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Uniform Space

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Summary. In this article, we formalize in Mizar [1] the notion of uniform space introduced by André Weil using the concepts of entourages [2].

We present some results between uniform space and pseudo metric space. We introduce the concepts of left-uniformity and right-uniformity of a topological group.

Next, we define the concept of the partition topology. Following the Vlach's works [11, 10], we define the semi-uniform space induced by a tolerance and the uniform space induced by an equivalence relation.

Finally, using mostly Gehrke, Grigorieff and Pin [4] works, a Pervin uniform space defined from the sets of the form $((X \setminus A) \times (X \setminus A)) \cup (A \times A)$ is presented.

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1. Preliminaries

From now on X denotes a set, D denotes a partition of X, T denotes a non empty topological group, and A denotes a subset of X.

Now we state the propositions:

(1) $A \times A \cup (X \setminus A) \times (X \setminus A) \subseteq (X \setminus A) \times X \cup X \times A$.

(2) $\{1, 2, 3\} \setminus \{1\} = \{2, 3\}$.

(3) Suppose $X = \{1, 2, 3\}$ and $A = \{1\}$. Then

(i) $\langle 2, 1\rangle \in (X \setminus A) \times X \cup X \times A$, and

(ii) $\langle 2, 1\rangle \notin A \times A \cup (X \setminus A) \times (X \setminus A)$.

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The theorem is a consequence of (2).

(4) Let us consider a subset A of X. Then $(A \times A \cup (X \setminus A) \times (X \setminus A))^\smile = A \times A \cup (X \setminus A) \times (X \setminus A)$.

(5) Let us consider subsets P_1, P_2 of D. If $\bigcup P_1 = \bigcup P_2$, then $P_1 = P_2$.

(6) Let us consider a subset P of D. Then $\bigcup (D \setminus P) = \bigcup D \setminus \bigcup P$.

(7) Let us consider an upper family S_1 of subsets of X, and an element S of S_1. Then $\bigcap S_1 \subseteq S$.

(8) Let us consider an additive group G, and subsets A, B, C, D of G. If $A \subseteq B$ and $C \subseteq D$, then $A + C \subseteq B + D$.

Let us consider an element e of T and a neighbourhood V of $\mathbf{1}_T$. Now we state the propositions:

(9) $\{e\} \cdot V$ is a neighbourhood of e.

(10) $V \cdot \{e\}$ is a neighbourhood of e.

(11) Let us consider a neighbourhood V of $\mathbf{1}_T$. Then V^{-1} is a neighbourhood of $\mathbf{1}_T$.

2. Uniform Space

A uniform space is an upper, $\cap$-closed uniform space structure satisfying axiom U1, axiom U2, and axiom U3. From now on Q denotes a uniform space.

Now we state the propositions:

(12) Q is a quasi-uniform space.

(13) Q is a semi-uniform space.

Let X be a set and $\mathcal{B}$ be a family of subsets of $X \times X$. We say that $\mathcal{B}$ satisfies axiom UP2 if and only if

(Def. 1) for every element B_1 of $\mathcal{B}$, there exists an element B_2 of $\mathcal{B}$ such that $B_2 \subseteq {B_1}^\smile$.

Now we state the proposition:

(14) Let us consider an empty set X. Then every empty family of subsets of $X \times X$ is quasi-basis and satisfies axiom UP1, axiom UP2, and axiom UP3.

One can verify that there exists a uniform space which is strict.

Now we state the proposition:

(15) Let us consider a set X, and a family S_1 of subsets of $X \times X$. Suppose $X = \{\emptyset\}$ and $S_1 = \{X \times X\}$. Then $\langle X, S_1 \rangle$ is a uniform space.

Let us observe that there exists a strict uniform space which is non empty.

Now we state the proposition:

(16) Let us consider a set X, and a family $\mathcal{B}$ of subsets of $X \times X$. Suppose $\mathcal{B}$ is quasi-basis and satisfies axiom UP1, axiom UP2, and axiom UP3. Then there exists a strict uniform space Q such that

(i) the carrier of $Q = X$, and

(ii) the entourages $Q = [\mathcal{B}]$.

