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QE-transitive abelian groups

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Abstract. Let G be a torsion-free abelian group and n be a (finite) cardinal. The group G is called n-QE-transitive if for any pure subgroups X, Y of rank n of G, there exists some endomorphism φ of G such that $\varphi(X) \subseteq Y$ and $Y/\varphi(X)$ is a torsion abelian group. We investigate these groups and obtain results similar to the ones for E-transitive groups.

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1 Introduction

The word 'group' in this paper will always mean 'torsion-free abelian group' and our notations are standard as in [9] or [12]. Let G be a group. Then $\mathrm{End}(G)$ denotes the ring of endomorphisms of G and, for any cardinal n, we use $\mathcal{L}_n(G)$ to denote the set of all pure subgroups of rank n of G. Recall that the group G is called strongly homogeneous if $\mathrm{Aut}(G)$ operates transitively on the set $\mathcal{L}_1(G)$, and G is called E-transitive if for all $X, Y \in \mathcal{L}_1(G)$ there exists some $\gamma \in \mathrm{End}(G)$ such that $\gamma(X) = Y$. We will generalize these notions by introducing the following:

Definition 1.1. Let G be a group and n be a cardinal. Then G is n-QE-transitive if for all $X, Y \in \mathcal{L}_n(G)$ there exists some $\gamma \in \mathrm{End}(G)$ such that $\gamma(X) \subseteq Y$ and $Y/(\gamma(X))$ is torsion, i.e. Y is the purification of $\gamma(X)$ in G. If G is 1-QE-transitive, we simply call the group G QE-transitive.

Strongly homogeneous groups G of finite rank were characterized in [1] and strongly homogeneous groups G of infinite rank were investigated by Krylov in [13]. It was shown that $G \cong A \otimes_{\mathbb{Z}} F$, where A is some group of rank 1 and F is an $\aleph_1$-free module over the strongly homogeneous ring $C = \mathrm{Center}(\mathrm{End}(G))$. Thus, if G is countable, then F is a free C-module. J. Hausen obtained the same conclusion for E-transitive groups in [10] and extensions of this result can be found in [11] and [6].

Note that "QE-transitive" means "E-transitive" in the quasi-category of torsion-free abelian groups. In Section 4 we present an example to show that QE-transitive is indeed weaker than E-transitive. Here is a list of some of our results:

- If G is QE-transitive, then G is homogeneous, i.e. all non-zero elements of G have the same type.
- We present examples of groups G which are QE-transitive but not E-transitive.
- G is n-QE-transitive if and only if $\mathbb{Q}\,\mathrm{End}(G)$ operates transitively on $\mathcal{L}_n(\mathbb{Q}G)$ ($\mathbb{Q}\_$ is short for $\mathbb{Q} \otimes_{\mathbb{Z}} \_$).
- If G is n-QE-transitive and $1 \leq n < \mathrm{rank}(G)$, then G is QE-transitive.
- If G is QE-transitive of infinite rank such that $D := \{\delta \in \mathrm{End}_{\mathbb{Q}}(\mathbb{Q}G) : \gamma\delta = \gamma\delta$ for all $\gamma \in \mathrm{End}(G)\} = \mathbb{Q}$, then G is n-QE-transitive for all $n \in \mathbb{N}$.
- Let G be a torsion-free 2-QE-transitive group of rank at least 4. Then G is n-QE-transitive for all $n \in \mathbb{N}$ and $D = \mathbb{Q}$. Moreover, pure, countable subgroups of G are homogeneous completely decomposable.
- Let G be a countable torsion-free 2-QE-transitive group of rank at least 4. Then G is homogeneous completely decomposable and thus strongly homogeneous.
- There exist 2-QE-transitive groups of rank 3 which are strongly indecomposable but not strongly homogeneous with $D = \mathbb{Q}\,\mathrm{End}(G)$ of rank 3.
- Let F be a free module over a Dedekind domain S. Let F_1, F_2 be infinite rank submodules of F such that F_1 is pure and $\mathrm{rank}(F_1) \geq \mathrm{rank}(F_2)$. Then there exists some $\varphi \in \mathrm{End}_S(F)$ such that $\varphi(F_1) = F_2 = \varphi(F)$.
- Let G be a torsion-free $\aleph_0$-QE-transitive group of infinite rank. Then one of the following holds:
  (1) G is n-QE-transitive for all $n \in \mathbb{N}$ and $D = \mathbb{Q}$.
  (2) There exists a pure, finite rank fully invariant subgroup $V \neq 0$ such that G/V belongs to case (1).
  (3) G is the union of a countable chain of pure finite rank fully invariant subgroups.
- We will give examples to show that all three cases in the previous item actually occur. Note that groups with properties (2) or (3) are *not* QE-transitive.

2 QE-transitive groups

For a group G, let $R = \mathrm{End}(G)$ denote the ring of endomorphisms of G. Note that if G is a group, then G is a faithful left R-module, and we set $\mathbb{Q}G = \mathbb{Q} \otimes_{\mathbb{Z}} G$ and we have that the vector space $\mathbb{Q}G$ is a faithful left module over the $\mathbb{Q}$-algebra $\mathbb{Q}\,\mathrm{End}(G)$. The next proposition follows directly from the definitions:

Proposition 2.1. *Let G be a group, n a cardinal and $R = \mathrm{End}(G)$. Then the following are equivalent:*

(1) *G is n-QE-transitive.*

(2) *$\mathbb{Q}R$ operates transitively on the set $\mathcal{L}_n(\mathbb{Q}G)$.*

It is easy to see that any QE-transitive group is homogeneous of some type τ: Let X, Y be pure subgroups of rank 1 of G. Then there is some $\varphi \in \mathrm{End}(G)$ such that $\varphi(X) \neq \{0\}$ is a subgroup of Y, which is the purification of $\varphi(X)$ in G. It follows that $\mathrm{type}(X) \leq \mathrm{type}(\varphi(X)) \leq \mathrm{type}(Y)$. By symmetry, $\mathrm{type}(X) = \mathrm{type}(Y)$.

Recall that a group G is called irreducible if $\{0\}$ and G are the only *pure*, fully invariant subgroups of G. Next we show:

Proposition 2.2. *Let G be a group. Then:*

(1) *G is QE-transitive if and only if G is irreducible.*

(2) *G is QE-transitive if and only if $\mathbb{Q}G$ is a simple $\mathbb{Q}R$-module.*

(3) *If G is n-QE-transitive and $1 \leq n < \mathrm{rank}(G)$, then G is QE-transitive.*

Proof. To show (1), assume that G is QE-transitive and let $\{0\} \neq P$ be a pure, fully invariant subgroup of G. There exists some $X \in \mathcal{L}_1(G)$ with $X \subseteq P$. For any $Y \in \mathcal{L}_1(G)$ there is some $\gamma \in R$ such that $\{0\} \neq \gamma(X) \subseteq Y$ and Y is the purification of $\gamma(X)$ in G. It follows that $Y \subseteq P$ and thus $P = G$. On the other hand, if G is irreducible, and $X \in \mathcal{L}_1(G)$, then the purification of $R(X)$ is a pure, fully invariant subgroup of G and thus equal to G. Let $Y \in \mathcal{L}_1(G)$. Then there exists some $\gamma \in \mathrm{End}(G)$ such that $\{0\} \neq \gamma(X) \subseteq Y$ and we have that G is QE-transitive.

