ON CERTAIN SUBCLASS OF MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

S.P. GOYAL, ONKAR SINGH, AND RAKESH KUMAR

Abstract
In this paper we introduce and investigate a certain subclass of functions which are analytic in the punctured unit disk and meromorphically close-to-convex. The subordination property, inclusion relationship, coefficient inequalities, distortion theorem and a sufficient condition for our subclass of functions are derived. The results presented here would provide extensions of those given in earlier works.

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1. INTRODUCTION
Let $\Sigma$ denote the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \quad (1.1)$$

which are analytic in the punctured open unit disk

$$U^* = \{ z : z \in \mathbb{C}; 0 < |z| < 1 \} = U \setminus \{0\}.$$ 

where $U$ is an open unit disk.

Let $\mathcal{P}$ denote the class of functions $p$ given by

$$p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in U) \quad (1.2)$$

which are analytic and convex in $U$ and satisfy the condition

$$p(z) < \frac{1 + Az}{1 + Bz} \quad (z \in U; -1 \leq B < A \leq 1) \quad (1.3)$$

Let $f, g \in \Sigma$, where $f$ is given by (1.1) and $g$ is defined by

$$f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \quad (1.4)$$

A function $f \in \Sigma$ is said to be in the class $\mathcal{MS}^*(\alpha)$ of meromorphic starlike of order $\alpha$ if it satisfies the inequality

$$R \left( \frac{-zf'(z)}{f(z)} \right) > \alpha \quad (z \in U; 0 \leq \alpha < 1) \quad (1.5)$$
Moreover, a function \( f \in \Sigma \) is said to be in the class \( \mathcal{MC} \) of meromorphic close-to-convex functions if it satisfies the condition
\[
R \left( \frac{zf'(z)}{g(z)} \right) < 0 \quad (z \in \mathcal{U}; g \in \mathcal{MS}^*(0) \equiv \mathcal{MS}^*)
\]  

Further let
\[
f(z) = z + a_2 z^2 + \ldots
\]
be analytic in \( \mathcal{U} \). If there exists a function \( g(z) \in S^*(\frac{1}{2}) \) such that
\[
R \left( \frac{zf'(z)}{g(z)g(-z)} \right) < 0 \quad (z \in \mathcal{U})
\]
then we say that \( f \in \mathcal{K}_s \), where \( S^*(\frac{1}{2}) \) denotes the usual class of starlike functions of order \( \frac{1}{2} \). The function class \( \mathcal{K}_s \) was introduced and studied by Gao and Zhou [3]. Also Srivastava et al. [10] considered the class \( \mathcal{MS}_s^* \) of meromorphic starlike functions with respect to symmetric points which satisfy the condition
\[
R \left( \frac{zf'(z)}{f(z) - f(-z)} \right) > 0
\]
where \( g(z) \in S^*(\frac{1}{2}) \).

Motivated by the class \( \mathcal{K}_s \), Seker [8] introduced a new class \( \mathcal{K}_s^{(k)}(\gamma) \) of analytic functions related to starlike functions. Let \( f \) be an analytic function defined by \( (1.3) \). Then \( f \in \mathcal{K}_s(\gamma) \) if it satisfies the inequality
\[
R \left( \frac{-zf'(z)}{g(z)g(-z)} \right) > \gamma \quad (z \in \mathcal{U}; 0 \leq \gamma < 1)
\]  

where \( g(z) \in S^*(\frac{1}{2}) \).

Recently Wang et al. [11] considered and investigated the class \( \mathcal{MK} \) of meromorphic close-to-convex function, if \( f \in \Sigma \) it satisfies the inequality
\[
R \left( \frac{f'(z)}{g(z)g(-z)} \right) > 0 \quad (z \in \mathcal{U})
\]  

where \( g \in \mathcal{MS}^*(\frac{1}{2}) \).