3. Open Set and Uniform Space

Now we state the propositions:

(17) Let us consider a non empty uniform space Q. Then

(i) the carrier of the topological space induced by $Q =$ the carrier of Q, and

(ii) the topology of the topological space induced by $Q =$ the open set family of the FMTinduced by Q.

(18) Let us consider a non empty uniform space Q, and a subset S of the FMT-induced by Q. Then S is open if and only if for every element x of Q such that $x \in S$ holds $S \in \text{Neighborhood}\,x$.

(19) Let us consider a non empty uniform space Q. Then the open set family of the FMTinduced by $Q =$ the set of all O where O is an open subset of the FMTinduced by Q.

Let us consider a non empty uniform space Q and a subset S of the FMTinduced by Q. Now we state the propositions:

(20) S is open if and only if $S \in$ the open set family of the FMTinduced by Q.

(21) $S \in$ the open set family of the FMTinduced by Q if and only if for every element x of Q such that $x \in S$ holds $S \in \text{Neighborhood}\,x$.

4. Pseudo Metric Space and Uniform Space

Let M be a non empty metric structure and r be a positive real number. The functor $\text{ent}(M, r)$ yielding a subset of (the carrier of M) $\times$ (the carrier of M) is defined by the term

(Def. 2) $\{\langle x, y\rangle$, where x, y are elements of $M : \rho(x, y) \leqslant r\}$.

Let M be a non empty, reflexive metric structure. Let us observe that $\text{ent}(M, r)$ is non empty.

Let M be a non empty metric structure. The functor $\text{ENT}(M)$ yielding a non empty family of subsets of (the carrier of M) $\times$ (the carrier of M) is defined by the term

(Def. 3) the set of all $\operatorname{ent}(M, r)$ where r is a positive real number.

The uniformity induced by M yielding a uniform space structure is defined by the term

(Def. 4) $\langle$the carrier of $M, [\operatorname{ENT}(M)]\rangle$.

Let M be a pseudo metric space. The uniformity induced by M yielding a non empty, strict uniform space is defined by the term

(Def. 5) $\langle$the carrier of $M, [\operatorname{ENT}(M)]\rangle$.

Let us consider a pseudo metric space M. Now we state the propositions:

(22) The open set family of the FMTinduced by the uniformity induced by $M =$ the open set family of M.
PROOF: Set $X =$ the open set family of the FMTinduced by the uniformity induced by M. Set $Y =$ the open set family of M. $X \subseteq Y$ by (18), (20), [5, (11)]. Reconsider $t_1 = t$ as a subset of M. For every element x of the uniformity induced by M such that $x \in t_1$ holds $t_1 \in \operatorname{Neighborhood} x$ by [5, (11)]. $\square$

(23) The topological space induced by the uniformity induced by $M = M_{\mathrm{top}}$.
The theorem is a consequence of (22).

5. Uniform Space and Topological Group

Let G be a topological group and Q be a neighbourhood of $\mathbf{1}_G$. The functor $\operatorname{leftU}(Q)$ yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 6) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : x^{-1} \cdot y \in Q\}$.

Let T be a non empty topological group. The functor $\operatorname{SleftU}(T)$ yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 7) the set of all $\operatorname{leftU}(Q)$ where Q is a neighbourhood of $\mathbf{1}_T$.

The left-uniformity T yielding a non empty uniform space is defined by the term

(Def. 8) $\langle$the carrier of $T, [\operatorname{SleftU}(T)]\rangle$.

Let G be a topological group and Q be a neighbourhood of $\mathbf{1}_G$. The functor $\operatorname{rightU}(Q)$ yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 9) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : y \cdot x^{-1} \in Q\}$.

Let T be a non empty topological group. The functor $\operatorname{SrightU}(T)$ yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 10) the set of all $\mathrm{rightU}(Q)$ where Q is a neighbourhood of $\mathbf{1}_T$.

The right-uniformity T yielding a non empty uniform space is defined by the term

(Def. 11) $\langle$the carrier of $T, [\mathrm{SrightU}(T)]\rangle$.

