

# The Axiomatization of Propositional Logic<sup>1</sup>

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**Summary.** This article introduces propositional logic as a formal system ([14], [10], [11]). The formulae of the language are as follows  $\phi ::= \perp \mid p \mid \phi \rightarrow \phi$ . Other connectives are introduced as abbreviations. The notions of model and satisfaction in model are defined. The axioms are all the formulae of the following schemes

- $\alpha \Rightarrow (\beta \Rightarrow \alpha)$ ,
- $(\alpha \Rightarrow (\beta \Rightarrow \gamma)) \Rightarrow ((\alpha \Rightarrow \beta) \Rightarrow (\alpha \Rightarrow \gamma))$ ,
- $(\neg\beta \Rightarrow \neg\alpha) \Rightarrow ((\neg\beta \Rightarrow \alpha) \Rightarrow \beta)$ .

Modus ponens is the only derivation rule. The soundness theorem and the strong completeness theorem are proved. The proof of the completeness theorem is carried out by a counter-model existence method. In order to prove the completeness theorem, Lindenbaum's Lemma is proved. Some most widely used tautologies are presented.

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## 1. PRELIMINARIES

Now we state the propositions:

- (1) Let us consider functions  $f, g$ . Suppose  $\text{dom } f \subseteq \text{dom } g$  and for every set  $x$  such that  $x \in \text{dom } f$  holds  $f(x) = g(x)$ . Then  $\text{rng } f \subseteq \text{rng } g$ .
- (2) Let us consider Boolean objects  $p, q$ . Then  $p \wedge q \Rightarrow p = \text{true}$ .
- (3) Let us consider a Boolean object  $p$ . Then  $\neg\neg p \Leftrightarrow p = \text{true}$ .

Let us consider Boolean objects  $p, q$ . Now we state the propositions:

- (4)  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q = \text{true}$ .
- (5)  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q = \text{true}$ .
- (6)  $p \Rightarrow q \Rightarrow (\neg q \Rightarrow \neg p) = \text{true}$ .

Let us consider Boolean objects  $p, q, r$ . Now we state the propositions:

- (7)  $p \Rightarrow q \Rightarrow (p \Rightarrow r \Rightarrow (p \Rightarrow q \wedge r)) = \text{true}$ .
- (8)  $p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r)) = \text{true}$ .

Let us consider Boolean objects  $p, q$ . Now we state the propositions:

- (9)  $p \wedge q \Leftrightarrow q \wedge p = \text{true}$ .
- (10)  $p \vee q \Leftrightarrow q \vee p = \text{true}$ .

Let us consider Boolean objects  $p, q, r$ . Now we state the propositions:

- (11)  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r) = \text{true}$ .
- (12)  $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r) = \text{true}$ .
- (13) Let us consider Boolean objects  $p, q$ . Then  $\neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q) = \text{true}$ .

Let us consider Boolean objects  $p, q, r$ . Now we state the propositions:

- (14)  $p \wedge (q \vee r) \Leftrightarrow p \wedge q \vee p \wedge r = \text{true}$ .
- (15)  $p \vee q \wedge r \Leftrightarrow (p \vee q) \wedge (p \vee r) = \text{true}$ .
- (16) Let us consider a finite set  $X$ , and a set  $Y$ . Suppose  $Y$  is  $\subseteq$ -linear and  $X \subseteq \bigcup Y$  and  $Y \neq \emptyset$ . Then there exists a set  $Z$  such that
  - (i)  $X \subseteq Z$ , and
  - (ii)  $Z \in Y$ .

## 2. THE SYNTAX

Let  $D$  be a set. We say that  $D$  has propositional variables if and only if

(Def. 1) for every element  $n$  of  $\mathbb{N}$ ,  $\langle 3 + n \rangle \in D$ .

We say that  $D$  is PL-closed if and only if

(Def. 2)  $D \subseteq \mathbb{N}^*$  and  $D$  has FALSUM, implication and propositional variables.

Let us note that every set which is PL-closed is also non empty and has also FALSUM, implication, and propositional variables and every subset of  $\mathbb{N}^*$  which has FALSUM, implication, and propositional variables is also PL-closed.

