

# Quasi-uniform Space

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**Summary.** In this article, using mostly Pervin [9], Kunzi [6], [8], [7], Williams [11] and Bourbaki [3] works, we formalize in Mizar [2] the notions of quasi-uniform space, semi-uniform space and locally uniform space.

We define the topology induced by a quasi-uniform space. Finally we formalize from the sets of the form  $((X \setminus \Omega) \times X) \cup (X \times \Omega)$ , the Csaszar-Pervin quasi-uniform space induced by a topological space.

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## 1. PRELIMINARIES

From now on  $X$  denotes a set,  $A$  denotes a subset of  $X$ , and  $R, S$  denote binary relations on  $X$ .

Now we state the propositions:

- (1)  $(X \setminus A) \times X \cup X \times A \subseteq X \times X$ .
- (2)  $(X \setminus A) \times X \cup X \times A = A \times A \cup (X \setminus A) \times X$ .

PROOF:  $(X \setminus A) \times X \cup X \times A \subseteq A \times A \cup (X \setminus A) \times X$  by (1), [4, (87)].  $\square$

- (3)  $R \cdot S = \{\langle x, y \rangle, \text{ where } x, y \text{ are elements of } X : \text{ there exists an element } z \text{ of } X \text{ such that } \langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S\}$ .

PROOF:  $R \cdot S \subseteq \{\langle x, y \rangle, \text{ where } x, y \text{ are elements of } X : \text{ there exists an element } z \text{ of } X \text{ such that } \langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S\}$  by [4, (87)].  $\{\langle x, y \rangle, \text{ where } x, y \text{ are elements of } X : \text{ there exists an element } z \text{ of } X \text{ such that } \langle x, z \rangle \in R \text{ and } \langle z, y \rangle \in S\} \subseteq R \cdot S$ .  $\square$

Let  $X$  be a set and  $\mathcal{B}$  be a family of subsets of  $X$ . One can check that  $[\mathcal{B}]$  is non empty.

Let  $\mathcal{B}$  be a family of subsets of  $X \times X$ . Note that every element of  $\mathcal{B}$  is relation-like.

Let  $B$  be an element of  $\mathcal{B}$ . We introduce the notation  $B[\sim]$  as a synonym of  $B^\sim$ .

Let us observe that the functor  $B[\sim]$  yields a subset of  $X \times X$ . Let  $B_1, B_2$  be elements of  $\mathcal{B}$ . We introduce the notation  $B_1 \otimes B_2$  as a synonym of  $B_1 \cdot B_2$ .

One can verify that the functor  $B_1 \otimes B_2$  yields a subset of  $X \times X$ . Now we state the propositions:

- (4) Let us consider a set  $X$ , and a family  $G$  of subsets of  $X$ . If  $G$  is upper, then  $\text{FinMeetCl}(G)$  is upper.
- (5) If  $R$  is symmetric in  $X$ , then  $R^\sim$  is symmetric in  $X$ .

## 2. UNIFORM SPACE STRUCTURE

We consider uniform space structures which extend 1-sorted structures and are systems

$$\langle \text{a carrier, entourages} \rangle$$

where the carrier is a set, the entourages constitute a family of subsets of  $(\text{the carrier}) \times (\text{the carrier})$ .

Let  $U$  be a uniform space structure. We say that  $U$  is void if and only if

(Def. 1) the entourages of  $U$  is empty.

Let  $X$  be a set. The functor  $\text{UniformSpace}(X)$  yielding a strict uniform space structure is defined by the term

(Def. 2)  $\langle X, \emptyset_{2^{X \times X}} \rangle$ .

The functors: the trivial uniform space and the non empty trivial uniform space yielding strict uniform space structures are defined by terms

(Def. 3)  $\langle \emptyset, 2_*^{\emptyset \times \emptyset} \rangle$ ,

(Def. 4) there exists a family  $S_1$  of subsets of  $\{\emptyset\} \times \{\emptyset\}$  such that  $S_1 = \{\{\emptyset\} \times \{\emptyset\}\}$  and the non empty trivial uniform space =  $\langle \{\emptyset\}, S_1 \rangle$ ,

respectively. Let  $X$  be an empty set. One can verify that  $\text{UniformSpace}(X)$  is empty.

Let  $X$  be a non empty set. One can check that  $\text{UniformSpace}(X)$  is non empty.

Let  $X$  be a set. Note that  $\text{UniformSpace}(X)$  is void and the trivial uniform space is empty and non void and the non empty trivial uniform space is non empty and non void and there exists a uniform space structure which is empty,

strict, and void and there exists a uniform space structure which is empty, strict, and non void and there exists a uniform space structure which is non empty, strict, and void and there exists a uniform space structure which is non empty, strict, and non void.