Statement (2) follows from (1). To show (3), let $\{0\} \neq P$ be a pure, fully invariant subgroup of G. Assume $\mathrm{rank}(P) \geq n$. Then each $Y \in \mathcal{L}_1(G)$ is contained in a pure subgroup C of G of rank n and P contains a pure subgroup A of rank n. There is some $\gamma \in R$ such that $\gamma(A) \subseteq C$ and C is the purification of $\gamma(A)$ which is contained in P. It follows that $Y \subseteq P$ and thus $P = G$ and G is irreducible. By (1), G is QE-transitive. Now assume that $k = \mathrm{rank}(P) < n$. Let X be a pure rank n subgroup of G such that $P \subseteq X$. Let Y be a pure rank n subgroup of G. Then there exists some $\varphi \in \mathrm{End}(G)$ such that $\varphi(X) \subseteq Y$ with $Y/\varphi(X)$ torsion.

The map $\varphi\restriction_X$ is injective and $\varphi(P) \subset P$, which implies that P is the purification of $\varphi(P)$ and thus $P \subseteq Y$. It follows that all pure, rank n subgroups of G contain P. This can only happen if $n = \text{rank}(G)$, a contradiction. □

We will fix the following:

Notation 2.3. Let G be a QE-transitive group and $R = \text{End}(G)$. We view G as a left R-module. Let $D = \{\delta \in \text{End}_{\mathbb{Q}}(\mathbb{Q}G) : \gamma\delta = \delta\gamma \text{ for all } \gamma \in R\}$ denote the centralizer of R in $\text{End}_{\mathbb{Q}}(\mathbb{Q}G)$. By Proposition 2.2 (2), Schur's lemma applies and D is a division ring containing $\mathbb{Q}$. It is natural to view $\mathbb{Q}G$ as a right D-vector space, since then $\mathbb{Q}G$ becomes a $\mathbb{Q}R$-D-bimodule and $\mathbb{Q}G = \bigoplus_{b\in B} bD$ is a right D-vector space with D-basis B. Moreover, all maps in $\mathbb{Q}R$ are D-linear transformations of the D-vector space $\mathbb{Q}G$. We may, and will, assume that B is a subset of G. For each $b \in B$, there is a subgroup T_b of D such that $G \cap bD = bT_b$.

Recall that the ring S with additive group S^+ is called an E-ring if for any endomorphism φ of S^+ there exists some $a \in S$ such that $\varphi(x) = ax$ for all $x \in S^+$. It is well known that E-rings are commutative with identity.

Proposition 2.4. *Let G be QE-transitive of infinite rank such that $D = \mathbb{Q}$. Then G is n-QE-transitive for all $n \in \mathbb{N}$. Moreover, if X, Y are pure finite rank subgroups of G with* $\text{rank}(X) \geq \text{rank}(Y)$*, then there exists some $\varphi \in \text{End}(G)$ with $\varphi(X) \subseteq Y$ and $Y/\varphi(X)$ torsion.*

Proof. Note that $\mathbb{Q}G$ is a simple, faithful $\mathbb{Q}R$-module. Let $\{u_i : 1 \leq i \leq n\}$ be a set of $\mathbb{Q}$-linearly independent elements of G and $\{v_i : 1 \leq i \leq n\}$ be a set of at most n elements of G. By the Jacobson Density Theorem [12], there exists some $\psi \in \mathbb{Q}R$ such that $\psi(u_i) = v_i$ for all $1 \leq i \leq n$. Thus there exists some $k \in \mathbb{N}$ such that $k\psi = \varphi \in R$ and $\varphi(u_i) = kv_i$ for all $1 \leq i \leq n$. The result follows. □

Let $b, b' \in B$. Since G is QE-transitive, there exists some $\varphi \in R$ and $m \in \mathbb{N}$ such that $\varphi(b) = mb'$. Then

$$\begin{aligned} b'mT_b &= \varphi(bT_b) = \varphi(G \cap bD) \\ &\subseteq G \cap \varphi(bD) = G \cap (\varphi(b)D) = G \cap b'mD \\ &\subseteq G \cap b'D = b'T_{b'}. \end{aligned}$$

We infer that $mT_b \subseteq T_{b'}$ and, by symmetry, T_b is quasi-equal to $T_{b'}$ for all elements $b, b' \in B$. Set $T = T_b$.

We are now ready for:

Proposition 2.5. *Let V be a finite dimensional subspace of $(\mathbb{Q}G)_D$ and consider $W = G \cap V$. Then W is quasi-isomorphic to a finite direct sum of copies of T.*

Proof. We will induct over $n = \dim_D(V)$. If $n = 1$, then we may assume that $V = bD$ for some $b \in B$ and $W = G \cap bD$ is quasi-isomorphic to T. Now let $V = U \oplus bD$ and $W_0 = G \cap U$. By induction hypothesis, W_0 is quasi-isomorphic to a finite direct sum of copies of T. By the Jacobson Density Theorem, there exists some $\rho \in \mathbb{Q}R$ such that $\rho(U) = \{0\}$ and $\rho(b) = b$. Thus there exists some $m \in \mathbb{N}$ such that $\sigma = m\rho \in R$ and $m = \sigma + (m - \sigma)$. It follows that

$$\begin{aligned} mW &\subseteq \sigma(W) + (m-\sigma)(W) \\ &\subseteq (G \cap bD) \oplus (G \cap U) = bT_b \oplus W_0 \\ &\subseteq G \cap V = W. \end{aligned}$$

This shows that W is quasi-isomorphic to $bT \oplus W_0$. □

Corollary 2.6. *Let G be QE-transitive such that $\dim((\mathbb{Q}\mathbb{G})_D) = n$ is finite. Then G is quasi-isomorphic to a direct sum of n copies of T and T is a strongly indecomposable QE-transitive group. Moreover, there exists a full rank subring S of D such that $S \subseteq T \subseteq \mathbb{Q}S = D$ and* $\operatorname{End}(T) = S$.

Proof. By Proposition 2.5, there exists a finite subset B of G such that G is quasi-isomorphic to $M = \bigoplus_{b \in B} bT_b$ and $bT_b = G \cap bD$ are quasi-isomorphic for all $b \in B$. Thus there exists some $m \in \mathbb{N}$ such that $mG \subseteq M \subseteq G$.

Let $\rho \in R$ such that $\rho(b) \in bT_b$. Then $\rho(bT_b) \subseteq G \cap bD = bT_b$. This shows that $\rho \in R_b = \{\sigma \in R : \sigma(bT_b) \subseteq bT_b\}$. Note that R_b is a subring of R.

Let $\theta \in \operatorname{End}(bT_b)$. Then θ extends to an endomorphism of M that we denote by θ again. Let $g \in G$. Then $\theta(mg) \in M \subseteq G$ and thus $m\theta \in R_b$. This shows that $m \operatorname{End}(bT_b) \subseteq (R_b)\restriction_{bT_b} \subseteq \operatorname{End}(bT_b)$.

Let $0 \neq bs, bt \in bT_b$. Since G is QE-transitive, there exists some $\alpha \in R$ and $k \in \mathbb{N}$ such that $\alpha(bs) = kbt$ and thus $\alpha(b) \in \alpha(bD) \cap G = bD \cap G = bT_b$. We infer $\alpha \in R_b$ and it follows that bT_b is QE-transitive. Moreover, D is contained in the centralizer D_b of $\operatorname{End}(bT_b)$.

Let $S_b = R_b/\operatorname{ann}_R(b) \cong (R_b)\restriction_{bT_b}$. Then

$$bS_b \subseteq b\operatorname{End}(bT_b) \subseteq bT_b \subseteq b\mathbb{Q}S_b = bD,$$

which shows that bT_b is strongly indecomposable. □

Remark 2.7. Zassenhaus has shown that if S is a ring with free additive group, then there exists some T satisfying the last sentence of Corollary 2.6. We refer to [5] and the literature referenced there for an extension of Zassenhaus' result to some rings S with free additive group of countable rank.

Next we deal with the case of countable D-dimension.