The functor PL-WFF yielding a set is defined by

(Def. 3)  $it$  is PL-closed and for every set  $D$  such that  $D$  is PL-closed holds  $it \subseteq D$ .

Observe that PL-WFF is PL-closed and there exists a set which is PL-closed and non empty and PL-WFF is functional and every element of PL-WFF is finite sequence-like.

The functor  $\perp_{PL}$  yielding an element of PL-WFF is defined by the term

(Def. 4)  $\langle 0 \rangle$ .

Let  $p, q$  be elements of PL-WFF. The functor  $p \Rightarrow q$  yielding an element of PL-WFF is defined by the term

(Def. 5)  $(\langle 1 \rangle \wedge p) \wedge q$ .

Let  $n$  be an element of  $\mathbb{N}$ . The functor  $\text{Prop } n$  yielding an element of PL-WFF is defined by the term

(Def. 6)  $\langle 3 + n \rangle$ .

The functor  $AP$  yielding a subset of PL-WFF is defined by

(Def. 7) for every set  $x, x \in it$  iff there exists an element  $n$  of  $\mathbb{N}$  such that  $x = \text{Prop } n$ .

From now on  $p, q, r, s, A, B$  denote elements of PL-WFF,  $F, G, H$  denote subsets of PL-WFF,  $k, n$  denote elements of  $\mathbb{N}$ , and  $f, f_1, f_2$  denote finite sequences of elements of PL-WFF.

Let  $D$  be a subset of PL-WFF. Observe that  $D$  has implication if and only if the condition (Def. 8) is satisfied.

(Def. 8) for every  $p$  and  $q$  such that  $p, q \in D$  holds  $p \Rightarrow q \in D$ .

The scheme  $PLInd$  deals with a unary predicate  $\mathcal{P}$  and states that

(Sch. 1) For every  $r, \mathcal{P}[r]$   
provided

- $\mathcal{P}[\perp_{PL}]$  and
- for every  $n, \mathcal{P}[\text{Prop } n]$  and
- for every  $r$  and  $s$  such that  $\mathcal{P}[r]$  and  $\mathcal{P}[s]$  holds  $\mathcal{P}[r \Rightarrow s]$ .

Now we state the proposition:

(17)  $PL\text{-WFF} \subseteq HP\text{-WFF}$ .

PROOF: Define  $\mathcal{P}[\text{element of PL-WFF}] \equiv \$1 \in HP\text{-WFF}$ . For every  $n, \mathcal{P}[\text{Prop } n]$ . For every  $r$  and  $s$  such that  $\mathcal{P}[r]$  and  $\mathcal{P}[s]$  holds  $\mathcal{P}[r \Rightarrow s]$ . For every  $A, \mathcal{P}[A]$  from  $PLInd$ .  $\square$

Let us consider  $p$ . The functor  $\neg p$  yielding an element of PL-WFF is defined by the term

$$\text{(Def. 9)} \quad p \Rightarrow \perp_{\text{PL}}.$$

The functor  $\top_{\text{PL}}$  yielding an element of PL-WFF is defined by the term

$$\text{(Def. 10)} \quad \neg \perp_{\text{PL}}.$$

Let us consider  $p$  and  $q$ . The functors:  $p \wedge q$  and  $p \vee q$  yielding elements of PL-WFF are defined by terms

$$\text{(Def. 11)} \quad \neg(p \Rightarrow \neg q),$$

$$\text{(Def. 12)} \quad \neg p \Rightarrow q,$$

respectively. The functor  $p \Leftrightarrow q$  yielding an element of PL-WFF is defined by the term

$$\text{(Def. 13)} \quad (p \Rightarrow q) \wedge (q \Rightarrow p).$$

### 3. THE SEMANTICS

A PL-model is a subset of  $AP$ . From now on  $M$  denotes a PL-model.

