N} \times \mathbb{N} \setminus$ the set of all $\langle 0, n \rangle$ where n is a natural number. $\uparrow^2(1) \subseteq X$ by (32), [1, (44)]. $X \notin \text{FrechetFilter}(\mathbb{N} \times \mathbb{N})$ by [12, (51)], [15, (5)], (9). \square

(47) $\text{FrechetFilter}(\mathbb{N} \times \mathbb{N}) \neq \langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$.

4. TOPOLOGICAL SPACE AND DOUBLE SEQUENCE

In the sequel T denotes a non empty topological space, s denotes a function from $\mathbb{N} \times \mathbb{N}$ into the carrier of T , M denotes a subset of the carrier of T , and $\mathcal{F}_1, \mathcal{F}_2$ denote filters of the carrier of T . Now we state the propositions:

(48) If \mathcal{F}_2 is finer than \mathcal{F}_1 , then $\text{LimFilter}(\mathcal{F}_1) \subseteq \text{LimFilter}(\mathcal{F}_2)$.

(49) Let us consider a function f from X into Y , and filters $\mathcal{F}_1, \mathcal{F}_2$ of X . Suppose \mathcal{F}_2 is finer than \mathcal{F}_1 . Then the image of filter \mathcal{F}_2 under f is finer than the image of filter \mathcal{F}_1 under f .

(50) $s^{-1}(M) \in \text{FrechetFilter}(\mathbb{N} \times \mathbb{N})$ if and only if there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(M) = \mathbb{N} \times \mathbb{N} \setminus A$.

(51) $s^{-1}(M) \in \langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$ if and only if there exists n such that $\uparrow^2(n) \subseteq s^{-1}(M)$. The theorem is a consequence of (43), (39), and (42).

(52) The image of filter $\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})$ under $s = \{M$, where M is a subset of the carrier of T : there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(M) = \mathbb{N} \times \mathbb{N} \setminus A\}$. The theorem is a consequence of (50).

(53) The image of filter $\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle$ under $s = \{M$, where M is a subset of the carrier of T : there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(M)\}$. The theorem is a consequence of (51).

Let us consider a point x of T . Now we state the propositions:

(54) $x \in \lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every neighbourhood A of x , there exists a finite subset B of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(A) = \mathbb{N} \times \mathbb{N} \setminus B$. The theorem is a consequence of (52).

(55) $x \in \lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every neighbourhood A of x , $\mathbb{N} \times \mathbb{N} \setminus s^{-1}(A)$ is finite. The theorem is a consequence of (54), (1), and (2).

- (56) $x \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every neighbourhood A of x , there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(A)$. The theorem is a consequence of (53).

Let us consider a point x of T and a generalized basis \mathcal{B} of BooleanFilter ToFilter(the neighborhood system of x). Now we state the propositions:

- (57) $x \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of \mathcal{B} , there exists a natural number n such that $\uparrow^2(n) \subseteq s^{-1}(B)$. The theorem is a consequence of (56).
- (58) $x \in \lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of \mathcal{B} , there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^{-1}(B) = \mathbb{N} \times \mathbb{N} \setminus A$. The theorem is a consequence of (54), (1), and (55).
- (59) $x \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of \mathcal{B} , there exists a natural number n such that $s^\circ(\uparrow^2(n)) \subseteq B$. The theorem is a consequence of (57).

- (60) $x \in \lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of \mathcal{B} , there exists a finite subset A of $\mathbb{N} \times \mathbb{N}$ such that $s^\circ(\mathbb{N} \times \mathbb{N} \setminus A) \subseteq B$.

PROOF: For every neighbourhood A of x , $\mathbb{N} \times \mathbb{N} \setminus s^{-1}(A)$ is finite by [4, (2)], [19, (143)], [9, (76)]. \square

- (61) $x \in \lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of \mathcal{B} , there exists n and there exists m such that $s^\circ(\mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_n \times \mathbb{Z}_m) \subseteq B$. The theorem is a consequence of (60) and (17).

- (62) $x \in s^\circ(\uparrow^2(n))$ if and only if there exists i and there exists j such that $n \leq i$ and $n \leq j$ and $x = s(i, j)$.

- (63) $x \in s^\circ(\mathbb{N} \times \mathbb{N} \setminus \mathbb{Z}_i \times \mathbb{Z}_j)$ if and only if there exist natural numbers n, m such that $(i \leq n$ or $j \leq m)$ and $x = s(n, m)$.