Proposition 2.8. *Let G be QE-transitive such that $\mathbb{Q}G = \bigoplus_{i\in\mathbb{N}} b_i D$ has countable D-dimension. Let $V_n = \bigoplus_{i=1}^{n} b_i D$ and $G_n = G \cap V_n$. Then the group G_n is QE-transitive and a quasi-summand of G_{n+1} for all n. Moreover, G_{n+1}/G_n is quasi-isomorphic to T. Note that $G = \bigcup_{n\geq 1} G_n$.*

Proof. Note that $V = \mathbb{Q}G$ is a simple, faithful $\mathbb{Q}R$-module. Let $0 \neq g, g' \in G_n$. We may assume that $\{g, b_2, \ldots, b_n\}$ is a D-basis of V_n. By the Jacobson Density Theorem there is some $\psi \in \mathbb{Q}R$ such that $\psi(g) = g'$ and $\psi(b_i) \in G_n$. This shows that $n\psi = \varphi \in R$ for some natural number n and $\varphi\restriction_{G_n} : G_n \to G_n$ such that $\varphi(g) = ng'$. We infer that G_n is QE-transitive for all $n \geq 1$, and the rest follows from Proposition 2.5. □

Later we will construct an example of such a group G that is not quasi-isomorphic to a direct sum of copies of T.

Let R be a torsion-free ring and let G be a group such that $R \subseteq G \subseteq \mathbb{Q}R$ and $\text{End}(G) = R\cdot$. Let $D = \{\alpha \in \text{End}_{\mathbb{Q}}(\mathbb{Q}R) : \alpha\varphi = \varphi\alpha \text{ for all } \varphi \in \text{End}(G)\}$. Let $\alpha \in D$. Then $\alpha(rx) = r\alpha(x)$ for all $r \in R$. It follows that $\alpha = \cdot(\alpha(1))$ is a multiplication and thus $D = \cdot(\mathbb{Q}R)$. If $\mathbb{Q}R$ is a division ring, then G is QE-transitive.

Proposition 2.9. *Let R be a torsion-free ring and let G be a group such that $R \subseteq G \subseteq \mathbb{Q}R$ and $\text{End}(G) = R\cdot$. Then G is QE-transitive if and only if $\mathbb{Q}R$ is a division ring. Moreover, G is E-transitive if and only if $G = AR$, where R is an E-ring that is a strongly homogeneous principal ideal domain and A is some subgroup of $\mathbb{Q}$.*

Proof. The first assertion is easy to show. Assume that the group G is E-transitive. By [10, Proposition 2.3], we have $D = \mathbb{Q}\mathcal{Z}(R)$ and $D = \cdot(\mathbb{Q}R)$ by the above. This shows that R is commutative. By [10, Theorem 3.4] we have that R is a strongly homogeneous principal ideal domain. By [10, Proposition 2.2], $G = AF$ where F is E-transitive and E-cyclic. By [10, Propostion 4.3], $R = \mathcal{Z}(R)$ is an E-ring and F is a free R-module of rank 1, i.e. $F = R$. The converse follows easily from the definitions since each element of R is an integer multiple of a unit of R. □

3 2-QE-transitive groups

Our goal in the section is to prove the following

Theorem 3.1. *Let G be a torsion-free, 2-QE-transitive abelian group with the property that $\text{rank}(G) \geq 4$. Then G is n-QE-transitive for all $n \in \mathbb{N}$. Moreover, if H is a pure subgroup of G with $\text{rank}(H) \leq \aleph_0$, then H is homogeneous completely decomposable.*

We maintain the notations of the previous section.

Proposition 3.2. *Let G be a 2-QE-transitive group. Then either $bD = \mathbb{Q}G$ for any $0 \neq b \in G$ or $D = \mathbb{Q}$. In other words,* $\dim_D(\mathbb{Q}G) = 1$ *or else* $\dim_{\mathbb{Q}}(D) = 1$.

Proof. Suppose $\dim_D(\mathbb{Q}G) > 1$ and $\dim_{\mathbb{Q}}(D) > 1$. Pick an element $d \in D - \mathbb{Q}$ and $b, b' \in G$ some D-linearly independent elements of G. Then there exists some $\gamma \in R = \mathrm{End}(G)$ such that $\gamma(\{b, bd\}) \subseteq \langle b, b' \rangle_*$ such that $\langle b, b' \rangle_* \,/\, \langle \gamma(\langle b, bd \rangle) \rangle$ is a torsion group. But $\gamma(bd) = \gamma(b)d \in \gamma(b)D$ and $\gamma(\{b, bd\})$ is contained in a 1-dimensional D-subspace of $\mathbb{Q}G$. A contradiction. □

Before we proceed, we recall a result due to Baer [3], see also [9, Proposition 86.5].

Proposition 3.3 ([3]). *Let C be a pure subgroup of the torsion-free group A such that:*

(a) *A/C is completely decomposable and homogeneous of type τ,*

(b) *all elements in $A - C$ are of type τ.*

Then C is a direct summand of A

This proposition can be applied to show that any finite extension A of a homogeneous, completely decomposable group C is completely decomposable and thus A and C are isomorphic. We are now ready for:

Theorem 3.4. *Let G be 2-QE-transitive. There are two cases:*

- *$R \subseteq G \subseteq D = \mathbb{Q}R$, $R = \mathrm{End}(G)$ and D is a field with $[D : \mathbb{Q}] \leq 3$,*
- *$D = \mathbb{Q}$ and each pure, countable rank subgroup of G is completely decomposable and homogeneous of type τ.*

In the latter case, G is n-QE-transitive for all finite $n <$ $\mathrm{rank}(G)$.

Proof. First we deal with the case of $\mathbb{Q}G = D$. Let $u, v \in R$ such that $\langle 1, u \rangle_*$ and $\langle 1, v \rangle_*$ are pure rank-2 subgroups of G. Then there exists an $r \in R$ such that $r \langle 1, u \rangle_* \subseteq \langle 1, v \rangle_*$ and has rank 2. It follows that $r \in \mathbb{Q}(v)$, the subfield of D generated by v over $\mathbb{Q}$ and $ru \in \mathbb{Q}(v)$. We infer that $u \in \mathbb{Q}(v)$ and thus $D = \mathbb{Q}(v)$ is a field. If $v^2 \in \mathbb{Q}$, then we are done. Now assume that $\{1, v^2\}$ and $\{1, v\}$ are linearly independent over $\mathbb{Q}$. Then there exists an $r \in \mathbb{Q} \oplus \mathbb{Q}v$ such that $rv^2 \in \mathbb{Q} \oplus \mathbb{Q}v$ since G is 2-QE-transitive. It follows that the minimal polynomial of v over $\mathbb{Q}$ has degree at most 3. In the second case, we have $D = \mathbb{Q}$ and $G \cap bD = bT_b$ for some subgroup T_b of $\mathbb{Q}$, all of which have the same type τ and thus are all isomorphic to each other. By Proposition 3.3, we get that each G_n is a direct summand in G_{n+1} and each G_n is completely decomposable of type τ. □

Corollary 3.5. *Let G be a 2-QE-transitive group such that $4 \leq \operatorname{rank}(G) \leq \aleph_0$. Then G is homogeneous completely decomposable.*

Corollary 3.6. *Let G be a 2-QE-transitive group of rank at least 4. Let $n \in \mathbb{N}$ and $\{u_i : 1 \leq i \leq n\}$ be a linearly independent set of elements of G. Let $\{v_i : 1 \leq i \leq n\}$ be a set of elements of G. Then there exists some $\varphi \in \operatorname{End}(G)$ and $0 \neq m \in \mathbb{Z}$ such that $\varphi(u_i) = m v_i$ for all $1 \leq i \leq n$.*

Proof. By Theorem 3.4, we have that G is QE-transitive and $D = \mathbb{Q}$. As in the proof of Proposition 2.4, we may apply Jacobson's density theorem and find some $\psi \in \mathbb{Q}R$ such that $\psi(u_i) = v_i$ for all $1 \leq i \leq n$. There exists some $m \in \mathbb{N}$ such that $\varphi = m\psi \in R$ has the desired property. □

Let G be a homogeneous group of type $\tau = (\tau_p)_{p \in \mathbb{P}}$. Let

$$P = \{p \in \mathbb{P} : \tau_p = \infty\} \quad \text{and} \quad Q = \{p \in \mathbb{P} : 1 \leq \tau_p < \infty\}.$$

Let $S = \mathbb{Z}[\frac{1}{p} : p \in P]$, a subring of $\mathbb{Q}$ of type σ, and $A = \sum_{p \in Q} p^{-\tau_p}\mathbb{Z}$. Then AS, which is short for $A \otimes_{\mathbb{Z}} S$, has type τ. Define $L(G) = \bigcap_{p \in Q} p^{\tau_p} G$.