Let  $M$  be a PL-model. The functor  $\text{SAT}_M$  yielding a function from PL-WFF into *Boolean* is defined by

$$\text{(Def. 14)} \quad it(\perp_{\text{PL}}) = 0 \text{ and for every } k, it(\text{Prop } k) = 1 \text{ iff } \text{Prop } k \in M \text{ and for every } p \text{ and } q, it(p \Rightarrow q) = it(p) \Rightarrow it(q).$$

Now we state the propositions:

$$(18) \quad \text{SAT}_M(A \Rightarrow B) = 1 \text{ if and only if } \text{SAT}_M(A) = 0 \text{ or } \text{SAT}_M(B) = 1.$$

$$(19) \quad \text{SAT}_M(\neg p) = \neg(\text{SAT}_M(p)).$$

$$(20) \quad \text{SAT}_M(\neg A) = 1 \text{ if and only if } \text{SAT}_M(A) = 0. \text{ The theorem is a consequence of (19).}$$

$$(21) \quad \text{SAT}_M(A \wedge B) = \text{SAT}_M(A) \wedge \text{SAT}_M(B). \text{ The theorem is a consequence of (19).}$$

$$(22) \quad \text{SAT}_M(A \wedge B) = 1 \text{ if and only if } \text{SAT}_M(A) = 1 \text{ and } \text{SAT}_M(B) = 1. \text{ The theorem is a consequence of (21).}$$

$$(23) \quad \text{SAT}_M(A \vee B) = \text{SAT}_M(A) \vee \text{SAT}_M(B). \text{ The theorem is a consequence of (19).}$$

$$(24) \quad \text{SAT}_M(A \vee B) = 1 \text{ if and only if } \text{SAT}_M(A) = 1 \text{ or } \text{SAT}_M(B) = 1. \text{ The theorem is a consequence of (23).}$$

$$(25) \quad \text{SAT}_M(A \Leftrightarrow B) = \text{SAT}_M(A) \Leftrightarrow \text{SAT}_M(B). \text{ The theorem is a consequence of (21).}$$

$$(26) \quad \text{SAT}_M(A \Leftrightarrow B) = 1 \text{ if and only if } \text{SAT}_M(A) = \text{SAT}_M(B). \text{ The theorem is a consequence of (25).}$$

Let us consider  $M$  and  $p$ . We say that  $M \models p$  if and only if

(Def. 15)  $\text{SAT}_M(p) = 1$ .

Let us consider  $F$ . We say that  $M \models F$  if and only if

(Def. 16) for every  $p$  such that  $p \in F$  holds  $M \models p$ .

Let us consider  $p$ . We say that  $F \models p$  if and only if

(Def. 17) for every  $M$  such that  $M \models F$  holds  $M \models p$ .

Let us consider  $A$ . We say that  $A$  is a tautology if and only if

(Def. 18) for every  $M$ ,  $\text{SAT}_M(A) = 1$ .

Now we state the propositions:

(27)  $A$  is a tautology if and only if  $\emptyset_{\text{PL-WFF}} \models A$ .

(28)  $p \Rightarrow (q \Rightarrow p)$  is a tautology.

(29)  $p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$  is a tautology.

(30)  $\neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q)$  is a tautology. The theorem is a consequence of (19) and (13).

(31)  $p \Rightarrow q \Rightarrow (\neg q \Rightarrow \neg p)$  is a tautology. The theorem is a consequence of (19) and (6).

(32)  $p \wedge q \Rightarrow p$  is a tautology. The theorem is a consequence of (21) and (2).

(33)  $p \wedge q \Rightarrow q$  is a tautology. The theorem is a consequence of (21) and (2).

(34)  $p \Rightarrow p \vee q$  is a tautology. The theorem is a consequence of (23).

(35)  $q \Rightarrow p \vee q$  is a tautology. The theorem is a consequence of (23).

(36)  $p \wedge q \Leftrightarrow q \wedge p$  is a tautology. The theorem is a consequence of (25), (21), and (9).

(37)  $p \vee q \Leftrightarrow q \vee p$  is a tautology. The theorem is a consequence of (25), (23), and (10).

(38)  $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$  is a tautology. The theorem is a consequence of (25), (21), and (11).

(39)  $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$  is a tautology. The theorem is a consequence of (25), (23), and (12).

(40)  $p \wedge (q \vee r) \Leftrightarrow p \wedge q \vee p \wedge r$  is a tautology. The theorem is a consequence of (25), (21), (23), and (14).

(41)  $p \vee q \wedge r \Leftrightarrow (p \vee q) \wedge (p \vee r)$  is a tautology. The theorem is a consequence of (25), (23), (21), and (15).

(42)  $\neg\neg p \Leftrightarrow p$  is a tautology. The theorem is a consequence of (25), (19), and (3).