Let  $X$  be a set and  $S_1$  be a family of subsets of  $X \times X$ . The functor  $S_1[\sim]$  yielding a family of subsets of  $X \times X$  is defined by the term

(Def. 5) the set of all  $S[\sim]$  where  $S$  is an element of  $S_1$ .

Let  $U$  be a uniform space structure. The functor  $U[\sim]$  yielding a uniform space structure is defined by the term

(Def. 6)  $\langle$ the carrier of  $U$ , (the entourages of  $U$ )  $[\sim]$  $\rangle$ .

Let  $U$  be a non empty uniform space structure. One can verify that  $U[\sim]$  is non empty.

### 3. AXIOMS

Let  $U$  be a uniform space structure. We say that  $U$  is upper if and only if

(Def. 7) the entourages of  $U$  is upper.

We say that  $U$  is  $\cap$ -closed if and only if

(Def. 8) the entourages of  $U$  is  $\cap$ -closed.

We say that  $U$  satisfies axiom U1 if and only if

(Def. 9) for every element  $S$  of the entourages of  $U$ ,  $\text{id}_\alpha \subseteq S$ , where  $\alpha$  is the carrier of  $U$ .

We say that  $U$  satisfies axiom U2 if and only if

(Def. 10) for every element  $S$  of the entourages of  $U$ ,  $S[\sim] \in$  the entourages of  $U$ .

We say that  $U$  satisfies axiom U3 if and only if

(Def. 11) for every element  $S$  of the entourages of  $U$ , there exists an element  $W$  of the entourages of  $U$  such that  $W \otimes W \subseteq S$ .

Let us consider a non void uniform space structure  $U$ . Now we state the propositions:

(6)  $U$  satisfies axiom U1 if and only if for every element  $S$  of the entourages of  $U$ , there exists a binary relation  $R$  on the carrier of  $U$  such that  $R = S$  and  $R$  is reflexive in the carrier of  $U$ .

(7)  $U$  satisfies axiom U1 if and only if for every element  $S$  of the entourages of  $U$ , there exists a total, reflexive binary relation  $R$  on the carrier of  $U$  such that  $R = S$ . The theorem is a consequence of (6).

Note that every uniform space structure which is void does not satisfy also axiom U2.

Now we state the proposition:

- (8) Let us consider a uniform space structure  $U$ . Suppose  $U$  satisfies axiom U2. Let us consider an element  $S$  of the entourages of  $U$ , and elements  $x, y$  of  $U$ . Suppose  $\langle x, y \rangle \in S$ . Then  $\langle y, x \rangle \in \bigcup(\text{the entourages of } U)$ .

Let us consider a non void uniform space structure  $U$ . Now we state the propositions:

- (9) Suppose for every element  $S$  of the entourages of  $U$ , there exists a binary relation  $R$  on the carrier of  $U$  such that  $S = R$  and  $R$  is symmetric in the carrier of  $U$ . Then  $U$  satisfies axiom U2. The theorem is a consequence of (5).
- (10) Suppose for every element  $S$  of the entourages of  $U$ , there exists a binary relation  $R$  on the carrier of  $U$  such that  $S = R$  and  $R$  is symmetric. Then  $U$  satisfies axiom U2. The theorem is a consequence of (9).
- (11) If for every element  $S$  of the entourages of  $U$ , there exists a tolerance  $R$  of the carrier of  $U$  such that  $S = R$ , then  $U$  satisfies axiom U1 and axiom U2. The theorem is a consequence of (7) and (10).

Let  $X$  be an empty set. Observe that  $\text{UniformSpace}(X)$  is upper and  $\cap$ -closed and satisfies axiom U1 and axiom U3 and does not satisfy axiom U2 and  $\text{UniformSpace}(\{\emptyset\})$  is upper and  $\cap$ -closed and does not satisfy axiom U2 and the trivial uniform space is upper and  $\cap$ -closed and satisfies axiom U1, axiom U2, and axiom U3 and the non empty trivial uniform space is upper and  $\cap$ -closed and satisfies axiom U1, axiom U2, and axiom U3.

There exists a uniform space structure which is strict, empty, non void, upper, and  $\cap$ -closed and satisfies axiom U1, axiom U2, and axiom U3 and every strict uniform space structure which is empty satisfies also axiom U1 and there exists a uniform space structure which is strict, non empty, non void, upper, and  $\cap$ -closed and satisfies axiom U1, axiom U2, and axiom U3.

