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Niven's Theorem[1]

Artur Korniłowicz
Institute of Informatics
University of Białystok
Poland

Adam Naumowicz
Institute of Informatics
University of Białystok
Poland

Summary. This article formalizes the proof of Niven's theorem [12] which states that if x/π and $sin(x)$ are both rational, then the sine takes values 0, $\pm 1/2$, and ± 1. The main part of the formalization follows the informal proof presented at Pr∞fWiki (`https://proofwiki.org/wiki/Niven's_Theorem#Source_of_Name`). For this proof, we have also formalized the rational and integral root theorems setting constraints on solutions of polynomial equations with integer coefficients [8, 9].

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From now on r, t denote real numbers, i denotes an integer, k, n denote natural numbers, p denotes a polynomial over $\mathbb{R}_{\rm F}$, e denotes an element of $\mathbb{R}_{\rm F}$, L denotes a non empty zero structure, and z, z_0, z_1, z_2 denote elements of L.

Now we state the propositions:

(1) Let us consider complexes a, b, c, d. If $b \neq 0$ and $\frac{a}{b} = \frac{c}{d}$, then $a = \frac{b \cdot c}{d}$.

(2) Let us consider real numbers a, b. If $|a| = b$, then $a = b$ or $a = -b$.

(3) If $|i| \leqslant 2$, then $i = -2$ or $i = -1$ or $i = 0$ or $i = 1$ or $i = 2$. The theorem is a consequence of (2).

(4) If $n \neq 0$, then $i \mid i^n$.

(5) If $t > 0$, then there exists i such that $t \cdot i \leqslant r \leqslant t \cdot (i + 1)$.

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PROOF: Define $\mathcal{P}[\text{integer}] \equiv t \cdot \$_1 \leqslant r$. There exists an integer i_1 such that $\mathcal{P}[i_1]$. Set $F = \lceil \frac{r}{t} \rceil$. For every integer i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leqslant F$. Consider i such that $\mathcal{P}[i]$ and for every integer i_1 such that $\mathcal{P}[i_1]$ holds $i_1 \leqslant i$ from [15, Sch. 6]. □

(6) Let us consider a finite sequence p of elements of $\mathbb{R}_{\mathrm{F}}$, and a real-valued finite sequence q. If $p = q$, then $\sum p = \sum q$.
PROOF: Define $\mathcal{P}[\text{finite sequence}] \equiv$ for every finite sequence p of elements of $\mathbb{R}_{\mathrm{F}}$ for every real-valued finite sequence q such that $p = q$ and $p = \$_1$ holds $\sum p = \sum q$. $\mathcal{P}[\emptyset]$ by [16, (43)], [4, (72)]. For every finite sequence f and for every object x such that $\mathcal{P}[f]$ holds $\mathcal{P}[f \frown \langle x \rangle]$ by [2, (36), (38)], [5, (31)], [16, (41), (44)]. For every finite sequence f, $\mathcal{P}[f]$ from [2, Sch. 3]. □

(7) Let us consider a natural number i, and an element r of $\mathbb{R}_{\mathrm{F}}$. Then $\mathrm{power}_{\mathbb{R}_{\mathrm{F}}}(r, i) = r^i$.
PROOF: Define $\mathcal{P}[\text{natural number}] \equiv \mathrm{power}_{\mathbb{R}_{\mathrm{F}}}(r, \$_1) = r^{\$_1}$. For every natural number n, $\mathcal{P}[n]$ from [1, Sch. 2]. □

(8) $\sin(\frac{5 \cdot \pi}{6}) = \frac{1}{2}$.

(9) $\sin(\frac{5 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = \frac{1}{2}$.

(10) $\sin(\frac{7 \cdot \pi}{6}) = -\frac{1}{2}$.

(11) $\sin(\frac{7 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$.

(12) $\sin(\frac{11 \cdot \pi}{6}) = -\frac{1}{2}$.

(13) $\sin(\frac{11 \cdot \pi}{6} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$.

(14) $\cos(\frac{4 \cdot \pi}{3}) = -\frac{1}{2}$.

(15) $\cos(\frac{4 \cdot \pi}{3} + 2 \cdot \pi \cdot i) = -\frac{1}{2}$.

(16) $\cos(\frac{5 \cdot \pi}{3}) = \frac{1}{2}$.

(17) $\cos(\frac{5 \cdot \pi}{3} + 2 \cdot \pi \cdot i) = \frac{1}{2}$.

(18) If $0 \leqslant r \leqslant \frac{\pi}{2}$ and $\cos r = \frac{1}{2}$, then $r = \frac{\pi}{3}$.