PROOF: Consider n, m being natural numbers such that $i \leq n$ or $j \leq m$ and $x = s(n, m)$. $\langle n, m \rangle \notin \mathbb{Z}_i \times \mathbb{Z}_j$ by [1, (44)]. \square

Let us consider a point x of T and a generalized basis \mathcal{B} of BooleanFilter ToFilter(the neighborhood system of x). Now we state the propositions:

- (64) $x \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every element B of \mathcal{B} , there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $s(n_1, n_2) \in B$. The theorem is a consequence of (62) and (59).

- (65) $x \in \lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s$ if and only if for every element B of \mathcal{B} , there exists i and there exists j such that for every m and n such that $i \leq m$ or $j \leq n$ holds $s(m, n) \in B$. The theorem is a consequence of (61).

- (66) $\lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} s \subseteq \lim_{[\uparrow^2_{\mathbb{N}}]}$. The theorem is a consequence of (42), (43), (45), (48), and (49).

5. METRIC SPACE AND DOUBLE SEQUENCE

Now we state the propositions:

(67) Let us consider a non empty metric space M , a point p of M , a point x of M_{top} , and a function s from $\mathbb{N} \times \mathbb{N}$ into M_{top} . Suppose $x = p$. Then $x \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every non zero natural number m , there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $s(n_1, n_2) \in \{q, \text{ where } q \text{ is a point of } M : \rho(p, q) < \frac{1}{m}\}$.

PROOF: $x \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} s$ iff for every non zero natural number m , there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $s(n_1, n_2) \in \{q, \text{ where } q \text{ is a point of } M : \rho(p, q) < \frac{1}{m}\}$ by [13, (6)], (64). \square

(68) Let us consider a non empty metric space M , a point p of M , a point x of M_{top} , a function s from $\mathbb{N} \times \mathbb{N}$ into M_{top} , and a function s_2 from $\mathbb{N} \times \mathbb{N}$ into M . Suppose $x = p$ and $s = s_2$. Then $x \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} s$ if and only if for every non zero natural number m , there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $s_2(n_1, n_2) \in \{q, \text{ where } q \text{ is a point of } M : \rho(p, q) < \frac{1}{m}\}$.

6. ONE-DIMENSIONAL EUCLIDEAN METRIC SPACE AND DOUBLE SEQUENCE

In the sequel R denotes a function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} .

Now we state the proposition:

(69) Let us consider a point x of $(\mathcal{E}^1)_{\text{top}}$, a point y of \mathcal{E}^1 , a generalized basis \mathcal{B} of $\text{BooleanFilterToFilter}(\text{the neighborhood system of } x)$, and an element b of \mathcal{B} . Suppose $x = y$ and $\mathcal{B} = \text{Balls } x$. Then there exists a natural number n such that $b = \{q, \text{ where } q \text{ is an element of } \mathcal{E}^1 : \rho(y, q) < \frac{1}{n}\}$.

Let s be a function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . The functor $\# s$ yielding a function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R}^1 is defined by the term

(Def. 7) s .

Now we state the propositions:

(70) Let us consider a function s from $\mathbb{N} \times \mathbb{N}$ into $(\mathcal{E}^1)_{\text{top}}$, and a point y of \mathcal{E}^1 . Then $s^\circ(\uparrow^2(n)) \subseteq \{q, \text{ where } q \text{ is an element of } \mathcal{E}^1 : \rho(y, q) < \frac{1}{m}\}$ if and only if for every object x such that $x \in s^\circ(\uparrow^2(n))$ there exist real numbers r_1, r_2 such that $x = \langle r_1 \rangle$ and $y = \langle r_2 \rangle$ and $|r_2 - r_1| < \frac{1}{m}$. The theorem is a consequence of (8).

(71) $r \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$ if and only if for every non zero natural number m , there exists a natural number n such that for every

natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $|R(n_1, n_2) - r| < \frac{1}{m}$.

PROOF: Reconsider $p = r$ as a point of the metric space of real numbers. for every non zero natural number m , there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $R(n_1, n_2) \in \{q, \text{ where } q \text{ is a point of the metric space of real numbers} : \rho(p, q) < \frac{1}{m}\}$ iff for every non zero natural number m , there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $|R(n_1, n_2) - r| < \frac{1}{m}$ by (6), [8, (60)]. \square

7. BASIC RELATIONS CONVERGENCE IN PRINGSHEIM’S SENSE AND FILTER CONVERGENCE

Now we state the propositions:

- (72) Suppose $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R \neq \emptyset$. Then there exists a real number x such that $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R = \{x\}$.
- (73) If R is P-convergent, then $\text{P-lim } R \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$. The theorem is a consequence of (71).
- (74) R is P-convergent if and only if $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R \neq \emptyset$. The theorem is a consequence of (71) and (5).
- (75) Suppose R is P-convergent. Then $\{\text{P-lim } R\} = \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$. The theorem is a consequence of (73) and (72).
- (76) Suppose $\lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$ is not empty. Then
 - (i) R is P-convergent, and
 - (ii) $\{\text{P-lim } R\} = \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# R$.