Let  $S_4$  be a non empty uniform space structure satisfying axiom U1,  $x$  be an element of  $S_4$ , and  $V$  be an element of the entourages of  $S_4$ . The functor  $\text{Nbh}(V, x)$  yielding a non empty subset of  $S_4$  is defined by the term

(Def. 12)  $\{y, \text{ where } y \text{ is an element of } S_4 : \langle x, y \rangle \in V\}$ .

Now we state the proposition:

- (12) Let us consider a non empty uniform space structure  $U$  satisfying axiom U1, an element  $x$  of the carrier of  $U$ , and an element  $V$  of the entourages of  $U$ . Then  $x \in \text{Nbh}(V, x)$ .

Let  $U$  be a  $\cap$ -closed uniform space structure and  $V_1, V_2$  be elements of the entourages of  $U$ . One can check that the functor  $V_1 \cap V_2$  yields an element of the entourages of  $U$ . Now we state the proposition:

- (13) Let us consider a non empty,  $\cap$ -closed uniform space structure  $U$  satisfying axiom U1, an element  $x$  of  $U$ , and elements  $V, W$  of the entourages

of  $U$ . Then  $\text{Nbh}(V, x) \cap \text{Nbh}(W, x) = \text{Nbh}(V \cap W, x)$ .

Let  $U$  be a non empty uniform space structure satisfying axiom U1. Let us observe that the entourages of  $U$  has non empty elements and the entourages of  $U$  is non empty.

Let  $x$  be a point of  $U$ . The functor Neighborhood  $x$  yielding a family of subsets of  $U$  is defined by the term

(Def. 13) the set of all  $\text{Nbh}(V, x)$  where  $V$  is an element of the entourages of  $U$ .

Let us note that Neighborhood  $x$  is non empty.

Now we state the proposition:

(14) Let us consider a non empty uniform space structure  $S_4$  satisfying axiom U1, an element  $x$  of the carrier of  $S_4$ , and an element  $V$  of the entourages of  $S_4$ . Then

- (i)  $\text{Nbh}(V, x) = V^\circ\{x\}$ , and
- (ii)  $\text{Nbh}(V, x) = \text{rng}(V \upharpoonright \{x\})$ , and
- (iii)  $\text{Nbh}(V, x) = V^\circ x$ , and
- (iv)  $\text{Nbh}(V, x) = [x]_V$ , and
- (v)  $\text{Nbh}(V, x) = \text{neighbourhood}(x, V)$ .

PROOF:  $\text{Nbh}(V, x) = V^\circ\{x\}$  by [4, (87)].  $\square$

Let  $U$  be a non empty uniform space structure satisfying axiom U1. The functor Neighborhood  $U$  yielding a function from the carrier of  $U$  into  $2^{2^{\text{(the carrier of } U\text{)}}}$  is defined by

(Def. 14) for every element  $x$  of  $U$ ,  $it(x) = \text{Neighborhood } x$ .

We say that  $U$  is topological if and only if

(Def. 15)  $\langle \text{the carrier of } U, \text{Neighborhood } U \rangle$  is a topology from neighbourhoods.

#### 4. QUASI-UNIFORM SPACE

A quasi-uniform space is an upper,  $\cap$ -closed uniform space structure satisfying axiom U1 and axiom U3. In the sequel  $Q$  denotes a quasi-uniform space.

Now we state the propositions:

- (15) If the entourages of  $Q$  is empty, then the entourages of  $Q[\sim] = \{\emptyset\}$ .
- (16) Suppose the entourages of  $Q[\sim] = \{\emptyset\}$  and the entourages of  $Q[\sim]$  is upper. Then the carrier of  $Q$  is empty.

Let  $Q$  be a non void quasi-uniform space. One can check that  $Q[\sim]$  is upper and  $\cap$ -closed and satisfies axiom U1 and axiom U3.

Let  $X$  be a set and  $\mathcal{B}$  be a family of subsets of  $X \times X$ . We say that  $\mathcal{B}$  satisfies axiom UP1 if and only if

(Def. 16) for every element  $B$  of  $\mathcal{B}$ ,  $\text{id}_X \subseteq B$ .

We say that  $\mathcal{B}$  satisfies axiom UP3 if and only if

(Def. 17) for every element  $B_1$  of  $\mathcal{B}$ , there exists an element  $B_2$  of  $\mathcal{B}$  such that  $B_2 \otimes B_2 \subseteq B_1$ .