Motivated essentially by the aforementioned function classes \( \mathcal{MK} \) and \( \mathcal{K}_s^{(k)}(\gamma) \), in this paper we introduce and investigate a new class \( \mathcal{MK}^{(k)}[A,B] \) of meromorphic functions.
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Definition. A function \( f \in \Sigma \) is said to be in the class \( \mathcal{MK}^{(k)}[A,B] \) if it satisfies the inequality

\[
\frac{-f'(z)}{z^{k-2}g_k(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}; -1 \leq B < A \leq 1) \tag{1.13}
\]

where \( g \in \mathcal{MS}^*\left(\frac{1}{k-1}\right) \), \( k \geq 1 \) is fixed positive integer and \( g_k(z) \) is defined by the following equality

\[
g_k(z) = \prod_{\nu=0}^{k-1} \rho^\nu g(\rho^\nu z) \quad (\rho = e^{\frac{2\pi i}{k}}) \tag{1.14}
\]

Remark 1. \( \mathcal{MK}^{(2)}[1,1] = \mathcal{MK} \), where \( \mathcal{MK} \) were studied by Wang et al.[11]. By simple calculations we see that the inequality (1.13) is equivalent to

\[
\left| \frac{f'(z)}{z^{k-2}g_k(z)} + 1 \right| < \left| \frac{Bf'(z)}{z^{k-2}g_k(z)} + A \right| \quad (z \in \mathcal{U}; -1 \leq B < A \leq 1)
\]

The class \( \mathcal{MK}^{(k)}[A,B] \) is generalization of \( \mathcal{MK}^{(2)}[A,B] \) which was defined by Sim and Kwon [9].

In this paper we prove that the class \( \mathcal{MK}^{(k)}[A,B] \) is a subclass of meromorphic close-to-convex functions. Furthermore, we investigate coefficient inequalities, distortion theorems and inclusion relationship for functions belonging to the class \( \mathcal{MK}^{(k)}[A,B] \).

2. Results Required

To prove our main results given in the next section, we shall require the results contained in following Lemmas:

Lemma 1. Let \( \varphi_i(z) \in \mathcal{MS}^*(\alpha_i) \) where \( 0 \leq \alpha_i < 1 \) \( (i = 0,1,2,...,k-1) \).

Then for \( k - 1 \leq \sum_{i=0}^{k-1} \alpha_i < k \), we have

\[
z^{k-1} \prod_{i=0}^{k-1} \varphi_i(z) \in \mathcal{MS}^* \left( \sum_{i=0}^{k-1} \alpha_i - (k-1) \right)
\]

Proof: Since \( \varphi_i(z) \in \mathcal{MS}^*(\alpha_i) \) where \( 0 \leq \alpha_i < 1 \) \( (i = 0,1,2,...,k-1) \), we have

\[
\text{Re} \left( \frac{-z \varphi_0'(z)}{\varphi_0(z)} \right) > \alpha_0, \text{Re} \left( \frac{-z \varphi_1'(z)}{\varphi_1(z)} \right) > \alpha_1, ..., \text{Re} \left( \frac{-z \varphi_{k-1}'(z)}{\varphi_{k-1}(z)} \right) > \alpha_{k-1} \tag{2.1}
\]

We now let

\[
F_k(z) = z^{k-1} \varphi_0(z) \varphi_1(z) ... \varphi_{k-1}(z) \tag{2.2}
\]

Differentiating (2.2) logarithmically, we have

\[
\frac{zF_k'(z)}{F_k(z)} = (k-1) + \frac{z \varphi_0'(z)}{\varphi_0(z)} + \frac{z \varphi_1'(z)}{\varphi_1(z)} + ... + \frac{z \varphi_{k-1}'(z)}{\varphi_{k-1}(z)} \tag{2.3}
\]

Therefore

\[
\text{Re} \left( \frac{-zF_k'(z)}{F_k(z)} \right) > -(k-1) + \alpha_0 + \alpha_1 + ... \alpha_{k-1}
\]
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\[ k - 1 \sum_{i=0}^{k-1} \alpha_i - (k - 1) \quad (2.4) \]

Thus, if

\[ 0 \leq \sum_{i=0}^{k-1} \alpha_i - (k - 1) < 1 \]

that is,

\[ (k - 1) \leq \sum_{i=0}^{k-1} \alpha_i < k \]

Then

\[ F_k(z) = z^{k-1} \prod_{i=0}^{k-1} \phi_i(z) \in \mathcal{MS}^* \left( \sum_{i=0}^{k-1} \alpha_i - (k - 1) \right) \quad (2.5) \]

