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Leibniz Series for π[1]

Karol Pąk
Institute of Informatics
University of Białystok
Ciołkowskiego 1M, 15-245 Białystok
Poland

Summary. In this article we prove the Leibniz series for π which states that

$$\frac{\pi}{4} = \sum_{n=0}^{\infty} \frac{(-1)^n}{2 \cdot n + 1}.$$

The formalization follows K. Knopp [8], [1] and [6]. *Leibniz's Series for Pi* is item `#26` from the "Formalizing 100 Theorems" list maintained by Freek Wiedijk at `http://www.cs.ru.nl/F.Wiedijk/100/` .

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1. Preliminaries

From now on i, n, m denote natural numbers, r, s denote real numbers, and A denotes a non empty, closed interval subset of $\mathbb{R}$.

Now we state the proposition:

(1) $\operatorname{rng}((\text{the function tan})\restriction]-\frac{\pi}{2}, \frac{\pi}{2}[) = \mathbb{R}$.

Proof: Set $P = \frac{\pi}{2}$. Set $I =]-P, P[$. $\mathbb{R} \subseteq \operatorname{rng}((\text{the function tan})\restriction I)$ by [4, (50)], [20, (30)], [14, (15)], [16, (1)]. $\square$

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One can verify that the function arctan is total and the function arctan is differentiable.

Now we state the propositions:

(2) (The function arctan)$'(r) = \frac{1}{1+r^{\mathbf{2}}}$.

(3) Let us consider an open subset Z of $\mathbb{R}$. Then

(i) the function arctan is differentiable on Z, and

(ii) for every r such that $r \in Z$ holds (the function arctan)$'_{\restriction Z}(r) = \frac{1}{1+r^{\mathbf{2}}}$.

The theorem is a consequence of (2).

Let us consider n. One can verify that $\square^n$ is continuous.

Now we state the propositions:

(4) (i) $\mathrm{dom}(\frac{\square^n}{\square^0+\square^2}) = \mathbb{R}$, and

(ii) $\frac{\square^n}{\square^0+\square^2}$ is continuous, and

(iii) $(\frac{\square^n}{\square^0+\square^2})(r) = \frac{r^n}{1+r^{\mathbf{2}}}$.

(5) $\displaystyle\int\limits_A (\frac{\square^0}{\square^0+\square^2})(x)dx =$ (the function arctan)(sup A) − (the function arctan)(inf A).

PROOF: Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $f = \frac{Z_0}{Z_0+Z_2}$. $\mathrm{dom}\, f = \mathbb{R}$. f is continuous. If $r \in \mathbb{R}$, then $f(r) = \frac{1}{1+r^{\mathbf{2}}}$ by [13, (4)], (4). For every element x of $\mathbb{R}$ such that $x \in$ dom(the function arctan)$'_{\restriction \mathbb{R}}$ holds (the function arctan)$'_{\restriction \mathbb{R}}(x) = f(x)$. □

(6) $\displaystyle\int\limits_A ((-1)^i \cdot (\frac{\square^{2\cdot n}}{\square^0+\square^2}))(x)dx = (-1)^i \cdot ((\frac{1}{2\cdot n+1}) \cdot (\sup A)^{2\cdot n+1} - (\frac{1}{2\cdot n+1}) \cdot (\inf A)^{2\cdot n+1}) + \int\limits_A ((-1)^{i+1} \cdot (\frac{\square^{2\cdot(n+1)}}{\square^0+\square^2}))(x)dx.$

PROOF: Set $I_1 = (-1)^i$. Set $i_1 = i+1$. Set $n_1 = n+1$. Set $I_2 = (-1)^{i_1}$. Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $Z_{2n} = \square^{2\cdot n}$. Set $f = I_1 \cdot Z_{2n}$. Set $g = I_2 \cdot (\frac{\square^{2\cdot n_1}}{Z_0+Z_2})$. $\mathrm{dom}\, g = \mathbb{R}$. For every element x of $\mathbb{R}$, $(I_1 \cdot (\frac{Z_{2n}}{Z_0+Z_2}))(x) = (f+g)(x)$ by [13, (6)], [17, (36)], (4). $f + g = I_1 \cdot (\frac{Z_{2n}}{Z_0+Z_2})$. $\frac{\square^{2\cdot n_1}}{Z_0+Z_2}$ is continuous. □

(7) Suppose $A = [0, r]$ and $r \geqslant 0$. Then $|\displaystyle\int\limits_A (\frac{\square^{2\cdot n}}{\square^0+\square^2})(x)dx| \leqslant (\frac{1}{2\cdot n+1}) \cdot r^{2\cdot n+1}$.