Claim 3.7. *We have $G = AL(G)$ and $L(G)$ is a S-module of type σ.*

Proof. Clearly, $AL(G) \subseteq G$. Let $0 \neq g \in G$. Let $Q_0 = \{p \in Q : h_p(g) = \tau_p\}$, $Q_1 = \{p \in Q : h_p(g) > \tau_p\}$, $Q_2 = \{p \in Q : h_p(g) < \tau_p\}$. Then Q_1 and Q_2 are finite sets. Let $m = \prod_{p \in Q_2} p^{\tau_p - e_p}$, where $e_p = h_p(g)$. Then

$$mg \in L(G) \quad \text{and} \quad \frac{1}{m} = \prod_{p \in Q_2} \frac{p^{e_p}}{p^{\tau_p}} \in A.$$

It follows that $g = \frac{1}{m}(mg) \in AL(G)$ and thus $G = AL(G)$. □

Note that $H = L(G)$ is a fully invariant subgroup of G and since $\mathbb{Z} = \operatorname{End}(A)$, it is easy to see that $\operatorname{End}(G) = \operatorname{End}(H)$.

Corollary 3.8. *Let G be a 2-QE-transitive group with $\operatorname{rank}(G) \geq 4$. Then one has $G = AH$, where A is a rank-1 group and H is a 2-QE-transitive group and an $\aleph_1$-free S-module for some subring S of $\mathbb{Q}$. Moreover, H is 2-QE-transitive with $\operatorname{End}(H) = \operatorname{End}(G)$ and S is the center of $\operatorname{End}(G)$.*

Proposition 3.9. *Let F be a field extension of $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(F) = 3$. Then $F\cdot$ operates transitively on $\mathcal{L}_2(F)$.*

Proof. Pick $X, Y \in \mathcal{L}_2(F)$. We want to find an element $t \in F$ such that $tX = Y$. We may assume that $X = 1\mathbb{Q} \oplus v\mathbb{Q}$ and $Y = 1\mathbb{Q} \oplus u\mathbb{Q}$. There exists $u' \in Y$ with minimal polynomial $m_{u'}(x) = x^3 + ax + b \in \mathbb{Q}[x]$. (See [12, proof of Proposition 4.8].) We may replace u by u' without changing Y. Note that $F = \mathbb{Q}[u]$ and we may write $v = \alpha u^2 + \beta u + \gamma$ for some rational numbers α, β, γ with $\alpha \neq 0$ since otherwise $X = Y$ and there is nothing to show in this case.

Define $t = -\beta + \alpha u \in Y$. We compute

$$\begin{aligned} tv &= (-\beta + \alpha u)(\alpha u^2 + \beta u + \gamma) \\ &= u^2 0 + u(-\beta^2 - a\alpha^2 + \alpha\gamma) - \beta\gamma - \alpha^2 b \in Y. \end{aligned}$$

This shows that $tX = Y$. □

Corollary 3.10. *Let F be a field extension of $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(F) = 3$ and let G be a subgroup of the additive group of F such that $\mathbb{Q}G = F$ and $\mathbb{Q}(\mathrm{End}(G)) \supseteq F\cdot$. Then G is 2-QE-transitive.*

Corollary 3.11. *Let F be a field extension of $\mathbb{Q}$ with $\dim_{\mathbb{Q}}(F) = 3$ and let R be a subring of F with $\mathbb{Q}R = F$. If $G \subseteq F$ has rank 3 and is an R-module, then G is 2-QE-transitive.*

Recall that the subgroup B is a quasi-summand of A if there exists some $m \in \mathbb{N}$ such that, for some subgroup C of A, we have $mA \subseteq B \oplus C \subseteq A$. Note that if B is a quasi-summand of A, then B_*, the purification of B in A, is also a quasi-summand of A. We call the group A 1-*Q-separable* if each pure rank-1 subgroup of A is a quasi-summand of A. We have:

Proposition 3.12. *Let B be a pure subgroup of the 1-Q-separable group A. Then B is 1-Q-separable.*

Proof. Let X be a pure rank-1 subgroup of B. Then X is also a pure rank-1 subgroup of A and thus there exists some $m \in \mathbb{N}$ and some subgroup $C \subseteq A$ such that $mA \subseteq X \oplus C \subseteq A$. Intersecting this with B yields

$$mB = mA \cap B \subseteq (X \oplus C) \cap B = X \oplus (C \cap B) \subseteq A \cap B = B.$$

This shows that B is 1-Q-separable. □

We call the group A *Q-separable* if each pure, finite rank subgroup of A is a quasi-summand.

Proposition 3.13. *Any 1-Q-separable group A is also Q-separable.*

Proof. Let B be a pure subgroup of A of rank n. We induct over n. If $n = 1$, there is nothing to show because A is 1-Q-separable. Assume that B has rank n and let B' be a pure subgroup of B of rank $n-1$. By induction hypothesis, there exists some $m \in \mathbb{N}$ and a pure subgroup $C \subseteq A$ such that $mA \subseteq B' \oplus C \subseteq A$. There exists some $b \in B$ such that $B = \langle B', b\rangle_*$. We have that $mb = b' + c$ for some $b' \in B'$ and $c \in C \cap B$. By Proposition 3.12, we have that $\langle c\rangle_*$ is a quasi-summand of C, i.e. $kC \subseteq \langle c\rangle_* \oplus C' \subseteq C$. It follows that

$$mkA \subseteq kB' \oplus kC \subseteq kB' \oplus (\langle c\rangle_* \oplus C') \subseteq B' \oplus \langle c\rangle_* \oplus C' \subseteq A.$$

This shows that $B' \oplus \langle c\rangle_*$ is a quasi-summand of A and so is B, its purification in A. □

Corollary 3.14. *Let A be a 1-Q-separable group that is homogeneous of type τ. Then each pure finite rank subgroup B is a quasi-summand of A and B is completely decomposable and homogeneous of type τ. If C is a pure subgroup of A of countable rank, then C is completely decomposable.*

Proof. The proof of the previous proposition shows that B is a quasi-summand of A and $mkB \subseteq B' \oplus \langle c\rangle_* \subseteq B$. By induction hypothesis, B' is completely decomposable of type τ. That B is completely decomposable is an immediate consequence of [2, Theorem 2.3]. Moreover, an inspection of the proof of this result shows that B' is a direct summand of B. Thus C is the union of a chain of finite rank subgroups B_n such that B_n is a direct summand of B_{n+1} for all n and it follows that C is completely decomposable. □

Theorem 3.15. *Let G be a 2-QE-transitive group of rank at least 4 such that there exists some $\gamma \in \mathrm{End}(G)$ such that $\gamma(G)$ is countable. Then G is Q-separable.*