Now we state the propositions:

(24) Let us consider a non empty, commutative topological group T, and a neighbourhood Q of $\mathbf{1}_T$. Then $\mathrm{leftU}(Q) = \mathrm{rightU}(Q)$.

(25) Let us consider a non empty, commutative topological group T. Then the left-uniformity T = the right-uniformity T. The theorem is a consequence of (24).

Let G be a semi additive topological group and Q be a neighbourhood of 0_G. The functor $\mathrm{leftU}(Q)$ yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 12) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : -x + y \in Q\}$.

Let T be a non empty semi additive topological group. The functor $\mathrm{SleftU}(T)$ yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 13) the set of all $\mathrm{leftU}(Q)$ where Q is a neighbourhood of 0_T.

Let T be a non empty topological additive group. The left-uniformity T yielding a non empty uniform space is defined by the term

(Def. 14) $\langle$the carrier of $T, [\mathrm{SleftU}(T)]\rangle$.

Let G be a semi additive topological group and Q be a neighbourhood of 0_G. The functor $\mathrm{rightU}(Q)$ yielding a subset of (the carrier of G) $\times$ (the carrier of G) is defined by the term

(Def. 15) $\{\langle x, y\rangle$, where x is an element of G, y is an element of $G : y + -x \in Q\}$.

Let T be a non empty semi additive topological group. The functor $\mathrm{SrightU}(T)$ yielding a non empty family of subsets of (the carrier of T) $\times$ (the carrier of T) is defined by the term

(Def. 16) the set of all $\mathrm{rightU}(Q)$ where Q is a neighbourhood of 0_T.

Let T be a non empty topological additive group. The right-uniformity T yielding a non empty uniform space is defined by the term

(Def. 17) $\langle$the carrier of $T, [\mathrm{SrightU}(T)]\rangle$.

Now we state the propositions:

(26) Let us consider an Abelian semi additive topological group T, and a neighbourhood Q of 0_T. Then $\mathrm{leftU}(Q) = \mathrm{rightU}(Q)$.

(27) Let us consider a non empty topological additive group T. Suppose T

is Abelian. Then the left-uniformity $T =$ the right-uniformity T. The theorem is a consequence of (26).

(28) The topology of the topological space induced by the left-uniformity $T =$ the topology of T.

PROOF: Set $X =$ the topology of FMT2TopSpace(the FMTinduced by the left-uniformity T). Set $Y =$ the topology of T. $X \subseteq Y$ by (9), [6, (7)]. $Y \subseteq X$ by [9, (3)], [6, (6)], [8, (6)]. □

(29) The topology of the topological space induced by the right-uniformity $T =$ the topology of T.

PROOF: Set $X =$ the topology of FMT2TopSpace(the FMTinduced by the right-uniformity T). Set $Y =$ the topology of T. $X \subseteq Y$ by (10), [6, (7)]. $Y \subseteq X$ by [9, (3)], [6, (6)], [8, (6)]. □

6. Function Uniformly Continuous

Let Q_1, Q_2 be uniform space structures and f be a function from Q_1 into Q_2. We say that f is uniformly continuous if and only if

(Def. 18) for every element V of the entourages Q_2, there exists an element Q of the entourages Q_1 such that for every objects x, y such that $\langle x, y\rangle \in Q$ holds $\langle f(x), f(y)\rangle \in V$.

Let Q_1, Q_2 be non empty uniform space structures satisfying axiom U1. One can check that there exists a function from Q_1 into Q_2 which is uniformly continuous.

7. Partition Topology

Now we state the propositions:

(30) the set of all $\bigcup P$ where P is a subset of $D =$ UniCl(D).

(31) $X \in$ UniCl(D). The theorem is a consequence of (30).

(32) If $D = \emptyset$, then X is empty and UniCl$(D) = \{\emptyset\}$.

Let X be a set and D be a partition of X. Let us note that UniCl(D) is $\cap$-closed and UniCl(D) is union-closed and every family of subsets of X which is union-closed is also $\cup$-closed.