(43)  $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$  is a tautology. The theorem is a consequence of (25), (19), (21), (23), and (4).

- (44)  $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$  is a tautology. The theorem is a consequence of (25), (19), (23), (21), and (5).
- (45)  $p \Rightarrow q \Rightarrow (p \Rightarrow r \Rightarrow (p \Rightarrow q \wedge r))$  is a tautology. The theorem is a consequence of (21) and (7).
- (46)  $p \Rightarrow r \Rightarrow (q \Rightarrow r \Rightarrow (p \vee q \Rightarrow r))$  is a tautology. The theorem is a consequence of (23) and (8).
- (47) If  $F \models A$  and  $F \models A \Rightarrow B$ , then  $F \models B$ .

#### 4. THE AXIOMS. DERIVABILITY.

Let  $D$  be a set. We say that  $D$  is with axioms of PL if and only if

- (Def. 19) for every  $p, q$ , and  $r$  holds  $p \Rightarrow (q \Rightarrow p)$ ,  $p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$ ,  $\neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q) \in D$ .

The functor PL-axioms yielding a subset of PL-WFF is defined by

- (Def. 20) *it* is with axioms of PL and for every subset  $D$  of PL-WFF such that  $D$  is with axioms of PL holds  $it \subseteq D$ .

One can check that PL-axioms is with axioms of PL.

Let us consider  $p, q$ , and  $r$ . We say that MP( $p, q, r$ ) if and only if

- (Def. 21)  $q = p \Rightarrow r$ .

Observe that PL-axioms is non empty.

Let us consider  $A$ . We say that  $A$  is the simplification axiom if and only if

- (Def. 22) there exists  $p$  and there exists  $q$  such that  $A = p \Rightarrow (q \Rightarrow p)$ .

We say that  $A$  is Frege axiom if and only if

- (Def. 23) there exists  $p$  and there exists  $q$  and there exists  $r$  such that  $A = p \Rightarrow (q \Rightarrow r) \Rightarrow (p \Rightarrow q \Rightarrow (p \Rightarrow r))$ .

We say that  $A$  is the explosion axiom if and only if

- (Def. 24) there exists  $p$  and there exists  $q$  such that  $A = \neg q \Rightarrow \neg p \Rightarrow (\neg q \Rightarrow p \Rightarrow q)$ .

Now we state the propositions:

- (48) Every element of PL-axioms is the simplification axiom or Frege axiom or the explosion axiom.
- (49) If  $A$  is the simplification axiom or Frege axiom or the explosion axiom, then  $F \models A$ . The theorem is a consequence of (28), (29), and (30).

Let  $i$  be a natural number. Let us consider  $f$  and  $F$ . We say that  $\text{prc}(f, F, i)$  if and only if

- (Def. 25)  $f(i) \in \text{PL-axioms}$  or  $f(i) \in F$  or there exist natural numbers  $j, k$  such that  $1 \leq j < i$  and  $1 \leq k < i$  and  $\text{MP}(f_j, f_k, f_i)$ .

Let us consider  $p$ . We say that  $F \vdash p$  if and only if

(Def. 26) there exists  $f$  such that  $f(\text{len } f) = p$  and  $1 \leq \text{len } f$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $\text{prc}(f, F, i)$ .

Now we state the propositions:

(50) Let us consider natural numbers  $i, n$ . Suppose  $n + \text{len } f \leq \text{len } f_2$  and for every natural number  $k$  such that  $1 \leq k \leq \text{len } f$  holds  $f(k) = f_2(k + n)$  and  $1 \leq i \leq \text{len } f$ . If  $\text{prc}(f, F, i)$ , then  $\text{prc}(f_2, F, i + n)$ .

(51) Suppose  $f_2 = f \wedge f_1$  and  $1 \leq \text{len } f$  and  $1 \leq \text{len } f_1$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $\text{prc}(f, F, i)$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f_1$  holds  $\text{prc}(f_1, F, i)$ . Let us consider a natural number  $i$ . If  $1 \leq i \leq \text{len } f_2$ , then  $\text{prc}(f_2, F, i)$ . The theorem is a consequence of (50).