Now we state the propositions:

(17) Let us consider a non empty set  $X$ , and an empty family  $\mathcal{B}$  of subsets of  $X \times X$ . Then  $\mathcal{B}$  does not satisfy axiom UP1.

(18) Let us consider a set  $X$ , and a family  $\mathcal{B}$  of subsets of  $X \times X$ . Suppose  $\mathcal{B}$  is quasi-basis and satisfies axiom UP1 and axiom UP3. Then  $\langle X, [\mathcal{B}] \rangle$  is a quasi-uniform space.

## 5. SEMI-UNIFORM SPACE

A semi-uniform space is an upper,  $\cap$ -closed uniform space structure satisfying axiom U1 and axiom U2. From now on  $S_4$  denotes a semi-uniform space.

Let us observe that every semi-uniform space is non void.

Now we state the proposition:

(19) If  $S_4$  is empty, then  $\emptyset \in$  the entourages of  $S_4$ .

Let  $S_4$  be an empty semi-uniform space. One can verify that the entourages of  $S_4$  has the empty element.

## 6. LOCALLY UNIFORM SPACE

Let  $S_4$  be a non empty semi-uniform space. We say that  $S_4$  satisfies axiom UL if and only if

(Def. 18) for every element  $S$  of the entourages of  $S_4$  and for every element  $x$  of  $S_4$ , there exists an element  $W$  of the entourages of  $S_4$  such that  $\{y, \text{ where } y \text{ is an element of } S_4 : \langle x, y \rangle \in W \otimes W\} \subseteq \text{Nbh}(S, x)$ .

One can verify that every non empty semi-uniform space which satisfies axiom U3 satisfies also axiom UL and there exists a non empty semi-uniform space which satisfies axiom UL.

A locally uniform space is a non empty semi-uniform space satisfying axiom UL. Now we state the propositions:

(20) Let us consider a non empty, upper uniform space structure  $U$  satisfying axiom U1, and an element  $x$  of the carrier of  $U$ . Then Neighborhood  $x$  is upper.

- (21) Let us consider a non empty uniform space structure  $U$  satisfying axiom U1, an element  $x$  of  $U$ , and an element  $V$  of the entourages of  $U$ . Then  $x \in \text{Nbh}(V, x)$ .
- (22) Let us consider a non empty,  $\cap$ -closed uniform space structure  $U$  satisfying axiom U1, and an element  $x$  of  $U$ . Then Neighborhood  $x$  is  $\cap$ -closed. The theorem is a consequence of (13).
- (23) Let us consider a non empty, upper,  $\cap$ -closed uniform space structure  $U$  satisfying axiom U1, and an element  $x$  of  $U$ . Then Neighborhood  $x$  is a filter of the carrier of  $U$ . The theorem is a consequence of (22) and (20).

Let us observe that every locally uniform space is topological.

### 7. TOPOLOGICAL SPACE INDUCED BY A UNIFORM SPACE STRUCTURE

Let  $U$  be a topological, non empty uniform space structure satisfying axiom U1. The FMT induced by  $U$  yielding a non empty, strict topology from neighbourhoods is defined by the term

(Def. 19)  $\langle \text{the carrier of } U, \text{Neighborhood } U \rangle$ .

The topological space induced by  $U$  yielding a topological space is defined by the term

(Def. 20)  $\text{FMT2TopSpace}(\text{the FMT induced by } U)$ .

### 8. THE QUASI-UNIFORM PERVIN SPACE INDUCED BY A TOPOLOGICAL SPACE

Let  $X$  be a set and  $A$  be a subset of  $X$ . The functor  $\text{qBlock}(A)$  yielding a subset of  $X \times X$  is defined by the term

(Def. 21)  $(X \setminus A) \times X \cup X \times A$ .

Now we state the proposition:

- (24) (i)  $\text{id}_X \subseteq \text{qBlock}(A)$ , and
- (ii)  $\text{qBlock}(A) \cdot \text{qBlock}(A) \subseteq \text{qBlock}(A)$ .

PROOF:  $\text{id}_X \subseteq \text{qBlock}(A)$  by [4, (96)].  $\square$

Let  $T$  be a topological space. The functor  $\text{qBlocks}(T)$  yielding a family of subsets of  $(\text{the carrier of } T) \times (\text{the carrier of } T)$  is defined by the term

(Def. 22) the set of all  $\text{qBlock}(O)$  where  $O$  is an element of the topology of  $T$ .

Let  $T$  be a non empty topological space. One can check that  $\text{qBlocks}(T)$  is non empty.