8. EXAMPLE: DOUBLE SEQUENCE CONVERGES IN PRINGSHEIM’S SENSE BUT NOT IN FRECHET FILTER OF $\mathbb{N} \times \mathbb{N}$ SENSE

The functor DbSeq-ex1 yielding a function from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} is defined by (Def. 8) for every natural numbers m, n , $it(m, n) = \frac{1}{m+1}$.

Now we state the propositions:

- (77) Let us consider a non zero natural number m . Then there exists a natural number n such that for every natural numbers n_1, n_2 such that $n \leq n_1$ and $n \leq n_2$ holds $|(\text{DbSeq-ex1})(n_1, n_2) - 0| < \frac{1}{m}$.
- (78) $0 \in \lim_{\langle \text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}) \rangle} \# \text{DbSeq-ex1}$.

- (79) $\lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} \# \text{DblSeq-ex1} = \emptyset$. The theorem is a consequence of (66), (42), (43), (72), (78), and (65).
- (80) $\lim_{(\text{FrechetFilter}(\mathbb{N}), \text{FrechetFilter}(\mathbb{N}))} \# \text{DblSeq-ex1} \neq \lim_{\text{FrechetFilter}(\mathbb{N} \times \mathbb{N})} \# \text{DblSeq-ex1}$.

9. CORRESPONDENCE WITH SOME DEFINITIONS FROM [14]

Let X_1, X_2 be non empty sets, \mathcal{F}_1 be a filter of X_1 , Y be a Hausdorff, non empty topological space, and f be a function from $X_1 \times X_2$ into Y . Assume for every element x of X_2 , $\lim_{\mathcal{F}_1} \text{curry}'(f, x) \neq \emptyset$. The functor $\lim_1(f, \mathcal{F}_1)$ yielding a function from X_2 into Y is defined by

(Def. 9) for every element x of X_2 , $\{it(x)\} = \lim_{\mathcal{F}_1} \text{curry}'(f, x)$.

Let \mathcal{F}_2 be a filter of X_2 . Assume for every element x of X_1 , $\lim_{\mathcal{F}_2} \text{curry}(f, x) \neq \emptyset$. The functor $\lim_2(f, \mathcal{F}_2)$ yielding a function from X_1 into Y is defined by

(Def. 10) for every element x of X_1 , $\{it(x)\} = \lim_{\mathcal{F}_2} \text{curry}(f, x)$.

Now we state the propositions:

- (81) Every function from X into \mathbb{R} is a function from X into \mathbb{R}^1 .
- (82) Every sequence of \mathbb{R} is a function from \mathbb{N} into \mathbb{R}^1 .

From now on f denotes a function from $\Omega_{\text{the ordered } \mathbb{N}}$ into \mathbb{R}^1 and s_1 denotes a function from \mathbb{N} into \mathbb{R} .

Now we state the propositions:

- (83) Suppose $f = s_1$ and $\text{LimF}(f) \neq \emptyset$. Then

- (i) s_1 is convergent, and
- (ii) there exists a real number z such that $z \in \text{LimF}(f)$ and for every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - z| < p$.

PROOF: Consider x being an object such that $x \in \text{LimF}(f)$. Reconsider $y = x$ as a point of (the metric space of real numbers)_{top}. Reconsider $z = y$ as a real number. Consider y_1 being a point of the metric space of real numbers such that $y_1 = y$ and $\text{Balls } y = \{\text{Ball}(y_1, \frac{1}{n})\}$, where n is a natural number : $n \neq 0$. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leq m$ holds $|s_1(m) - z| < p$ by (5), [12, (84), (50)], [2, (18)]. \square

- (84) If $f = s_1$ and $\text{LimF}(f) \neq \emptyset$, then $\text{LimF}(f) = \{\lim s_1\}$.