Let D be a partition of X. Let us note that UniCl(D) is closed for complement operator and UniCl(D) is $\cup$-closed and $\setminus$-closed.

Now we state the proposition:

(33) UniCl(D) is a ring of sets. The theorem is a consequence of (30).

Let us consider X and D. One can verify that $\mathrm{UniCl}(D)$ has the empty element.

Let X be a set and D be a partition of X. Let us observe that $\mathrm{UniCl}(D)$ is non empty.

Now we state the proposition:

(34) $\mathrm{UniCl}(D)$ is a field of subsets of X.

Let X be a set and D be a partition of X. Observe that $\mathrm{UniCl}(D)$ is σ-additive and $\mathrm{UniCl}(D)$ is σ-multiplicative.

Now we state the proposition:

(35) $\mathrm{UniCl}(D)$ is a σ-field of subsets of X.

Let X be a set and D be a partition of X. Observe that $\mathrm{UniCl}(D)$ is closed for countable unions and closed for countable meets.

Now we state the proposition:

(36) Let us consider a non empty set Ω, and a partition D of Ω. Then $\mathrm{UniCl}(D)$ is a Dynkin system of Ω.

Let X be a set and D be a partition of X. The partition topology D yielding a topological space is defined by the term

(Def. 19) $\langle X, \mathrm{UniCl}(D)\rangle$.

Now we state the propositions:

(37) Every open subset of the partition topology D is closed.

(38) Every closed subset of the partition topology D is open.

(39) Let us consider a subset S of the partition topology D. Then S is open if and only if S is closed.

Let X be a non empty set and D be a partition of X. Observe that the partition topology D is non empty.

Let us consider a non empty set X and a partition D of X. Now we state the propositions:

(40) LC(the partition topology D) $= \mathrm{UniCl}(D)$. The theorem is a consequence of (38) and (31).

(41) OpenClosedSet(the partition topology D) $=$ the topology of the partition topology D. The theorem is a consequence of (37).

8. Uniform Space and Partition Topology

In the sequel R denotes a binary relation on X.

Let X be a set and R be a binary relation on X. The functor $\rho(R)$ yielding a non empty family of subsets of $X \times X$ is defined by the term

(Def. 20) $\{S$, where S is a subset of $X \times X : R \subseteq S\}$.

Now we state the propositions:

(42) $[\rho(R)] = \rho(R)$.

(43) $[\{R\}] = \rho(R)$.

(44) $\rho(R)$ is upper and $\cap$-closed.

Let us consider X and R. Observe that $\rho(R)$ is quasi-basis.

Now we state the propositions:

(45) Let us consider a total, reflexive binary relation R on X. Then $\rho(R)$ satisfies axiom UP1.

(46) Let us consider a symmetric binary relation R on X. Then $\rho(R)$ satisfies axiom UP2.

(47) Let us consider a total, transitive binary relation R on X. Then $\rho(R)$ satisfies axiom UP3.

Let X be a set and R be a binary relation on X. The uniformity induced by R yielding an upper, $\cap$-closed, strict uniform space structure is defined by the term

(Def. 21) $\langle X, \rho(R) \rangle$.

Now we state the propositions:

(48) Let us consider a set X, and a total, reflexive binary relation R on X. Then the uniformity induced by R satisfies axiom U1. The theorem is a consequence of (45).

(49) Let us consider a set X, and a symmetric binary relation R on X. Then the uniformity induced by R satisfies axiom U2. The theorem is a consequence of (46).

(50) Let us consider a set X, and a total, transitive binary relation R on X. Then the uniformity induced by R satisfies axiom U3. The theorem is a consequence of (47).

Let X be a set and R be a tolerance of X. Note that the uniformity induced by R yields a strict semi-uniform space. Now we state the proposition:

(51) Let us consider a set X, and an equivalence relation R of X. Then the uniformity induced by R is a uniform space.

Let X be a set and R be an equivalence relation of X. Observe that the uniformity induced by R yields a strict uniform space. Let X be a non empty set and R be a tolerance of X. Let us note that the uniformity induced by R is non empty and every non empty uniform space is topological.