Proof. Let $G = H \otimes L(G)$, where H is a subgroup of $\mathbb{Q}$ where all elements have finite p-height for all primes p and $L(G)$ is an S-module for some subring S of $\mathbb{Q}$ such that G is homogeneous of type type$(H \otimes S)$. Then $L(G)$ is 2-QE-transitive and an $\aleph_1$-free S-module. This shows that $\gamma(L(G))$ is contained in some free S-module and is thus free because S is a principal ideal domain. It follows that G has a quasi-direct summand U of rank 1, i.e. $mG \subseteq U \oplus G' \subseteq G$ for suitable m, G'. Let W be a pure rank-1 subgroup of G. Since G is also 1-QE-transitive, there exists some $\psi \in \mathrm{End}(G)$ such that $kU \subseteq \psi(W) \subseteq U$ for some natural number k. Let π be the endomorphism of $U \oplus G'$ such that $\pi(G') = \{0\}$ and $\pi\restriction_U = \mathrm{id}_U$. Then π induces a map from mG to G. Let $g \in W \cap \ker(\psi\pi)$. Then $0 = (g\psi)\pi \in U\pi$. It follows that $g\psi = 0$ and thus $g = 0$ because $\psi\restriction_W$ is injective. We have that $W \cap \ker(\psi\pi) = \{0\}$. Note that $kU = kU\pi \subseteq W\psi \subseteq G\psi$ and thus $mkU = mkU\pi \subseteq mW\psi\pi \subseteq mG\psi\pi \subseteq U$. This shows that there exist natural numbers a, b that are multiples of m such that $aW\psi\pi = bG\psi\pi$ and thus for all $g \in G$ there is some $w \in W$ with $bg - aw \in G \cap \ker(\psi\pi)$. It follows that $bG \subseteq W \oplus (G \cap \ker(\psi\pi)) \subseteq G$ and we infer that W is a quasi-summand of G and G is 1-Q-separable. By Proposition 3.13, G is Q-separable. □

Corollary 3.16. *Let G be a countable 2-QE-transitive group of rank at least 4. Then G is homogeneous completely decomposable.*

4 An example

Assume that S is an E-ring with torsion-free additive group S^+. It is easy to see that S^+ is QE-transitive if and only if $\mathbb{Q}S$ is a field. On the other hand, assume that R is a strongly homogeneous E-ring, i.e. each element in R in an integer multiple of a unit of R. Then R^+ is strongly homogeneous. The ring J_p of p-adic integers is an example of such a ring.

We will construct a class of groups to illustrate the difference of the notions of E-transitive and QE-transitive.

Let S be a ring such that the additive group of S is torsion-free and p-reduced for some fixed prime number p. Let $B = \{f\} \cup \{f_i : i = 0, 1, 2, \dots\}$ be a basis of a free right S-module F of countable rank. Pick a p-adic number

$$\lambda = \sum_{\alpha=0}^{\infty} p^{\alpha}\lambda_{\alpha}$$

such that $\lambda_0 \neq 0$ and $0 \leq \lambda_\alpha < p$ for all α. Let

$$a_i = \sum_{\alpha=0}^{i-1} p^{\alpha}\lambda_{\alpha} \quad \text{and} \quad x_i = p^{-i}\left(fa_i - \sum_{\alpha=0}^{i-1} p^{\alpha} f_{\alpha}\right) \in \mathbb{Q}F.$$

Now we define

$$H = H(\lambda, S) = F + \sum_{i=1}^{\infty} x_i\mathbb{Z}.$$

It is easy to see:

Observation 4.1. *One has $H/F \cong \mathbb{Z}(p^\infty)$, which implies that H and F are not quasi-equal.*

Observation 4.2. *Let X be a finite subset of B, $b \in B - X$, and $v \in F$. Then there exists some S-linear map $\varphi \in \mathrm{End}(F_S)$ and some $m \in \mathbb{N}$ such that $\varphi(b) = vp^m$, $\varphi(X) = \{0\}$ and $\varphi(H) \subseteq F$, i.e. φ induces an endomorphism of H.*

First we consider the case $b = f$. We may assume that $X = \{f_0, f_1, \ldots, f_{k-1}\}$. There exists $\psi \in \mathrm{End}(F_S)$ such that $\psi(f) = vp^k$, $\psi(f_i) = 0$ for $0 \le i \le k-1$ and $\psi(f_\alpha) = v\lambda_{\alpha-k}$ for all $\alpha \ge k$. Now we compute for $i > k$

$$p^i\psi(x_i) = vp^k a_i - \sum_{\alpha=k}^{i-1} vp^\alpha \lambda_{\alpha-k} = v\left(\sum_{\alpha=0}^{i-1} p^{\alpha+k}\lambda_\alpha - \sum_{\alpha=0}^{i-1-k} p^{\alpha+k}\lambda_\alpha\right)$$

$$= v\sum_{\alpha=i-k}^{i-1} p^{\alpha+k} = v\sum_{\alpha=i}^{i+k-1} p^\alpha \in p^i F.$$

This shows that $\psi(x_i) \in F$ for all $i > k$. Now let $\varphi = p^k\psi$ and $m = 2k$.

Let now $b = f_i$ for some index $i \ge 0$. Pick some $k \in \mathbb{N}$ such that $i < k$ and $X \subseteq \{f\} \cup \{f_\alpha : 0 \le \alpha \le k-1, \alpha \ne i\}$. Again we define some $\varphi \in \mathrm{End}(F_S)$ by $\varphi(f_i) = vp^k$, $\varphi(f_k) = -vp^i$ and $\varphi(f) = 0 = \varphi(f_\alpha\}$ for all $\alpha \notin \{i, k\}$. For $j \ge k$ we compute

$$p^j\varphi(x_j) = -(p^i\varphi(f_i) + p^k\varphi(f_k)) = -(vp^{i+k} - vp^{k+i}) = 0.$$

This shows that $\psi = p^k\varphi$ has the desired properties.

Observation 4.3. *Assume that $\mathbb{Q}S$ is a division ring. Then H is QE-transitive.*

Let $0 \ne u \in F$ and $u = bt + x$, where $t \ne 0$, $b \in B$ and $x \in \bigoplus_{b \ne c \in B} cS$. By Observation 4.2 there exists some $\gamma \in \mathrm{End}(H) \cap \mathrm{End}(F_S)$ such that $\gamma(x) = 0$ and $\gamma(b) = bsp^k$ for some $k \in \mathbb{N}$ where $st = n \in \mathbb{N}$. Such an element s exists because $\mathbb{Q}S$ is a division ring. Let $v \in F$. There exists $\delta \in \mathrm{End}(H) \cap \mathrm{End}(F_S)$ with $\delta(b) = vp^\ell$ for some $\ell \in \mathbb{N}$. We have

$$(\delta \circ \gamma)(u) = \delta(\gamma(bt)) = \delta(\gamma(b)t) = \delta(bsp^k t) = \delta(bnp^k) = vnp^{k+\ell} \in v\mathbb{Z}.$$

This shows that H is QE-transitive.

Naturally, $F =_S F_S$ is a S-bimodule. Let

$$D = \{\alpha \in \mathrm{End}(\mathbb{Q}F) : \alpha\varphi = \varphi\alpha \text{ for all } \varphi \in \mathrm{End}(H)\}.$$

Observation 4.4. *We have $D \subseteq \cdot(\mathbb{Q}S)$.*

Let $\alpha \in D$ and $b \in B$ with the property that $\alpha(b) = bt_b + x$. Then there exists some $\varphi \in \mathrm{End}(H) \cap \mathrm{End}(F_S)$ with $\varphi(x) = 0$ and $\varphi(b) = bp^k$ for some $k \in \mathbb{N}$. Then

$$bt_b p^k = \varphi(bt_b + x) = \alpha(bp^k) = bt_b p^k + xp^k$$

and it follows that

$$\alpha(b) = bt_b \in bS = Sb.$$

Moreover, there exists some $\psi \in \mathrm{End}(H) \cap \mathrm{End}(F_S)$ such that $\psi(b) = cp^k$ and $\psi(c) = bp^k$ for any $b, c \in B$. It follows that

$$ct_b p^k = \psi(bt_b) = \psi(\alpha(b)) = \alpha(\psi(b)) = \alpha(cp^k) = ct_c p^k$$

and thus $t_b = t$ for all $b \in B$. Let $r \in S$. There exists some $\rho \in \mathrm{End}(H) \cap \mathrm{End}(F_S)$ with $\rho(b) = brp^k$. Then $brp^k t = \rho(bt) = \rho(\alpha(b)) = \alpha(\rho(b)) = \alpha(brp^k)$ and thus $\alpha(rb) = \alpha(br) = brt = r(bt) = r\alpha(b)$ for all $b \in B$ and $r \in S$. Now it is easy to see that $\alpha \in \mathrm{End}(_S(\mathbb{Q}F))$ is a S-linear map of the left S-module and $\alpha = \cdot t \in \cdot(\mathbb{Q}S)$.