(19) Let us consider an add-associative, right zeroed, right complementable, left distributive, non empty double loop structure L, and a sequence p of L. Then $\mathbf{0}.\, L * p = \mathbf{0}.\, L$.

Let us consider L, z, and n. One can verify that $\mathbf{0}.\, L +\cdot (n, z)$ is finite-Support as a sequence of L.

Let us consider a polynomial p over L. Now we state the propositions:

(20) If $z \neq 0_L$, then if $p = \mathbf{0}.\, L +\cdot (n, z)$, then $\mathrm{len}\, p = n + 1$.
PROOF: the length of p is at most $n + 1$ by [1, (13)], [3, (32)], [14, (7)]. For every natural number m such that the length of p is at most m holds $n + 1 \leqslant m$ by [14, (13)], [3, (31)], [1, (13)]. □

(21) If $z \neq 0_L$, then if $p = \mathbf{0}.\,L +\cdot (n, z)$, then $\deg p = n$. The theorem is a consequence of (20).

Note that $\mathbf{0}.\,\mathbb{R}_{\mathrm{F}}$ is $\mathbb{Z}$-valued and $\mathbf{1}.\,\mathbb{R}_{\mathrm{F}}$ is $\mathbb{Z}$-valued and there exists an element of $\mathbb{R}_{\mathrm{F}}$ which is integer.

Now we state the proposition:

(22) $\mathrm{rng}\langle z\rangle = \{z, 0_L\}$.

Proof: Set $p = \langle z\rangle$. $\mathrm{rng}\,p \subseteq \{z, 0_L\}$ by [11, (32)], [1, (14)]. $\square$

Let us consider L, z_0, z_1, and z_2. The functor $\langle z_0, z_1, z_2\rangle$ yielding a sequence of L is defined by the term

(Def. 1) $((\mathbf{0}.\,L +\cdot (0, z_0)) +\cdot (1, z_1)) +\cdot (2, z_2)$.

Now we state the propositions:

(23) $\langle z_0, z_1, z_2\rangle(0) = z_0$.

(24) $\langle z_0, z_1, z_2\rangle(1) = z_1$.

(25) $\langle z_0, z_1, z_2\rangle(2) = z_2$.

(26) If $3 \leqslant n$, then $\langle z_0, z_1, z_2\rangle(n) = 0_L$.

Let us consider L, z_0, z_1, and z_2. Let us observe that $\langle z_0, z_1, z_2\rangle$ is finite-Support.

Now we state the propositions:

(27) $\mathrm{len}\langle z_0, z_1, z_2\rangle \leqslant 3$. The theorem is a consequence of (26).

(28) If $z_2 \neq 0_L$, then $\mathrm{len}\langle z_0, z_1, z_2\rangle = 3$. The theorem is a consequence of (25) and (26).

(29) Let us consider a right zeroed, non empty additive loop structure L, and elements z_0, z_1 of L. Then $\langle z_0\rangle + \langle z_1\rangle = \langle z_0 + z_1\rangle$.

(30) Let us consider a right zeroed, non empty additive loop structure L, and elements z_0, z_1, z_2, z_3 of L. Then $\langle z_0, z_1\rangle + \langle z_2, z_3\rangle = \langle z_0 + z_2, z_1 + z_3\rangle$.

(31) Let us consider a right zeroed, non empty additive loop structure L, and elements z_0, z_1, z_2, z_3, z_4, z_5 of L. Then $\langle z_0, z_1, z_2\rangle + \langle z_3, z_4, z_5\rangle = \langle z_0 + z_3, z_1 + z_4, z_2 + z_5\rangle$. The theorem is a consequence of (23), (24), (25), and (26).

(32) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and an element z_0 of L. Then $-\langle z_0\rangle = \langle -z_0\rangle$.

(33) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and elements z_0, z_1 of L. Then $-\langle z_0, z_1\rangle = \langle -z_0, -z_1\rangle$.

(34) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and elements z_0, z_1, z_2 of L. Then

$-\langle z_0, z_1, z_2\rangle = \langle -z_0, -z_1, -z_2\rangle$. The theorem is a consequence of (23), (24), (25), and (26).

(35) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and elements z_0, z_1 of L. Then $\langle z_0\rangle - \langle z_1\rangle = \langle z_0 - z_1\rangle$. The theorem is a consequence of (32) and (29).

(36) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and elements z_0, z_1, z_2, z_3 of L. Then $\langle z_0, z_1\rangle - \langle z_2, z_3\rangle = \langle z_0 - z_2, z_1 - z_3\rangle$. The theorem is a consequence of (33) and (30).