Let Q be a non empty uniform space. The functor ${}^{@}Q$ yielding a topological, non empty uniform space structure satisfying axiom U1 is defined by the term

(Def. 22) Q.

Now we state the proposition:

(52) Let us consider a non empty set X, and an equivalence relation R of X. Then the topological space induced by $^{@}$(the uniformity induced by R) = the partition topology Classes R. The theorem is a consequence of (30) and (18).

9. Uniformity Induced by a Tolerance or by an Equivalence

Now we state the proposition:

(53) Let us consider an upper uniform space structure Q. Suppose $\bigcap$(the entourages Q) $\in$ the entourages Q. Then there exists a binary relation R on the carrier of Q such that

(i) $\bigcap$(the entourages Q) $= R$, and

(ii) the entourages $Q = \rho(R)$.

PROOF: Reconsider $R = \bigcap$(the entourages Q) as a binary relation on the carrier of Q. $\rho(R) \subseteq$ the entourages Q. The entourages $Q \subseteq \rho(R)$ by [7, (3)]. $\square$

Let Q be a uniform space structure. The functor Uniformity2InternalRel(Q) yielding a binary relation on the carrier of Q is defined by the term

(Def. 23) $\bigcap$(the entourages Q).

The functor UniformSpaceStr2RelStr(Q) yielding a relational structure is defined by the term

(Def. 24) $\langle$the carrier of Q, Uniformity2InternalRel(Q)$\rangle$.

Let R_1 be a relational structure. The functor InternalRel2Uniformity(R_1) yielding a family of subsets of (the carrier of R_1) $\times$ (the carrier of R_1) is defined by the term

(Def. 25) $\{R$, where R is a binary relation on the carrier of R_1 : the internal relation of $R_1 \subseteq R\}$.

The functor RelStr2UniformSpaceStr(R_1) yielding a strict uniform space structure is defined by the term

(Def. 26) $\langle\!\langle$the carrier of R_1, InternalRel2Uniformity(R_1)$\rangle\!\rangle$.

The functor InternalRel2Element(R_1) yielding an element of the entourages RelStr2UniformSpaceStr(R_1) is defined by the term

(Def. 27) the internal relation of R_1.

Now we state the propositions:

(54) Let us consider a binary relation R on X. Then $\bigcap \rho(R) = R$.

(55) Let us consider a strict relational structure R_1. Then UniformSpaceStr2−RelStr(RelStr2UniformSpaceStr(R_1)) $= R_1$. The theorem is a consequence of (54).

(56) Let us consider a uniform space structure Q. Then

   (i) the carrier of RelStr2UniformSpaceStr(UniformSpaceStr2RelStr(Q)) = the carrier of Q, and

   (ii) the entourages RelStr2UniformSpaceStr(UniformSpaceStr2RelStr (Q)) $= \rho(\bigcap$(the entourages Q)).

(57) Let us consider a family S_1 of subsets of $X \times X$, and a binary relation R on X. If $S_1 = \rho(R)$, then $S_1 \subseteq \rho(\bigcap S_1)$.

(58) Let us consider an upper family S_1 of subsets of $X \times X$. If $\bigcap S_1 \in S_1$, then $\rho(\bigcap S_1) \subseteq S_1$.

(59) Let us consider an upper family S_1 of subsets of $X \times X$, and a binary relation R on X. Suppose $R \in S_1$ and $S_1 = \rho(R)$ and $\bigcap S_1 \in S_1$. Then $\rho(\bigcap S_1) = S_1$.

(60) Let us consider an upper uniform space structure Q. Suppose there exists a binary relation R on the carrier of Q such that the entourages $Q = \rho(R)$ and $\bigcap$(the entourages Q) $\in$ the entourages Q. Then the entourages $Q = \rho(\bigcap$(the entourages Q)). The theorem is a consequence of (57) and (58).

(61) Let us consider an upper uniform space structure Q, and a binary relation R on the carrier of Q. Suppose the entourages $Q = \rho(R)$ and $\bigcap$(the entourages Q) $\in$ the entourages Q.
Then the entourages $Q = \rho(\bigcap$(the entourages Q)).