We may consider Observation 4.2 by letting all φ be in $\mathrm{End}(H) \cap \mathrm{End}(_S F)$. Then we get:

Observation 4.5. *One has $D \subseteq \mathbb{Q}\mathcal{Z}(S)$, where $\mathcal{Z}(S)$ denotes the center of the ring S.*

There exists some $\psi \in \mathrm{End}(H) \cap \mathrm{End}(_S F)$ such that $\psi(b) = sp^k b$. Maintaining the above notation, we have

$$sp^k tb = \alpha(sp^k b) = \alpha(\psi(b)) = \psi(\alpha(b)) = \psi(tb) = t\psi(b) = tsp^k b.$$

It follows that $ts = st$ for all $s \in S$.

Observation 4.6. *If $\mathrm{End}_{\mathcal{Z}(S)}(S) = \mathrm{End}_{\mathbb{Z}}(S)$, then one has $D = \mathbb{Q}(\mathcal{Z}(S))$ and $\mathcal{Z}(\mathrm{End}(H)) = S_0$, where S_0 is the p-adic closure of $\mathbb{Z}$ in S. Note that $S_0 \subseteq \mathcal{Z}(S)$.*

By hypothesis, all maps in $\mathrm{End}(H)$ are $\mathcal{Z}(S)$-linear and by Obseration 4.5 we have $D = \mathbb{Q}\mathcal{Z}(S)$. Any element $s \in S_0$ induces an endomorphism of H because $S_0 = \bigcap_{n=1}^{\infty}(\mathbb{Z} + p^n S)$. Now let $s \in S$ such that $\cdot s \in \mathrm{End}(H)$. Then it is easy to see that $p^i x_i s \in p^i F + p^i x_i \mathbb{Z}$. We infer that $fa_i s \in fp^i S + fa_i \mathbb{Z}$. Since a_i is not divisible by p, we have that $s \equiv z_i \bmod p^i S$ for some integers z_i. This shows that $s \in S_0$.

Observation 4.7. *If $S = S_0$, then H is a free S-module of countable rank.*

By Obseration 4.6, H is a module over S and $H = fS \oplus \bigoplus_{i=1}^{\infty} x_i S$.

Observation 4.8. *Assume that*

(a) $\mathrm{End}_{\mathcal{Z}(S)}(S) = \mathrm{End}_{\mathbb{Z}}(S)$,

(b) *$\mathcal{Z}(S) \neq S_0$ and $\mathbb{Q}S$ is a division ring.*

Then H is QE-transitive but not E-transitive.

If H is E-transitive, then by [10, Proposition 2.3]

$$D = \mathbb{Q}(\mathcal{Z}(\mathrm{End}(H))) = \mathbb{Q}S_0 = \mathbb{Q}\mathcal{Z}(S).$$

Since S_0 and $\mathcal{Z}(S)$ are pure in S, we get the contradiction $S_0 = \mathcal{Z}(S)$.

Observation 4.9. *The subring S_0 is (isomorphic to) a p-pure subring of J_p, the ring of p-adic integers.*

Observation 4.10. *Let $H_n = fS \oplus \bigoplus_{i=0}^{n} f_i S + x_{n+1}\mathbb{Z}$. Then H_n is pure in H, $H_n \subset H_{n+1}$ and $H = \bigcup_{n=1}^{\infty} H_n$.*

Since S is p-reduced, the subring $1\mathbb{Z}$ is p-pure in S. Let $s \in S_0$. Then there exist elements $z_n \in \mathbb{Z}$ such that $s - z_n \in p^n S$. It follows that $z_{n+1} - z_n$ is an element of $p^n S \cap \mathbb{Z} = p^n\mathbb{Z}$ for all $n \in \mathbb{N}$. Thus we have a p-adic Cauchy sequence $\{z_n\}$ which converges to a limit $s^* \in J_p$. It is easy to see that $^* : S_0 \to J_p$ is an injective ring homomorphism with $1^* = 1 \in J_p$. It follows that $(S_0)^*$ is a p-pure subring of J_p. We have:

Proposition 4.11. *Let S be an E-ring of p-rank larger that 1 such that $\mathbb{Q}S$ is a field. Then H is QE-transitive but not E-transitive. Each finite rank S-submodule is quasi-contained in a finite rank free S-module. If S is a strongly homogeneous E-ring of p-rank larger than 1, then H is QE-transitive, but not E-transitive.*

Rings S with the properties mentioned in Proposition 4.11 exist in abundance. For example, let $\mathcal{O}$ be the ring of Gaussian integers, i.e. the ring of algebraic integers in the quadratic field $\mathbb{Q}[i]$. Let $p = 13$ and $P = (3 + 2i)\mathcal{O}$. Then $S = \mathcal{O}_P$, the localization of $\mathcal{O}$ at the maximal ideal P, is a strongly homogeneous E-ring by results due to Mader and Vinsonhaler [14].

5 Free modules over Dedekind domains

Remark 5.1. Let S be an integral domain and M an S-module. Then M is called separable, if for any finite subset F of M there exists a direct summand S of M such that $F \subseteq S$ and S is a free S-module of finite rank. Let S be a subring of $\mathbb{Q}$ and M a separable S-module. Let X, Y be pure submodules of M of rank n. It is easy to see that there is some automorphism α of M such that $\alpha(X) = Y$.

In this context, the following is a classical result due to Erdős [7] (see also [8, Corollary 51.5]): Let F, F' be free abelian groups and H, H' pure subgroups of F, resp. F'. The following are equivalent:

(1) $F/H \cong F'/H'$ and $\mathrm{rank}(H) = \mathrm{rank}(H')$.

(2) There exists an isomorphism $\alpha : F \to F'$ such that $\alpha(H) = H'$.

This shows that if F is a free abelian group of infinite rank, then there are plenty of pure, countable rank subgroups H, H' of F such that there is no automorphism of F mapping H onto H'. We will show that this is quite different for endomorphisms of F. If V is some vector space and X, Y subspaces of V with $\dim(X) \geq \dim(Y)$, then there exists a linear transformation τ of V such that $\tau(X) = Y = \tau(V)$. We will show that a similar result holds for free modules over Dedekind domains.

The following is well known: Let M be a finitely generated, torsion-free module over a Dedekind domain D. Then there exists a natural number $k \geq 0$ such that $M \cong D^k \oplus J$, where J is a fractional ideal of D and $\operatorname{rank}(M) = k + 1$ if $J \neq 0$. Thus, if $\operatorname{rank}(M) \geq 2$, then M contains a free direct summand.

For abelian groups, the following lemma is due to Erdős [7] and can be found in [8, Lemma 51.2].

Lemma 5.2. *Let R be a Dedekind domain and F be a free R-module of infinite rank. Let H be a pure submodule of F of infinite rank. Then there exists a direct summand T of F such that T is a free submodule of H and* $\operatorname{rank}(H) = \operatorname{rank}(T)$.