(37) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and elements z_0, z_1, z_2, z_3, z_4, z_5 of L. Then $\langle z_0, z_1, z_2\rangle - \langle z_3, z_4, z_5\rangle = \langle z_0 - z_3, z_1 - z_4, z_2 - z_5\rangle$. The theorem is a consequence of (34) and (31).

(38) Let us consider an add-associative, right zeroed, right complementable, left distributive, unital, associative, non empty double loop structure L, and elements z_0, z_1, z_2, x of L. Then $\mathrm{eval}(\langle z_0, z_1, z_2\rangle, x) = z_0 + z_1 \cdot x + z_2 \cdot x \cdot x$. The theorem is a consequence of (23), (24), (27), and (25).

Let a be an integer element of $\mathbb{R}_{\mathrm{F}}$. Note that $\langle a\rangle$ is $\mathbb{Z}$-valued.

Let a, b be integer elements of $\mathbb{R}_{\mathrm{F}}$. One can verify that $\langle a, b\rangle$ is $\mathbb{Z}$-valued.

Let a, b, c be integer elements of $\mathbb{R}_{\mathrm{F}}$. Observe that $\langle a, b, c\rangle$ is $\mathbb{Z}$-valued and there exists a polynomial over $\mathbb{R}_{\mathrm{F}}$ which is monic and $\mathbb{Z}$-valued and there exists a finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$ which is $\mathbb{Z}$-valued.

Let F be a $\mathbb{Z}$-valued finite sequence of elements of $\mathbb{R}_{\mathrm{F}}$. One can check that $\sum F$ is integer.

Let f be a $\mathbb{Z}$-valued sequence of $\mathbb{R}_{\mathrm{F}}$. Let us note that $-f$ is $\mathbb{Z}$-valued.

Let g be a $\mathbb{Z}$-valued sequence of $\mathbb{R}_{\mathrm{F}}$. Observe that $f+g$ is $\mathbb{Z}$-valued and $f-g$ is $\mathbb{Z}$-valued and $f * g$ is $\mathbb{Z}$-valued.

Now we state the proposition:

(39) Let us consider a non degenerated, non empty double loop structure L, and an element z of L. Then $\mathrm{LC}\langle z, 1_L\rangle = 1_L$.

Let L be a non degenerated, non empty double loop structure and z be an element of L. One can check that $\langle z, 1_L\rangle$ is monic.

Now we state the proposition:

(40) Let us consider a non degenerated, non empty double loop structure L, and elements z_1, z_2 of L. Then $\mathrm{LC}\langle z_1, z_2, 1_L\rangle = 1_L$. The theorem is a consequence of (28) and (25).

Let L be a non degenerated, non empty double loop structure and z_1, z_2 be elements of L. Let us observe that $\langle z_1, z_2, 1_L\rangle$ is monic.

Let p be a $\mathbb{Z}$-valued polynomial over $\mathbb{R}_{\mathrm{F}}$. Let us note that $\mathrm{LC}\, p$ is integer.

Now we state the proposition:

(41) Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L, and a polynomial p over L. Then $\deg(-p) = \deg p$.

Let us consider an add-associative, right zeroed, right complementable, non empty additive loop structure L and polynomials p, q over L. Now we state the propositions:

(42) If $\deg p > \deg q$, then $\deg(p + q) = \deg p$.

(43) If $\deg p > \deg q$, then $\deg(p - q) = \deg p$.

(44) If $\deg p < \deg q$, then $\deg(p - q) = \deg q$.

(45) Let us consider an add-associative, right zeroed, right complementable, distributive, non degenerated double loop structure L, and a polynomial p over L. Then $\mathrm{LC}\, p = -\mathrm{LC}(-p)$.

(46) Let us consider an add-associative, right zeroed, right complementable, associative, commutative, well unital, almost left invertible, distributive, non degenerated double loop structure L, and polynomials p, q over L. Then $\mathrm{LC}(p * q) = \mathrm{LC}\, p \cdot \mathrm{LC}\, q$. The theorem is a consequence of (19).

Let us consider an add-associative, right zeroed, right complementable, distributive, non degenerated double loop structure L, a monic polynomial p over L, and a polynomial q over L. Now we state the propositions:

(47) If $\deg p > \deg q$, then $p + q$ is monic. The theorem is a consequence of (42).

(48) If $\deg p > \deg q$, then $p - q$ is monic. The theorem is a consequence of (43).