(52) Suppose  $f = f_1 \wedge \langle p \rangle$  and  $1 \leq \text{len } f_1$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f_1$  holds  $\text{prc}(f_1, F, i)$  and  $\text{prc}(f, F, \text{len } f)$ . Then

(i) for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $\text{prc}(f, F, i)$ ,  
and

(ii)  $F \vdash p$ .

The theorem is a consequence of (50).

(53) If  $p \in \text{PL-axioms}$  or  $p \in F$ , then  $F \vdash p$ .

PROOF: Define  $\mathcal{P}[\text{set}, \text{set}] \equiv \$_2 = p$ . Consider  $f$  such that  $\text{dom } f = \text{Seg } 1$  and for every natural number  $k$  such that  $k \in \text{Seg } 1$  holds  $\mathcal{P}[k, f(k)]$  from [3, Sch. 5]. For every natural number  $j$  such that  $1 \leq j \leq \text{len } f$  holds  $\text{prc}(f, F, j)$ .  $\square$

(54) If  $F \vdash p$  and  $F \vdash p \Rightarrow q$ , then  $F \vdash q$ .

PROOF: Consider  $f$  such that  $f(\text{len } f) = p$  and  $1 \leq \text{len } f$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $\text{prc}(f, F, i)$ . Consider  $f_1$  such that  $f_1(\text{len } f_1) = p \Rightarrow q$  and  $1 \leq \text{len } f_1$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f_1$  holds  $\text{prc}(f_1, F, i)$ . Set  $g = (f \wedge f_1) \wedge \langle q \rangle$ . For every natural number  $i$  such that  $1 \leq i \leq \text{len } f_1$  holds  $g(\text{len } f + i) = f_1(i)$  by [3, (22), (39)], [1, (12)], [3, (65), (64)]. For every natural number  $i$  such that  $1 \leq i \leq \text{len}(f \wedge f_1)$  holds  $\text{prc}(f \wedge f_1, F, i)$ .  $\square$

(55) If  $F \subseteq G$ , then if  $F \vdash p$ , then  $G \vdash p$ .

PROOF: Consider  $f$  such that  $f(\text{len } f) = p$  and  $1 \leq \text{len } f$  and for every natural number  $k$  such that  $1 \leq k \leq \text{len } f$  holds  $\text{prc}(f, F, k)$ . Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $1 \leq \$_1 \leq \text{len } f$ , then  $G \vdash f_{\$_1}$ . For every natural number  $k$ ,  $\mathcal{P}[k]$  from [1, Sch. 4].  $\square$

## 5. SOUNDNESS THEOREM. DEDUCTION THEOREM.

Now we state the propositions:

(56) If  $F \vdash A$ , then  $F \models A$ .

PROOF: Consider  $f$  such that  $f(\text{len } f) = A$  and  $1 \leq \text{len } f$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $\text{prc}(f, F, i)$ . Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $1 \leq s_1 \leq \text{len } f$ , then  $F \models f_{s_1}$ . For every natural number  $i$  such that for every natural number  $j$  such that  $j < i$  holds  $\mathcal{P}[j]$  holds  $\mathcal{P}[i]$  by [1, (14)], [9, (1)], (48), (49). For every natural number  $i$ ,  $\mathcal{P}[i]$  from [1, Sch. 4].  $\square$

(57)  $F \vdash A \Rightarrow A$ . The theorem is a consequence of (53) and (54).

(58) DEDUCTION THEOREM:

If  $F \cup \{A\} \vdash B$ , then  $F \vdash A \Rightarrow B$ .

PROOF: Consider  $f$  such that  $f(\text{len } f) = B$  and  $1 \leq \text{len } f$  and for every natural number  $i$  such that  $1 \leq i \leq \text{len } f$  holds  $\text{prc}(f, F \cup \{A\}, i)$ . Define  $\mathcal{P}[\text{natural number}] \equiv$  if  $1 \leq s_1 \leq \text{len } f$ , then  $F \vdash A \Rightarrow f_{s_1}$ . For every natural number  $i$  such that for every natural number  $j$  such that  $j < i$  holds  $\mathcal{P}[j]$  holds  $\mathcal{P}[i]$  by [1, (14)], (53), [9, (1)], (54). For every natural number  $i$ ,  $\mathcal{P}[i]$  from [1, Sch. 4].  $\square$

(59) If  $F \vdash A \Rightarrow B$ , then  $F \cup \{A\} \vdash B$ . The theorem is a consequence of (53), (55), and (54).