PROOF: Consider x being an object such that $x \in \text{LimF}(f)$. Consider u being an object such that $\text{LimF}(f) = \{u\}$. $\text{LimF}(f) = \{\lim s_1\}$ by (83), [11, (3)]. \square

- (85) Let us consider a function f from Ω_α into T , and a sequence s of T . If $f = s$, then $\text{LimF}(f) = \text{LimF}(s)$, where α is the ordered \mathbb{N} .
- (86) Let us consider a function f from Ω_α into T , and a function g from \mathbb{N} into T . If $f = g$, then $\text{LimF}(f) = \text{LimF}(g)$, where α is the ordered \mathbb{N} .
- (87) Let us consider a function f from \mathbb{N} into \mathbb{R}^1 . Suppose $f = s_1$ and $\text{LimF}(f) \neq \emptyset$. Then $\text{LimF}(f) = \{\lim s_1\}$. The theorem is a consequence of (84).
- (88) for every element x of \mathbb{N} , $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}'(\# R, x) \neq \emptyset$ if and only if R is convergent in the first coordinate. The theorem is a consequence of (5).
- (89) for every element x of \mathbb{N} , $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}(\# R, x) \neq \emptyset$ if and only if R is convergent in the second coordinate. The theorem is a consequence of (5).

Let us consider an element t of \mathbb{N} , a function f from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R}^1 , and a function s_1 from $\mathbb{N} \times \mathbb{N}$ into \mathbb{R} . Now we state the propositions:

- (90) Suppose $f = s_1$ and for every element x of \mathbb{N} , $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}(f, x) \neq \emptyset$. Then $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}(f, t) = \{\lim \text{curry}(s_1, t)\}$. The theorem is a consequence of (87).
- (91) Suppose $f = s_1$ and for every element x of \mathbb{N} , $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}'(f, x) \neq \emptyset$. Then $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}'(f, t) = \{\lim \text{curry}'(s_1, t)\}$. The theorem is a consequence of (87).
- (92) Let us consider a Hausdorff, non empty topological space Y , and a function f from $\mathbb{N} \times \mathbb{N}$ into Y . Suppose for every element x of \mathbb{N} , $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}'(f, x) \neq \emptyset$ and $f = R$ and $Y = \mathbb{R}^1$. Then $\lim_1(f, \text{FrechetFilter}(\mathbb{N})) =$ the lim in the first coordinate of R . The theorem is a consequence of (91).
- (93) Let us consider a non empty, Hausdorff topological space Y , and a function f from $\mathbb{N} \times \mathbb{N}$ into Y . Suppose for every element x of \mathbb{N} , $\lim_{\text{FrechetFilter}(\mathbb{N})} \text{curry}(f, x) \neq \emptyset$ and $f = R$ and $Y = \mathbb{R}^1$. Then $\lim_2(f, \text{FrechetFilter}(\mathbb{N})) =$ the lim in the second coordinate of R . The theorem is a consequence of (90).

10. REGULAR SPACE, DOUBLE LIMIT AND ITERATED LIMIT

From now on Y denotes a non empty topological space, x denotes a point of Y , and f denotes a function from $X_1 \times X_2$ into Y .

Now we state the proposition:

- (94) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1] = \mathcal{F}_1$ and $[\mathcal{B}_2] = \mathcal{F}_2$. Let us consider a subset V of Y . Suppose V is open and $x \in V$. Then there exists an ele-

ment B_1 of \mathcal{B}_1 and there exists an element B_2 of \mathcal{B}_2 such that $f^\circ(B_1 \times B_2) \subseteq V$.

Let us consider a neighbourhood U of x . Now we state the propositions:

- (95) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1] = \mathcal{F}_1$ and $[\mathcal{B}_2] = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of \mathcal{B}_1 and there exists an element B_2 of \mathcal{B}_2 such that $f^\circ(B_1 \times B_2) \subseteq \text{Int } U$.
- (96) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1] = \mathcal{F}_1$ and $[\mathcal{B}_2] = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of \mathcal{B}_1 and there exists an element B_2 of \mathcal{B}_2 such that for every element y of B_1 , $f^\circ(\{y\} \times B_2) \subseteq \text{Int } U$. The theorem is a consequence of (95).
- (97) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1] = \mathcal{F}_1$ and $[\mathcal{B}_2] = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of \mathcal{B}_1 and there exists an element B_2 of \mathcal{B}_2 such that for every element z of X_1 for every element y of Y such that $z \in B_1$ and $y \in \lim_{\mathcal{F}_2} \text{curry}(f, z)$ holds $y \in \overline{\text{Int } U}$.