PROOF: Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $N = 2 \cdot n$. Set $Z_n = \square^N$. Set $f = \frac{Z_n}{Z_0+Z_2}$. f is continuous and $\mathrm{dom}\, f = \mathbb{R}$. Reconsider $f_1 = f \restriction A$ as a function from A into $\mathbb{R}$. Reconsider $Z_1 = Z_n \restriction A$ as a function from A into $\mathbb{R}$. For every r such that $r \in A$ holds $f_1(r) \leqslant Z_1(r)$ by [4, (49)], [17, (36)], [18, (3)], (4). For every object x such that $x \in \mathbb{R}$ holds $f(x) = |f|(x)$ by [13, (8)], (4). □

2. EULER TRANSFORMATION

Let a be a sequence of real numbers. The alternating series of a yielding a sequence of real numbers is defined by

(Def. 1) $it(i) = (-1)^i \cdot a(i)$.

Now we state the proposition:

(8) Let us consider a sequence a of real numbers. Suppose a is non-negative yielding, non-increasing, and convergent and $\lim a = 0$. Then

(i) the alternating series of a is summable, and

(ii) for every n, $(\sum_{\alpha=0}^{\kappa}(\text{the alternating series of } a)(\alpha))_{\kappa\in\mathbb{N}}(2\cdot n) \geqslant \sum(\text{the alternating series of } a) \geqslant (\sum_{\alpha=0}^{\kappa}(\text{the alternating series of } a)(\alpha))_{\kappa\in\mathbb{N}}(2\cdot n+1)$.

PROOF: Set $A =$ the alternating series of a. Set $P = (\sum_{\alpha=0}^{\kappa} A(\alpha))_{\kappa\in\mathbb{N}}$. Define $\mathcal{T}[\text{natural number}, \text{object}] \equiv \$_2 = P(2 \cdot \$_1)$. Define $\mathcal{S}[\text{natural number}, \text{object}] \equiv \$_2 = P(2 \cdot \$_1 + 1)$. Consider T being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element x of $\mathbb{N}$, $\mathcal{T}[x, T(x)]$ from [5, Sch. 3]. Consider S being a function from $\mathbb{N}$ into $\mathbb{R}$ such that for every element x of $\mathbb{N}$, $\mathcal{S}[x, S(x)]$ from [5, Sch. 3]. For every natural number n, $S(n) \leqslant S(n+1)$. For every natural number n, $T(n) \geqslant T(n+1)$. For every natural number n, $T(n) \geqslant S(n)$. For every natural number n, $T(n) > S(0) - 1$ by [10, (6)]. For every natural number n, $S(n) < T(0) + 1$ by [10, (8)].
Define $\mathcal{D}(\text{natural number}) = 2 \cdot \$_1 + 1$. Consider D being a function from $\mathbb{N}$ into $\mathbb{N}$ such that for every element x of $\mathbb{N}$, $D(x) = \mathcal{D}(x)$ from [5, Sch. 8]. Reconsider $D_1 = D$ as a many sorted set indexed by $\mathbb{N}$. For every natural number n, $D(n) < D(n + 1)$ by [2, (13)]. Reconsider $a_2 = a \cdot D_1$ as a sequence of real numbers.
For every object x such that $x \in \mathbb{N}$ holds $a_2(x) = (T - S)(x)$ by [4, (12)]. For every real number p such that $0 < p$ there exists a natural number n such that for every natural number m such that $n \leqslant m$ holds $|P(m) - \lim T| < p$ by [19, (9)]. □

3. Main Theorem

Let us consider r. The Leibniz series of r yielding a sequence of real numbers is defined by

(Def. 2) $it(n) = \frac{(-1)^n \cdot r^{2 \cdot n+1}}{2 \cdot n+1}$.

The Leibniz series yielding a sequence of real numbers is defined by the term

(Def. 3) the Leibniz series of 1.

Now we state the propositions:

(9) Suppose $r \in [-1, 1]$. Then

(i) |the Leibniz series of r| is non-negative yielding, non-increasing, and convergent, and

(ii) lim|the Leibniz series of r| $= 0$.