Let L be an add-associative, right zeroed, right complementable, associative, commutative, well unital, almost left invertible, distributive, non degenerated double loop structure and p, q be monic polynomials over L. Let us note that $p * q$ is monic.

Now we state the propositions:

(49) Let us consider an Abelian, add-associative, right zeroed, right complementable, unital, distributive, non empty double loop structure L, elements z_1, z_2 of L, and a polynomial p over L. Suppose $\mathrm{eval}(p, z_1) = z_2$. Then $\mathrm{eval}(p - \langle z_2 \rangle, z_1) = 0_L$.

(50) Rational root theorem:
Let us consider a $\mathbb{Z}$-valued polynomial p over $\mathbb{R}_{\mathrm{F}}$, and an element e of $\mathbb{R}_{\mathrm{F}}$. Suppose e is a root of p. Let us consider integers k, l. Suppose $l \neq 0$ and $e = \frac{k}{l}$ and k and l are relatively prime. Then

(i) $k \mid p(0)$, and

(ii) $l \mid \mathrm{LC}\, p$.

The theorem is a consequence of (7), (6), and (4).

(51) Integral root theorem:
Let us consider a monic, $\mathbb{Z}$-valued polynomial p over $\mathbb{R}_{\mathrm{F}}$, and a rational element e of $\mathbb{R}_{\mathrm{F}}$. If e is a root of p, then e is integer. The theorem is a consequence of (50).

(52) Suppose $1 \leqslant n$ and $e = 2 \cdot \cos t$. Then there exists a monic, $\mathbb{Z}$-valued polynomial p over $\mathbb{R}_{\mathrm{F}}$ such that

(i) $\mathrm{eval}(p, e) = 2 \cdot \cos(n \cdot t)$, and

(ii) $\deg p = n$, and

(iii) if $n = 1$, then $p = \langle 0_{\mathbb{R}_{\mathrm{F}}}, 1_{\mathbb{R}_{\mathrm{F}}} \rangle$, and

(iv) if $n = 2$, then there exists an element r of $\mathbb{R}_{\mathrm{F}}$ such that $r = -2$ and $p = \langle r, 0_{\mathbb{R}_{\mathrm{F}}}, 1_{\mathbb{R}_{\mathrm{F}}} \rangle$.

Proof: Define $\mathcal{P}[\text{natural number}] \equiv$ if $1 \leqslant \$_1$, then there exists a monic, $\mathbb{Z}$-valued polynomial p over $\mathbb{R}_{\mathrm{F}}$ such that $\mathrm{eval}(p, e) = 2 \cdot \cos(\$_1 \cdot t)$ and $\deg p = \$_1$ and if $\$_1 = 1$, then $p = \langle 0_{\mathbb{R}_{\mathrm{F}}}, 1_{\mathbb{R}_{\mathrm{F}}} \rangle$ and if $\$_1 = 2$, then there exists an element r of $\mathbb{R}_{\mathrm{F}}$ such that $r = -2$ and $p = \langle r, 0_{\mathbb{R}_{\mathrm{F}}}, 1_{\mathbb{R}_{\mathrm{F}}} \rangle$. $\mathcal{P}[1]$ by [11, (48), (40)]. $\mathcal{P}[2]$ by [6, (7)], (38), (28). For every non zero natural number k such that $\mathcal{P}[k]$ and $\mathcal{P}[k+1]$ holds $\mathcal{P}[k+2]$ by [1, (13)], [13, (38)], (48), [10, (24)]. For every non zero natural number k, $\mathcal{P}[k]$ from [7, Sch. 1]. $\square$

(53) If $0 \leqslant r \leqslant \frac{\pi}{2}$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational, then $r \in \{0, \frac{\pi}{3}, \frac{\pi}{2}\}$. The theorem is a consequence of (52), (1), (49), (48), (51), (3), and (18).

(54) Suppose $2 \cdot \pi \cdot i \leqslant r \leqslant \frac{\pi}{2} + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{2 \cdot \pi \cdot i, \frac{\pi}{3} + 2 \cdot \pi \cdot i, \frac{\pi}{2} + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (53).

(55) If $\frac{\pi}{2} \leqslant r \leqslant \pi$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational, then $r \in \{\frac{\pi}{2}, \frac{2 \cdot \pi}{3}, \pi\}$. The theorem is a consequence of (53).

(56) Suppose $\frac{\pi}{2} + 2 \cdot \pi \cdot i \leqslant r \leqslant \pi + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\frac{\pi}{2} + 2 \cdot \pi \cdot i, \frac{2 \cdot \pi}{3} + 2 \cdot \pi \cdot i, \pi + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (55).