PROOF: Consider B_1 being an element of \mathcal{B}_1 , B_2 being an element of \mathcal{B}_2 such that $f^\circ(B_1 \times B_2) \subseteq \text{Int } U$. For every element y of B_1 , $f^\circ(\{y\} \times B_2) \subseteq \text{Int } U$ by [11, (95)], [19, (125)]. For every element z of B_1 and for every element y of Y such that $y \in \lim_{\mathcal{F}_2} \text{curry}(f, z)$ holds the image of filter \mathcal{F}_2 under $\text{curry}(f, z)$ is a proper filter of $2_{\subseteq}^{\Omega_Y}$ and $\text{Int } U \in$ the image of filter \mathcal{F}_2 under $\text{curry}(f, z)$ and y is a cluster point of the image of filter \mathcal{F}_2 under $\text{curry}(f, z)$, Y by (18), [19, (132)], [10, (95)], (20). For every element z of B_1 and for every element y of Y such that $y \in \lim_{\mathcal{F}_2} \text{curry}(f, z)$ holds $y \in \overline{\text{Int } U}$ by [4, (25)]. \square

- (98) Suppose $x \in \lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f$ and $[\mathcal{B}_1] = \mathcal{F}_1$ and $[\mathcal{B}_2] = \mathcal{F}_2$. Then suppose U is closed. Then there exists an element B_1 of \mathcal{B}_1 and there exists an element B_2 of \mathcal{B}_2 such that for every element z of X_2 for every element y of Y such that $z \in B_2$ and $y \in \lim_{\mathcal{F}_1} \text{curry}'(f, z)$ holds $y \in \overline{\text{Int } U}$.

PROOF: Consider B_1 being an element of \mathcal{B}_1 , B_2 being an element of \mathcal{B}_2 such that $f^\circ(B_1 \times B_2) \subseteq \text{Int } U$. For every element y of B_2 , $f^\circ(B_1 \times \{y\}) \subseteq \text{Int } U$ by [11, (95)], [19, (125)]. For every element z of B_2 and for every element y of Y such that $y \in \lim_{\mathcal{F}_1} \text{curry}'(f, z)$ holds the image of filter \mathcal{F}_1 under $\text{curry}'(f, z)$ is a proper filter of $2_{\subseteq}^{\Omega_Y}$ and $\text{Int } U \in$ the image of filter \mathcal{F}_1 under $\text{curry}'(f, z)$ and y is a cluster point of the image of filter \mathcal{F}_1 under $\text{curry}'(f, z)$, Y by (18), [19, (132)], [10, (95)], (20). For every element z of B_2 and for every element y of Y such that $y \in \lim_{\mathcal{F}_1} \text{curry}'(f, z)$ holds $y \in \overline{\text{Int } U}$ by [4, (25)]. \square

Let us consider a Hausdorff, regular, non empty topological space Y and a function f from $X_1 \times X_2$ into Y . Now we state the propositions:

- (99) Suppose for every element x of X_2 , $\lim_{\mathcal{F}_1} \text{curry}'(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle}$

$f \subseteq \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (19) and (98).

- (100) Suppose for every element x of X_1 , $\lim_{\mathcal{F}_2} \text{curry}(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \subseteq \lim_{\mathcal{F}_1} \lim_2(f, \mathcal{F}_2)$. The theorem is a consequence of (19) and (97).

Let us consider non empty sets X_1 , X_2 , a filter \mathcal{F}_1 of X_1 , a filter \mathcal{F}_2 of X_2 , a Hausdorff, regular, non empty topological space Y , and a function f from $X_1 \times X_2$ into Y . Now we state the propositions:

- (101) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_1 , $\lim_{\mathcal{F}_2} \text{curry}(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f = \lim_{\mathcal{F}_1} \lim_2(f, \mathcal{F}_2)$. The theorem is a consequence of (100).
- (102) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_2 , $\lim_{\mathcal{F}_1} \text{curry}'(f, x) \neq \emptyset$. Then $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f = \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (99).
- (103) Suppose $\lim_{\langle \mathcal{F}_1, \mathcal{F}_2 \rangle} f \neq \emptyset$ and for every element x of X_1 , $\lim_{\mathcal{F}_2} \text{curry}(f, x) \neq \emptyset$ and for every element x of X_2 , $\lim_{\mathcal{F}_1} \text{curry}'(f, x) \neq \emptyset$. Then $\lim_{\mathcal{F}_1} \lim_2(f, \mathcal{F}_2) = \lim_{\mathcal{F}_2} \lim_1(f, \mathcal{F}_1)$. The theorem is a consequence of (102) and (101).

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