Proof: Set $r_1 =$ the Leibniz series of r. Set $A = |r_1|$. $A(n) = \frac{|r|^{2 \cdot n+1}}{2 \cdot n+1}$ by [15, (1)], [3, (67), (65)]. $A(n) \geqslant A(n+1)$ by [3, (46)], [15, (1)], [13, (6)], [2, (13)]. Set $C = \{0\}_{n \in \mathbb{N}}$. Define $\mathcal{F}(\text{natural number}) = \frac{\frac{1}{2}}{\$_1 + \frac{1}{2}}$. Consider f being a sequence of real numbers such that $f(n) = \mathcal{F}(n)$ from [11, Sch. 1]. $C(n) \leqslant A(n) \leqslant f(n)$ by [11, (57)], [3, (46)], [13, (11)], [2, (11)]. $\square$

(10) (i) if $r \geqslant 0$, then the alternating series of |the Leibniz series of r| $=$ the Leibniz series of r, and

(ii) if $r < 0$, then $(-1) \cdot$ (the alternating series of |the Leibniz series of r|) $=$ the Leibniz series of r.

Proof: Set $r_1 =$ the Leibniz series of r. Set $A = |r_1|$. Set $a_1 =$ the alternating series of A. $a_1(n) = (-1)^n \cdot (\frac{|r|^{2 \cdot n+1}}{2 \cdot n+1})$ by [15, (1)], [3, (67), (65)]. If $r \geqslant 0$, then $a_1 = r_1$. $\square$

(11) If $r \in [-1, 1]$, then the Leibniz series of r is summable. The theorem is a consequence of (9), (8), and (10).

(12) Suppose $A = [0, r]$ and $r \geqslant 0$. Then (the function arctan)$(r) = (\sum_{\alpha=0}^{\kappa}$(the Leibniz series of $r)(\alpha))_{\kappa \in \mathbb{N}}(n) + \int\limits_A ((-1)^{n+1} \cdot (\frac{\square^{2 \cdot (n+1)}}{\square^0 + \square^2}))(x) dx$.

Proof: Set $Z_0 = \square^0$. Set $Z_2 = \square^2$. Set $r_1 =$ the Leibniz series of r. Define $\mathcal{P}[\text{natural number}] \equiv$ (the function arctan)$(r) = (\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa \in \mathbb{N}}(\$_1) + \int\limits_A ((-1)^{\$_1+1} \cdot (\frac{\square^{2 \cdot (\$_1+1)}}{Z_0 + Z_2}))(x) dx$. $\mathcal{P}[0]$ by (5), [14, (43)], [13, (4)], [9, (21)]. If $\mathcal{P}[i]$, then $\mathcal{P}[i+1]$ by [13, (11)], [2, (11)], (6). $\mathcal{P}[i]$ from [2, Sch. 2]. $\square$

(13) If $0 \leqslant r \leqslant 1$, then (the function arctan)$(r) = \sum$(the Leibniz series of r).

PROOF: Set r_1 = the Leibniz series of r. Set $P = (\sum_{\alpha=0}^{\kappa} r_1(\alpha))_{\kappa\in\mathbb{N}}$. Set A = (the function arctan)(r). Define $\mathcal{I}$(natural number) $= \frac{\square^{2\cdot\$_1}}{\square^0+\square^2}$. P is convergent. For every s such that $0 < s$ there exists n such that for every m such that $n \leqslant m$ holds $|P(m) - A| < s$ by [12, (3)], (4), [7, (11), (10)]. $\square$

(14) LEIBNIZ SERIES FOR π:
$\frac{\pi}{4} = \sum$(the Leibniz series).

(15) $(\sum_{\alpha=0}^{\kappa}$(the Leibniz series)$(\alpha))_{\kappa\in\mathbb{N}}(2 \cdot n + 1) \leqslant \sum$(the Leibniz series) $\leqslant$ $(\sum_{\alpha=0}^{\kappa}$(the Leibniz series)$(\alpha))_{\kappa\in\mathbb{N}}(2 \cdot n)$. The theorem is a consequence of (9), (10), and (8).

(16) (i) $(\sum_{\alpha=0}^{\kappa}$(the Leibniz series)$(\alpha))_{\kappa\in\mathbb{N}}(1) = \frac{2}{3}$, and
(ii) if n is odd, then $(\sum_{\alpha=0}^{\kappa}$(the Leibniz series)$(\alpha))_{\kappa\in\mathbb{N}}(n+2) = (\sum_{\alpha=0}^{\kappa}$(the Leibniz series)$(\alpha))_{\kappa\in\mathbb{N}}(n) + \frac{2}{4\cdot n^{\mathbf{2}}+16\cdot n+15}$.

(17) π APPROXIMATION:
$\frac{313}{100} < \pi < \frac{315}{100}$. The theorem is a consequence of (16), (14), and (15).

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