(57) Suppose $\pi \leqslant r \leqslant \frac{3 \cdot \pi}{2}$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\pi, \frac{4 \cdot \pi}{3}, \frac{3 \cdot \pi}{2}\}$. The theorem is a consequence of (53).

(58) Suppose $\pi + 2 \cdot \pi \cdot i \leqslant r \leqslant \frac{3 \cdot \pi}{2} + 2 \cdot \pi \cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\pi + 2 \cdot \pi \cdot i, \frac{4 \cdot \pi}{3} + 2 \cdot \pi \cdot i, \frac{3 \cdot \pi}{2} + 2 \cdot \pi \cdot i\}$. The theorem is a consequence of (57).

(59) Suppose $\frac{3 \cdot \pi}{2} \leqslant r \leqslant 2 \cdot \pi$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\frac{3 \cdot \pi}{2}, \frac{5 \cdot \pi}{3}, 2 \cdot \pi\}$. The theorem is a consequence of (53).

(60) Suppose $\frac{3\cdot\pi}{2}+2\cdot\pi\cdot i \leqslant r \leqslant 2\cdot\pi+2\cdot\pi\cdot i$ and $\frac{r}{\pi}$ is rational and $\cos r$ is rational. Then $r \in \{\frac{3\cdot\pi}{2}+2\cdot\pi\cdot i, \frac{5\cdot\pi}{3}+2\cdot\pi\cdot i, 2\cdot\pi+2\cdot\pi\cdot i\}$. The theorem is a consequence of (59).

(61) If $\frac{r}{\pi}$ is rational and $\cos r$ is rational, then $\cos r \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$.

(62) If $0 \leqslant r \leqslant \frac{\pi}{2}$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational, then $r \in \{0, \frac{\pi}{6}, \frac{\pi}{2}\}$. The theorem is a consequence of (53).

(63) Suppose $2\cdot\pi\cdot i \leqslant r \leqslant \frac{\pi}{2}+2\cdot\pi\cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{2\cdot\pi\cdot i, \frac{\pi}{6}+2\cdot\pi\cdot i, \frac{\pi}{2}+2\cdot\pi\cdot i\}$. The theorem is a consequence of (62).

(64) If $\frac{\pi}{2} \leqslant r \leqslant \pi$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational, then $r \in \{\frac{\pi}{2}, \frac{5\cdot\pi}{6}, \pi\}$. The theorem is a consequence of (62).

(65) Suppose $\frac{\pi}{2}+2\cdot\pi\cdot i \leqslant r \leqslant \pi+2\cdot\pi\cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\frac{\pi}{2}+2\cdot\pi\cdot i, \frac{5\cdot\pi}{6}+2\cdot\pi\cdot i, \pi+2\cdot\pi\cdot i\}$. The theorem is a consequence of (64).

(66) Suppose $\pi \leqslant r \leqslant \frac{3\cdot\pi}{2}$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\pi, \frac{7\cdot\pi}{6}, \frac{3\cdot\pi}{2}\}$. The theorem is a consequence of (62).

(67) Suppose $\pi+2\cdot\pi\cdot i \leqslant r \leqslant \frac{3\cdot\pi}{2}+2\cdot\pi\cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\pi+2\cdot\pi\cdot i, \frac{7\cdot\pi}{6}+2\cdot\pi\cdot i, \frac{3\cdot\pi}{2}+2\cdot\pi\cdot i\}$. The theorem is a consequence of (66).

(68) Suppose $\frac{3\cdot\pi}{2} \leqslant r \leqslant 2\cdot\pi$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\frac{3\cdot\pi}{2}, \frac{11\cdot\pi}{6}, 2\cdot\pi\}$. The theorem is a consequence of (62).

(69) Suppose $\frac{3\cdot\pi}{2}+2\cdot\pi\cdot i \leqslant r \leqslant 2\cdot\pi+2\cdot\pi\cdot i$ and $\frac{r}{\pi}$ is rational and $\sin r$ is rational. Then $r \in \{\frac{3\cdot\pi}{2}+2\cdot\pi\cdot i, \frac{11\cdot\pi}{6}+2\cdot\pi\cdot i, 2\cdot\pi+2\cdot\pi\cdot i\}$. The theorem is a consequence of (68).

(70) NIVEN'S THEOREM:
If $\frac{r}{\pi}$ is rational and $\sin r$ is rational, then $\sin r \in \{0, 1, -1, \frac{1}{2}, -\frac{1}{2}\}$.

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