**Lemma 2** (see [2]). Suppose that

\[ h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{MS}^* \quad (2.6) \]

Then

\[ |c_n| \leq \frac{2}{n + 1} \quad (n \in \mathbb{N}) \quad (2.7) \]

Each of these inequality is sharp, with the extremal function given by

\[ h(z) = z^{-1} \left( 1 + (z^{n+1})^{\frac{2}{n+1}} \right) \quad (2.8) \]

**Lemma 3** (see [1]). Let \( p \in P[A, B] \) and \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \)

Then

\[ |c_n| \leq A - B \]

This result is sharp.

**Lemma 4** (see [4]). Let \( p \in P[A, B] \), then for \( |z| = r < 1 \)

\[ \frac{1 - Ar}{1 - Br} \leq \text{Re}p(z) \leq |p(z)| \leq \frac{1 + Ar}{1 + Br} \quad (2.9) \]

These bounds are sharp.

**Lemma 5** (see [7]). Suppose that \( g \in \mathcal{MS}^* \), then

\[ \frac{(1-r)^2}{r} \leq |g(z)| \leq \frac{(1+r)^2}{r} \quad (|z| = r; 0 < r < 1) \quad (2.10) \]

**Lemma 6** (see [6]). Let \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\). Then

\[ \frac{1 + A_1 z}{1 + B_1 z} < \frac{1 + A_2 z}{1 + B_2 z} \quad (2.11) \]
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3. Main Results

Theorem 1 Let \( g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{M}^* \left( \frac{k-1}{k} \right) \), then

\[
G_k(z) = z^{k-1}g_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_n z^n \in \mathcal{M}^* 
\]  

(3.1)

Proof. From (1.14), we know that

\[
z^{k-1}g_k(z) = z^{k-1} \prod_{\nu=0}^{k-1} \rho^{\nu} (\rho^\nu z) 
\]

\[
= z^{k-1} \left[ \prod_{\nu=0}^{k-1} \left( \frac{1}{z} + \sum_{n=2}^{\infty} b_n \rho^{\nu(n+1)} z^n \right) \right] 
\]  

(3.2)

Now since

\[
g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{M}^* \left( \frac{k-1}{k} \right) 
\]

Then by above Lemma 1 and equality (3.2), we can get \( G_k(z) \in \mathcal{M}^* \).

Corollary For \( k = 2 \) in Theorem 1, we get the result of Theorem 2 obtained by Wang et al. [11].

Remark 2 By Theorem 1 we see that \( G_k(z) \) given by (3.1) belongs to \( \mathcal{M}^* \). Thus by (1.13), we find that our class \( \mathcal{M} C^{(k)}(A, B) \) is a subclass of the class \( \mathcal{M} C \) of meromorphic close-to-convex functions.

Theorem 2. Let \( f(z) \) given by (1.1) and \( -1 \leq B < A \leq 1 \). if

\[
\sum_{n=1}^{\infty} \left\{ (1 + |B|) n |a_n| + (1 + |A|) \frac{2}{n+1} \right\} \leq A - B 
\]  

(3.3)

then \( f \in \mathcal{M} C^{(k)}(A, B) \).

Proof. Let the function \( f(z) \) and \( g_k(z) \) be given by (1.1) and (1.14) respectively. Furthermore, let \( g(z) \in \mathcal{M}^* \left( \frac{k-1}{k} \right) \).

Then by Theorem 1 and Lemma 2, we have

\[
G_k(z) = z^{k-1}g_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_n z^n \in \mathcal{M}^* 
\]

where \( |B_n| \leq \frac{2}{n+1} \).