Proof. Fix a basis B of F. Let S be a set of finite subsets of B.

We say that S is a disjoint system for H if

(1) for all $X, Y \in S$ we have $X = Y$ or $X \cap Y = \varnothing$,

(2) for all $X \in S$ we have $H \cap \operatorname{span}_R(X) \neq \{0\}$.

Let Δ denote the set of all disjoint systems for H. Then Δ is partially ordered by set inclusion and it is obvious that Δ satisfies the hypothesis of Zorn's lemma. Let S be a maximal element in Δ.

Assume that $|S| < \operatorname{rank}(H) \leq |B|$. Let

$$M = \sum_{X \in S} \operatorname{span}_R(X) = \bigoplus_{X \in S} \operatorname{span}_R(X)$$

since the elements of S are disjoint. Note that $\operatorname{rank}(M) = |S| < \operatorname{rank}(H)$. Let C be the compliment of $\bigcup_{X \in S} X$ in B and $N = \operatorname{span}_R(C)$. Then $F = M \oplus N$. Assume $H \cap N = \{0\}$. Then

$$H \cong H/\{0\} = H/(H \cap N) \cong (H + N)/N \subseteq F/N \cong M,$$

a contradiction to $\operatorname{rank}(M) < \operatorname{rank}(H)$. Thus there exists some $0 \neq g \in H \cap N$ and thus $g \in \operatorname{span}_R(Y)$ for some finite subset Y of C. This shows $S \cup \{Y\} \in \Delta$, a contradiction to the maximality of S in Δ and we have that $|S| = \operatorname{rank}(H)$.

Now let

$$T = \sum_{X \in S} (H \cap \operatorname{span}_R(X)) = \bigoplus_{X \in S} (H \cap \operatorname{span}_R(X)).$$

Then $\operatorname{rank}(T) = |S| = \operatorname{rank}(H)$ and $T \subseteq H$. Moreover,

$$M/T = \bigoplus_{X \in S} \operatorname{span}_R(X)/(H \cap \operatorname{span}_R(X)).$$

Since H is a pure submodule of F, each module $\operatorname{span}_R(X)/(H \cap \operatorname{span}_R(X)$ is a finitely generated, torsion-free R-module for all $X \in S$ and thus projective since R is a Dedekind domain. This shows that M/T is projective and thus T is a direct summand of M and thus of F. Note that $H \cap \operatorname{span}_R(X)$ is finitely generated since $X \in S$ is finite and R is a Noetherian ring.

Now we decompose the set S into two disjoint subsets $S = S_1 \cup S_2$ such that $S_2 = \{X' : X \in S_1\}$ and we write

$$T = \bigoplus_{X \in S_1} [(H \cap \operatorname{span}_R(X)) \oplus (H \cap \operatorname{span}_R(X'))],$$

where each module $T_X = (H \cap \operatorname{span}_R(X)) \oplus (H \cap \operatorname{span}_R(X'))$ is a torsion-free, projective module of rank at least two. This shows that each T_X contains a copy R_X of R as a direct summand. Let $T' = \bigoplus_{X \in S_1} R_X$. Then $T' \subseteq H$ is a free summand of F with $\operatorname{rank}(T') = \operatorname{rank}(H)$. □

Theorem 5.3. *Let R be a Dedekind domain, F be a free R-module and F_1, F_2 be infinite rank submodules of F such that F_1 is pure in F and* $\operatorname{rank}(F_1) \geq \operatorname{rank}(F_2)$. *Then there exists $\varphi \in \operatorname{End}_R(F)$ such that $\varphi(F_1) = F_2 = \varphi(F)$.*

Proof. Since F_1 is pure in F, Lemma 5.2 yields a free direct summand T of F such that $T \subseteq F_1$ and $\operatorname{rank}(T) = \operatorname{rank}(F_1) \geq \operatorname{rank}(F_2)$. Let $F = T \oplus N$ and pick an $\eta \in \operatorname{Hom}(T, F_2)$ such that η is surjective. Then η extends to a $\varphi \in \operatorname{End}(F)$ by setting $\varphi\restriction_N = 0$. □

6 $\aleph_0$-QE-transitive groups

In this section, let G denote a torsion-free $\aleph_0$-QE-transitive abelian group of infinite rank, i.e. for any two pure, countable rank subgroups X and Y of G, there exists some $\varphi \in \operatorname{End}(G)$ such that $\varphi(X) \subseteq Y$ and $Y/\varphi(X)$ is torsion. Equivalently, the algebra $\mathbb{Q}\operatorname{End}(G)$ operates transitively on the set of all $\mathbb{Q}$-subspaces of $\mathbb{Q}G$ of countable dimension. As before, let $D = \{\delta \in \operatorname{End}_{\mathbb{Q}}(\mathbb{Q}G) : \gamma\delta = \delta\gamma$ for all $\gamma \in \operatorname{End}(G)\}$.

Proposition 6.1. *Let G be a torsion-free $\aleph_0$-QE-transitive abelian group of infinite rank. If P is a pure, fully invariant subgroup of G of infinite rank, then $P = G$.*

Proof. The subgroup P contains a pure subgroup P' of countable rank. Let Y be a pure, rank-1 subgroup of G. There exists a pure, countable rank subgroup V of G with $Y \subseteq V$ and there exists some $\varphi \in \mathrm{End}(G)$ such that $\varphi(P') \subseteq V$ with $V/\varphi(P')$ torsion. Since $\varphi(P') \subseteq P$ and $Y \cap \varphi(P') \neq \{0\}$, it follows that $Y \subseteq P$. This shows that $P = G$. □

Corollary 6.2. *Let G be a torsion-free $\aleph_0$-QE-transitive abelian group of infinite rank such that $\{0\}$ is the only pure, finite rank fully invariant subgroup of G. Then G is irreducible and thus QE-transitive.*

Proposition 6.3. *Let G be a torsion-free $\aleph_0$-QE-transitive abelian group of infinite rank and let V be the purification of the sum of all fully invariant subgroups of finite rank of G.*

(1) *If* rank(V) *is finite, then $G' = G/V$ is a torsion-free $\aleph_0$-QE-transitive abelian group of infinite rank and G' is irreducible.*

(2) *If* rank(V) *is infinite, then G is countable and G is the union of an ascending chain of finite rank, pure, fully invariant subgroups.*

Proof. Since V is fully invariant in G, every $\varphi \in \mathrm{End}(G)$ induces a $\varphi' \in \mathrm{End}(G')$. Let X/V and Y/V be pure subgroups of G' of countable rank. Then there exists some $\varphi \in \mathrm{End}(G)$ with $\varphi(X) \subseteq Y$ and $Y/\varphi(X)$ torsion. Then $\varphi'(X/V) \subseteq Y/V$ and $(Y/V)/\varphi'(X/V)$ is torsion. This shows that G' is $\aleph_0$-QE-transitive. Assume that F/V is a finite rank fully invariant subgroup of G'. Then $\varphi'(F/V) \subseteq F/V$ for all $\varphi \in \mathrm{End}(G)$, i.e. $\varphi(F) \subseteq F$ for all $\varphi \in \mathrm{End}(G)$. This means that F is a finite rank, fully invariant subgroup of G and thus $F \subseteq V$. By Proposition 6.1, G' is irreducible and thus QE-transitive. This shows that (1) holds. If V has infinite rank, then G contains a fully invariant, pure subgroup P of countable rank. By Proposition 6.1, $P = G$ has countable rank. □

Theorem 6.4. *Let G be a torsion-free $\aleph_0$-QE-transitive abelian group of infinite rank. If G is irreducible, then $D = \mathbb{Q}$.*

Proof. Assume that $\dim_D(\mathbb{Q}G)$ is infinite. Consider

$$X = \bigoplus\{b_i\mathbb{Z} \oplus b_i d\mathbb{Z} : i < \omega\} \quad \text{and} \quad Y = \bigoplus\{b_i\mathbb{Z} : i < \omega\},$$

where $\{b_i : i < \omega\}$ is a D-linearly independent subset of the group G. Note that we have $Y \cap Yd = \{0\}$, where $\{b_i : i < \omega\} \subset G$ is linearly independent over D and $d \in D - \mathbb{Q}$. Now there exists some $\varphi \in \mathrm{End}(G)$ such that $\varphi(X_*) \subseteq Y_*$ with a torsion quotient. Then $\varphi(b_i d) = \varphi(b_i)d \in Y_* \cap Y_*d = \{0\}$ and we have the contradiction $\varphi(X) = \{0\}$. Thus we have $D = \mathbb{Q}$ or else $\dim_D(\mathbb{Q}G)$ is finite.