Let us consider a tolerance R of X. Now we state the propositions:

(62) (i) the uniformity induced by R is a semi-uniform space, and

   (ii) the entourages the uniformity induced by $R = \rho(R)$, and

   (iii) $\bigcap$(the entourages the uniformity induced by R) $= R$.

(63) RelStr2UniformSpaceStr(UniformSpaceStr2RelStr(the uniformity induced by R)) = the uniformity induced by R. The theorem is a consequence of (54).

(64) Let us consider an equivalence relation R of X. Then RelStr2UniformSpaceStr(UniformSpaceStr2RelStr(the uniformity induced by R)) = the uniformity induced by R. The theorem is a consequence of (54).

10. Uniform Pervin Space

Let X be a set, S_1 be a family of subsets of X, and A be an element of S_1. The functor Block(A) yielding a subset of $X \times X$ is defined by the term

(Def. 28) $(X \setminus A) \times (X \setminus A) \cup A \times A$.

From now on S_1 denotes a family of subsets of X and A denotes an element of S_1.

Now we state the propositions:

(65) If $A = \emptyset$, then $\mathrm{Block}(A) = X \times X$.

(66) Suppose X is not empty. Then $\mathrm{Block}(A) = \{\langle x, y\rangle$, where x, y are elements of $X : x \in A$ iff $y \in A\}$.
Proof: Set $S = \{\langle x, y\rangle$, where x, y are elements of $X : x \in A$ iff $y \in A\}$. $\mathrm{Block}(A) \subseteq S$ by [3, (87)]. $S \subseteq \mathrm{Block}(A)$ by [3, (87)]. □

(67) (i) $\mathrm{id}_X \subseteq \mathrm{Block}(A)$, and

(ii) $\mathrm{Block}(A) \cdot \mathrm{Block}(A) \subseteq \mathrm{Block}(A)$.

Let X be a set and S_1 be a family of subsets of X. The functor Blocks(S_1) yielding a family of subsets of $X \times X$ is defined by the term

(Def. 29) the set of all Block(A) where A is an element of S_1.

Let us observe that Blocks(S_1) is non empty.

The functor FMCBlocks(S_1) yielding a family of subsets of $X \times X$ is defined by the term

(Def. 30) FinMeetCl(Blocks(S_1)).

Now we state the propositions:

(68) FMCBlocks(S_1) is $\cap$-closed.

(69) FMCBlocks(S_1) is quasi-basis. The theorem is a consequence of (68).

(70) FMCBlocks(S_1) satisfies axiom UP1.

(71) Let us consider an element A of S_1, and a binary relation R on X. If $R = \mathrm{Block}(A)$, then $R^{\smile} = \mathrm{Block}(A)$. The theorem is a consequence of (65) and (4).

(72) Let us consider a binary relation R on X. Suppose R is an element of Blocks(S_1). Then $R^{\smile}$ is an element of Blocks(S_1). The theorem is a consequence of (71).

Let us consider a non empty family Y of subsets of $X \times X$. Now we state the propositions:

(73) If $Y \subseteq \mathrm{Blocks}(S_1)$, then $Y\,[\sim] = Y$. The theorem is a consequence of (71).

(74) If $Y \subseteq \text{Blocks}(S_1)$, then $(\bigcap Y)^{\smile} = \bigcap Y\,[\sim]$. The theorem is a consequence of (73) and (71).

(75) If $Y \subseteq \text{Blocks}(S_1)$, then $\bigcap Y = (\bigcap Y)^{\smile}$. The theorem is a consequence of (73) and (74).

(76) $\text{FMCBlocks}(S_1)$ satisfies axiom UP2. The theorem is a consequence of (73) and (75).

(77) $\text{FMCBlocks}(S_1)$ satisfies axiom UP3. The theorem is a consequence of (67).

Let X be a set and S_1 be a family of subsets of X. The Pervin uniform space of S_1 yielding a strict uniform space is defined by the term

(Def. 31) $\langle X, [\text{FMCBlocks}(S_1)]\rangle$.

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