Now we obtain

\[
\Delta = |zf'(z) + G_k(z)| - |Bzf'(z) + AG_k(z)| 
\]

\[
= \left| \sum_{n=1}^{\infty} n a_n z^n + \sum_{n=1}^{\infty} B_n z^n \right| - \left| (A - B) \frac{1}{z} + A \sum_{n=1}^{\infty} B_n z^n + B \sum_{n=1}^{\infty} n a_n z^n \right| 
\]

Thus for \( |z| = r \ (0 \leq r < 1) \), we have

\[
\Delta \leq \sum_{n=1}^{\infty} n |a_n| r^n + \sum_{n=1}^{\infty} |B_n| r^n - \left( (A - B) \frac{1}{r} - |A| \sum_{n=1}^{\infty} |B_n| r^n - |B| \sum_{n=1}^{\infty} n |a_n| r^n \right) 
\]
\[
= -(A - B) \frac{1}{r} + (1 + |A|) \sum_{n=1}^{\infty} |B_n| r^n + (1 + |B|) \sum_{n=1}^{\infty} n |a_n| r^n
\]

\[
\leq -(A - B) \frac{1}{r} + (1 + |A|) \sum_{n=1}^{\infty} \frac{2}{n+1} r^n + (1 + |B|) \sum_{n=1}^{\infty} n |a_n| r^n
\]

\[
\leq 0. \quad \text{(By the given condition)}
\]

Thus we have
\[
|zf'(z) + G_k(z)| < |Bzf'(z) + AG_k(z)|
\]

which is equivalent to
\[
\left| \frac{f'(z)}{z^{k-2} g_k(z)} + 1 \right| < \left| \frac{Bf'(z)}{z^{k-2} g_k(z)} + A \right| \quad (z \in U)
\]

which implies that \( f \in \mathcal{M}^{(k)}[A, B] \).

Next, we give the coefficient estimates of functions belonging to the class \( \mathcal{M}^{(k)}[A, B] \).

**Theorem 3.** Let \( f \in \mathcal{M}^{(k)}[A, B] \) \((-1 \leq B < A \leq 1)\) and \( g_k(z) \) is given by (1.1) and (1.14) respectively. Then for \( k \geq 1 \), we have
\[
\sum_{k=1}^{n} |ka_k + B_k|^2 - \sum_{k=1}^{n-1} |A.B_k + kBa_k|^2 < (A - B)^2 \quad (3.4)
\]

**Proof** Let \( f \in \mathcal{M}^{(k)}[A, B] \). Then we have
\[
\frac{-zf'(z)}{G_k(z)} = \frac{1 + Aw(z)}{1 + Bw(z)}
\]

where \( w \) is an analytic function in \( U \), \(|w(z)| < 1\) for \( z \in U \) and \( G_k(z) = z^{k-1}g_k(z) \).

Then,
\[
-zf'(z) - G_k(z) = (A.G_k(z) + Bzf'(z)) w(z).
\]

Thus, putting
\[
w(z) = \sum_{n=1}^{\infty} t_n z^n,
\]

we obtain
\[
-\sum_{n=1}^{\infty} na_n z^n - \sum_{n=1}^{\infty} B_n z^n = \left\{ (A - B) \frac{1}{z} + \sum_{n=1}^{\infty} A.B_n z^n + \sum_{n=1}^{\infty} nBa_n z^n \right\} \left( \sum_{n=1}^{\infty} t_n z^n \right) \quad (3.5)
\]

Now equating the coefficient of \( z^n \), we get
\[
-na_n - B_n = (A - B) t_{n+1} + (A.B_1 + Ba_1) t_{n-1} + \ldots + \{ A.B_{n-1} + (n - 1)Ba_{n-1} \} t_1
\]

and thus the coefficient combination on the R.H.S. of (3.5) depends only upon the coefficient combinations
\[
(A.B_1 + Ba_1), \ldots, \{ A.B_{n-1} + (n - 1)Ba_{n-1} \}
\]
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Hence for \( n \geq 1 \)
\[
\left[ (A - B) \frac{1}{z} + \sum_{k=1}^{n-1} (A.B_k + kBa_k) z^k \right] w(z) = \sum_{k=1}^{n} (-ka_k - B_k) z^k + \sum_{k=n+1}^{\infty} d_k z^k
\]  
(3.6)

Then squaring the modulus of both sides of the above equality and integrating along \( |z| = r \) and using the fact that \( |w(z)| < 1 \), we obtain
\[
\sum_{k=1}^{n} |ka_k + B_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} < (A - B)^2 \frac{1}{r^2} + \sum_{k=1}^{n-1} |A.B_k + kBa_k|^2 r^{2k}
\]
(3.7)

Letting \( r \to 1 \) on both sides of (3.7), we obtain
\[
\sum_{k=1}^{n} |ka_k + B_k|^2 < (A - B)^2 + \sum_{k=1}^{n-1} |A.B_k + kBa_k|^2
\]

Hence we have
\[
\sum_{k=1}^{n} |ka_k + B_k|^2 - \sum_{k=1}^{n-1} |A.B_k + kBa_k|^2 < (A - B)^2
\]

which implies the required inequality.