In the latter case, $\dim_D(\mathbb{Q}G) = 1$, because in this case $\dim_Q(D)$ is infinite and there are no surjective D-linear maps from a 1-dimensional space onto a 2-dimensional space.

Now assume that $\mathbb{Q}G = D$. We have that $\dim_Q(D)$ is infinite. Let $1 \in X, Y$ be pure countable rank subgroups of $G \subseteq D$. Then there exists some $\varphi \in \mathrm{End}(G)$ with $\varphi(X) \subseteq Y$ and φ induces a D-linear map from D to D. Thus there exists some $d \in D$ with $Xd \subseteq Y$ and $d \in Y$. This means that X is contained in the sub-division ring S generated by Y. It follows that $S = D$ and D is countable as well as G. Suppose there is some $x \in D$ that is transcendental over $\mathbb{Q}$. It follows that D is the rational number field $\mathbb{Q}(x)$. Let $X = \mathbb{Q}[x^2]$ and $Y = \mathbb{Q}[x^4]$. Obviously, X is not contained in the subfield $\mathbb{Q}[x^4]$. A contradiction. This shows that all elements $d \in D$ are algebraic over $\mathbb{Q}$. Let Y be a pure, countable subgroup of G. Then there exists some $d \in D \cap \mathrm{End}(G)$ with $Gd \subseteq Y$. It follows that $D = Dd = \mathbb{Q}Gd \subseteq \mathbb{Q}Y \subseteq \mathbb{Q}G = D$ and thus $Y = G$ for any pure, countable rank subgroup of G, which is absurd. □

Theorem 6.5. *Let G be a torsion-free $\aleph_0$-QE-transitive abelian group of infinite rank. Then one of the following holds:*

(a) *G is irreducible and $D = \mathbb{Q}$.*

(b) *There exists a pure, finite rank fully invariant subgroup V of G such that $G' = G/V$ is irreducible and $\aleph_0$-QE-transitive.*

(c) *G is countable and is a union of an ascending chain of pure, finite rank fully invariant subgroups.*

Proof. Let P be a pure, fully invariant subgroups of G of infinite rank. Let L be a pure subgroup of countable rank of P. Let Y be a pure subgroup of rank 1 of G. Then there exists a pure, countable rank subgroup X of G such that $Y \subseteq X$. We infer that there exists some $\varphi \in \mathrm{End}(G)$ such that $\varphi(L) \subseteq X$ and $X/\varphi(L)$ torsion and thus $Y/(Y \cap \varphi(L))$ is torsion. It follows that $Y \subseteq \varphi(L) \subseteq P$ and we infer that $P = G$ and G is irreducible unless G has pure, fully invariant subgroups of finite rank. By Theorem 6.4, we have that $D = \mathbb{Q}$ if G is irreducible.

From now on, we may assume that whenever $P \neq G$ is a pure, fully invariant subgroup of G, then $\mathrm{rank}(P)$ is finite. Let $\mathcal{F}$ be the set of all pure, fully invariant subgroups of G. Then $\sum \mathcal{F}$ is fully invariant and so is its purification V.

If V has finite rank, then (b) follows from Proposition 6.3 (1).

If V has infinite rank, then (c) follows from Proposition 6.3 (2). □

It is easy to see that case (b) actually occurs: Let $G = \mathbb{Q} \oplus F$, where F is a free abelian group of infinite rank. Then G is $\aleph_0$-QE-transitive but not irreducible while $G/\mathbb{Q}$ is $\aleph_0$-QE-transitive and irreducible.

Now we present an example for case (c): Let $P = \{p_i : i < \omega\}$ be a countable list of distinct primes. Let $P_n = \{p_i : n \leq i < \omega\}$ and $S_n = \mathbb{Z}[\frac{1}{p} : p \in P_n]$, a subring of $\mathbb{Q}$. Let

$$G = \bigoplus_{k<\omega} e_k S_k.$$

Let τ_k denote the type of S_k. Then $G(\tau_n) = \bigoplus_{0 \leq i \leq n} e_i S_i$ is fully invariant in G and $G = \bigcup_{n<\omega} G(\tau_n)$. We need to show that G is $\aleph_0$-QE-transitive:

Let X be a pure, countable rank subgroup of G. By [15, Theorem 1] there exists an infinite subset I of $\mathbb{N}$ such that

$$X = (X \cap e_0 S_0) \oplus \left(\bigoplus_{i \in I} \langle a_i \rangle_* \right)$$

with $\langle a_i \rangle_* \cong S_i$. This means that

$$a_i = \sum_{0 \leq j \leq i} e_j s_{ji}$$

for some $s_{ji} \in S_j$ and $s_{ii} \neq 0$. We call I the pivot set of X. There exist suitable natural numbers n_i such that

$$a_i' = \sum_{0 \leq j \leq i} e_j s_{ji} n_i = \sum_{0 \leq j \leq i} e_j s_{ji}'$$

and all $s_{ji}' \in \mathbb{Z}$. Let $i > j$. Then $b_i = a_i'(-s_{jj}') + a_i' s_{ji}' \in X$ has entry zero at the e_j. An easy induction shows that for all $i \in I$ there exist

$$b_i = e_i r_i + \sum_{i > j \notin I} e_j r j i$$

such that X/X' is torsion, where $X' = \bigoplus_{i \in I} b_i S_i$. There exists a surjective function $\alpha : I \to \omega$ such that for all $k < \omega$ there is some $k^\# \in I$ with $k = \alpha(k^\#)$ and $k \leq k^\#$. We define $\varphi \in \mathrm{End}(G)$ by

$$\varphi(e_i) = \begin{cases} e_{\alpha(i)} & \text{for } i \in I, \\ 0 & \text{for } i \notin I. \end{cases}$$

By our choice of the functions α, the homomorphism φ is surjective and for $i \in I$ we have $\varphi(b_i) = \varphi(e_i r_i) = e_{\alpha(i)} r_i$. This shows that $G/\varphi(X)$ is torsion. Now let Y be a pure, countable rank subgroup of G. By Wang's result [15], there exists some $\psi \in \mathrm{Hom}(G, Y) \subseteq \mathrm{End}(G)$ such that $\psi(G) \subseteq Y$ with $Y/\psi(G)$ torsion. The map $\eta = \psi \circ \varphi$ has the property that $\eta(X) \subseteq Y$ with $Y/\eta(X)$ torsion. This shows that G is $\aleph_0$-QE-transitive. Of course, G is not QE-transitive since G is not homogeneous.

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Remark 6.6. (1) For an uncountable cardinal n, nothing is known about n-QE-transitive groups.

(2) In [4, Definition 2.3] the notion of near isomorphism, a familiar term for finite rank groups, was defined for torsion-free abelian groups of infinite rank. This can be used to strengthen the notion of "n-QE-transitivity", since near isomorphism implies quasi-isomorphism. It might be worthwhile to take a look at this.

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