(60)  $F \vdash \neg A \Rightarrow (A \Rightarrow B)$ . The theorem is a consequence of (53), (54), and (58).

(61)  $F \vdash \neg A \Rightarrow A \Rightarrow A$ . The theorem is a consequence of (53), (57), and (54).

## 6. STRONG COMPLETENESS THEOREM

Let us consider  $F$ . We say that  $F$  is consistent if and only if

(Def. 27) there exists no  $p$  such that  $F \vdash p$  and  $F \vdash \neg p$ .

Now we state the propositions:

(62)  $F$  is consistent if and only if there exists  $A$  such that  $F \not\vdash A$ . The theorem is a consequence of (60) and (54).

(63) If  $F \not\vdash A$ , then  $F \cup \{\neg A\}$  is consistent. The theorem is a consequence of (58), (62), (61), and (54).

(64)  $F \vdash A$  if and only if there exists  $G$  such that  $G \subseteq F$  and  $G$  is finite and  $G \vdash A$ . The theorem is a consequence of (55).



(65) If  $F$  is not consistent, then there exists  $G$  such that  $G$  is finite and  $G$  is not consistent and  $G \subseteq F$ . The theorem is a consequence of (64) and (55).

Let us consider  $F$ . We say that  $F$  is maximal if and only if

(Def. 28) for every  $p$  holds  $p \in F$  or  $\neg p \in F$ .

Now we state the propositions:

(66) If  $F \subseteq G$  and  $F$  is not consistent, then  $G$  is not consistent. The theorem is a consequence of (55).

(67) If  $F$  is consistent and  $F \cup \{A\}$  is not consistent, then  $F \cup \{\neg A\}$  is consistent. The theorem is a consequence of (58), (62), (61), and (54).

In the sequel  $x, y$  denote sets. Now we state the propositions:

(68) LINDENBAUM'S LEMMA:

If  $F$  is consistent, then there exists  $G$  such that  $F \subseteq G$  and  $G$  is consistent and maximal.

PROOF: Set  $L = \text{PL-WFF}$ . Consider  $R$  being a binary relation such that  $R$  well orders  $L$ . Reconsider  $R_2 = R \upharpoonright^2 L$  as a binary relation on  $L$ . Reconsider  $R_1 = \langle L, R_2 \rangle$  as a non empty relational structure. Set  $c =$  the carrier of  $R_1$ . Define  $\mathcal{H}[\text{object}, \text{object}, \text{object}] \equiv$  for every  $p$  for every partial function  $f$  from  $c$  to  $2^L$  such that  $\$1 = p$  and  $\$2 = f$  holds if  $(\bigcup \text{rng}(f \text{ qua } (2^L)\text{-valued binary relation}) \cup F) \cup \{p\}$  is consistent, then  $\$3 = (\bigcup \text{rng } f \cup F) \cup \{p\}$  and if  $(\bigcup \text{rng}(f \text{ qua } (2^L)\text{-valued binary relation}) \cup F) \cup \{p\}$  is not consistent, then  $\$3 = \bigcup \text{rng } f \cup F$ . For every objects  $x, y$  such that  $x \in c$  and  $y \in c \rightarrow 2^L$  there exists an object  $z$  such that  $z \in 2^L$  and  $\mathcal{H}[x, y, z]$  by [8, (46)]. Consider  $h$  being a function from  $c \times (c \rightarrow 2^L)$  into  $2^L$  such that for every objects  $x, y$  such that  $x \in c$  and  $y \in c \rightarrow 2^L$  holds  $\mathcal{H}[x, y, h(x, y)]$  from [5, Sch. 1]. Consider  $f$  being a function from  $c$  into  $2^L$  such that  $f$  is recursively expressed by  $h$ . Reconsider  $G = \bigcup \text{rng}(f \text{ qua } (2^L)\text{-valued binary relation})$  as a subset of PL-WFF. Set  $i_1 =$  the internal relation of  $R_1$ . For every  $A$  and  $B$  such that  $\langle A, B \rangle \in R_2$  holds  $f(A) \subseteq f(B)$  by [4, (1)], [2, (4), (29), (9)].  $\text{rng } f$  is  $\subseteq$ -linear. Define  $\mathcal{S}[\text{element of } R_1] \equiv f(\$1)$  is consistent. For every element  $x$  of  $R_1$  such that for every element  $y$  of  $R_1$  such that  $y \neq x$  and  $\langle y, x \rangle \in i_1$  holds  $\mathcal{S}[y]$  holds  $\mathcal{S}[x]$  by [2, (9)], [7, (32)], [2, (1)], [15, (42)]. For every element  $A$  of  $R_1$ ,  $\mathcal{S}[A]$  from [12, Sch. 3].  $F \subseteq G$  by [6, (3)].  $G$  is consistent by (65), (16), [15, (42)], (66).  $G$  is maximal by [6, (3)], (17), [13, (16)], (66).  $\square$