Let  $T$  be a topological space. The functor  $\text{FMCqBlocks}(T)$  yielding a family of subsets of  $(\text{the carrier of } T) \times (\text{the carrier of } T)$  is defined by the term

(Def. 23)  $\text{FinMeetCl}(\text{qBlocks}(T))$ .

Let  $X$  be a set. One can check that every non empty family of subsets of  $X \times X$  which is  $\cap$ -closed is also quasi-basis.

In the sequel  $T$  denotes a topological space.

Let us consider  $T$ . One can check that  $\text{FMCqBlocks}(T)$  is  $\cap$ -closed and  $\text{FMCqBlocks}(T)$  is quasi-basis and  $\text{FMCqBlocks}(T)$  satisfies axiom UP1 and  $\text{FMCqBlocks}(T)$  satisfies axiom UP3.

Let  $T$  be a topological space. The Pervin quasi-uniformity of  $T$  yielding a strict quasi-uniform space is defined by the term

(Def. 24)  $\langle \text{the carrier of } T, [\text{FMCqBlocks}(T)] \rangle$ .

Now we state the propositions:

(25) Let us consider a non empty topological space  $T$ , and an element  $V$  of the entourages of the Pervin quasi-uniformity of  $T$ . Then there exists an element  $b$  of  $\text{FinMeetCl}(\text{qBlocks}(T))$  such that  $b \subseteq V$ .

(26) Let us consider a non empty topological space  $T$ , and a subset  $V$  of  $(\text{the carrier of } T) \times (\text{the carrier of } T)$ . Suppose there exists an element  $b$  of  $\text{FinMeetCl}(\text{qBlocks}(T))$  such that  $b \subseteq V$ . Then  $V$  is an element of the entourages of the Pervin quasi-uniformity of  $T$ .

(27)  $\text{qBlocks}(T) \subseteq$  the entourages of the Pervin quasi-uniformity of  $T$ .

Let us consider a non void quasi-uniform space  $Q$ . Now we state the propositions:

(28)  $(\text{The carrier of } Q) \times (\text{the carrier of } Q) \in$  the entourages of  $Q$ .

(29) Suppose the carrier of  $T =$  the carrier of  $Q$  and  $\text{qBlocks}(T) \subseteq$  the entourages of  $Q$ . Then the entourages of the Pervin quasi-uniformity of  $T \subseteq$  the entourages of  $Q$ .

PROOF: The entourages of the Pervin quasi-uniformity of  $T \subseteq$  the entourages of  $Q$  by (28), [1, (1)].  $\square$

Let  $T$  be a non empty topological space. One can check that the Pervin quasi-uniformity of  $T$  is non empty and the Pervin quasi-uniformity of  $T$  is topological.

Now we state the propositions:

(30) Let us consider a non empty topological space  $T$ , an element  $x$  of  $\text{qBlocks}(T)$ , and an element  $y$  of the Pervin quasi-uniformity of  $T$ . Then  $\{z, \text{ where } z \text{ is an element of the Pervin quasi-uniformity of } T : \langle y, z \rangle \in x\} \in$  the topology of  $T$ .

(31) Let us consider a non empty topological space  $T$ , an element  $x$  of the carrier of the Pervin quasi-uniformity of  $T$ , and an element  $b$  of  $\text{FinMeetCl}(\text{qBlocks}(T))$ . Then  $\{y, \text{ where } y \text{ is an element of } T : \langle x, y \rangle \in b\} \in$  the to-



polology of  $T$ . The theorem is a consequence of (30).

- (32) Let us consider a non empty, strict quasi-uniform space  $U$ , a non empty, strict formal topological space  $F$ , and an element  $x$  of  $F$ . Suppose  $F = \langle \text{the carrier of } U, \text{Neighborhood } U \rangle$ . Then there exists an element  $y$  of  $U$  such that
- (i)  $x = y$ , and
  - (ii)  $U_F(x) = \text{Neighborhood } y$ .
- (33) Let us consider a non empty topological space  $T$ . Then the open set family of the FMT induced by the Pervin quasi-uniformity of  $T =$  the topology of  $T$ .
- PROOF: The open set family of the FMT induced by the Pervin quasi-uniformity of  $T \subseteq$  the topology of  $T$  by (32), [5, (18)], (31), [12, (25)]. The topology of  $T \subseteq$  the open set family of the FMT induced by the Pervin quasi-uniformity of  $T$  by (32), [10, (4)], [5, (18)], [4, (87)].  $\square$
- (34) Let us consider a non empty, strict topological space  $T$ . Then the topological space induced by the Pervin quasi-uniformity of  $T = T$ . The theorem is a consequence of (33).

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