(69) If  $F$  is maximal and consistent, then for every  $p$ ,  $F \vdash p$  iff  $p \in F$ . The theorem is a consequence of (53).

(70) If  $F \models A$ , then  $F \vdash A$ .

PROOF: Consider  $G$  such that  $F \cup \{\neg A\} \subseteq G$  and  $G$  is consistent and  $G$  is maximal. Set  $M = \{\text{Prop } n, \text{ where } n \text{ is an element of } \mathbb{N} : \text{Prop } n \in G\}$ .

$M \subseteq AP$ . Define  $\mathcal{P}[\text{element of PL-WFF}] \equiv \$1 \in G$  iff  $M \models \$1$ .  $\mathcal{P}[\perp_{\text{PL}}]$ . For every  $n$ ,  $\mathcal{P}[\text{Prop } n]$ . For every  $r$  and  $s$  such that  $\mathcal{P}[r]$  and  $\mathcal{P}[s]$  holds  $\mathcal{P}[r \Rightarrow s]$ . For every  $B$ ,  $\mathcal{P}[B]$  from  $PLInd$ .  $M \not\models A$ .  $\square$

(71)  $A$  is a tautology if and only if  $\emptyset_{\text{PL-WFF}} \vdash A$ .

## REFERENCES

- [1] Grzegorz Bancerek. The fundamental properties of natural numbers. *Formalized Mathematics*, 1(1):41–46, 1990.
- [2] Grzegorz Bancerek. The well ordering relations. *Formalized Mathematics*, 1(1):123–129, 1990.
- [3] Grzegorz Bancerek and Krzysztof Hryniewiecki. Segments of natural numbers and finite sequences. *Formalized Mathematics*, 1(1):107–114, 1990.
- [4] Leszek Borys. On paracompactness of metrizable spaces. *Formalized Mathematics*, 3(1):81–84, 1992.
- [5] Czesław Byliński. Binary operations. *Formalized Mathematics*, 1(1):175–180, 1990.
- [6] Czesław Byliński. Functions and their basic properties. *Formalized Mathematics*, 1(1):55–65, 1990.
- [7] Czesław Byliński. Functions from a set to a set. *Formalized Mathematics*, 1(1):153–164, 1990.
- [8] Czesław Byliński. Partial functions. *Formalized Mathematics*, 1(2):357–367, 1990.
- [9] Mariusz Giero. Propositional linear temporal logic with initial validity semantics. *Formalized Mathematics*, 23(4):379–386, 2015. doi:10.1515/forma-2015-0030.
- [10] Witold Pogorzelski. *Dictionary of Formal Logic*. Wydawnictwo UWB - Białystok, 1992.
- [11] Witold Pogorzelski. *Notions and theorems of elementary formal logic*. Wydawnictwo UWB - Białystok, 1994.
- [12] Piotr Rudnicki and Andrzej Trybulec. On some equivalents of well-foundedness. *Formalized Mathematics*, 6(3):339–343, 1997.
- [13] Andrzej Trybulec. Defining by structural induction in the positive propositional language. *Formalized Mathematics*, 8(1):133–137, 1999.
- [14] Anita Wasilewska. *An Introduction to Classical and Non-Classical Logics*. SUNY Stony Brook, 2005.
- [15] Edmund Woronowicz. Relations and their basic properties. *Formalized Mathematics*, 1(1):73–83, 1990.

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