**Theorem 4.** Suppose that
\[
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in M^{(k)}(A, B)
\]

Then
\[
|a_n| \leq \frac{(A - B)}{n} \left[ -1 + 2 \sum_{m=1}^{n} \frac{1}{m} \right] + \frac{2}{n(n + 1)}
\]
(3.8)

**Proof** Suppose that \( f \in M^{(k)}[A, B] \). Then we know that
\[
-\frac{zf'(z)}{z^{k-2}g_k(z)} \prec \frac{1 + Az}{1 + Bz}
\]

this implies that
\[
-\frac{zf'(z)}{G_k(z)} \prec \frac{1 + Az}{1 + Bz}
\]

If we set
\[
q(z) = \frac{-zf'(z)}{G_k(z)}
\]
(3.9)

it follows that
\[
q(z) = 1 + d_1 z + d_2 z^2 + \ldots \in \mathcal{P}
\]

In view of Lemma 3, we know that
\[
|d_n| \leq A - B \quad (n \in N)
\]
By substituting the series expressions of functions \( f, G_k \) and \( q \) in (3.9), we obtain
\[
(1 + d_1 z + d_2 z^2 + \ldots + d_n z^n + \ldots) \left( \frac{1}{z} + B_1 z + B_2 z^2 + \ldots + B_n z^n + \ldots \right)
\]
\[
= \frac{1}{z} - a_1 z - 2a_2 z^2 - \ldots - na_n z^n - \ldots
\]
Comparing like coefficients of \( z^n \) in (3.10), we get
\[
-na_n = B_1 d_{n-1} + B_2 d_{n-2} + \ldots + B_{n-2} d_2 + B_{n-1} d_1 + B_n + d_{n+1}
\]
By using Lemma 2 and 3, we get
\[
n |a_n| \leq 2 (A - B) \left( 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right) + \frac{2}{n+1} - (A - B)
\]
Thus
\[
|a_n| \leq \frac{(A - B)}{n} \left[ -1 + 2 \sum_{m=1}^{n} \frac{1}{m} \right] + \frac{2}{n(n+1)}
\]
which implies the required inequality. The proof of Theorem 4 is completed.

**Theorem 5.** Let \( f \in \mathcal{MK}^{(k)}[A, B] \). Then
\[
\frac{(1-r)^2}{r^2} \left( \frac{1-Ar}{1-Br} \right) \leq |f'(z)| \leq \frac{(1+r)^2}{r^2} \left( \frac{1+Ar}{1+Br} \right) \quad (|z| = r, 0 < r < 1)
\]

**Proof.** Suppose \( f \in \mathcal{MK}^{(k)}[A, B] \). By definition we know that
\[
-zf'(z) \leq \frac{G_k(z)}{1+Bz}
\]
and since \( G_k(z) \in \mathcal{MS}^* \), thus by Lemma 5, we have
\[
\frac{(1-r)^2}{r^2} \leq |G_k(z)| \leq \frac{(1+r)^2}{r^2}
\]
and also by Lemma 4, we have
\[
\frac{(1-Ar)}{(1-Br)} \leq |q(z)| \leq \frac{(1+Ar)}{(1+Br)}
\]
Thus by virtue of (3.9) and Lemma 5, we obtain
\[
\frac{(1-r)^2}{r^2} \left( \frac{1-Ar}{1-Br} \right) \leq |f'(z)| \leq \frac{(1+r)^2}{r^2} \left( \frac{1+Ar}{1+Br} \right)
\]
Thus the proof is complete.

**Remark** For \( A = 1 \) and \( B = -1 \) in Theorem 5, we obtain result of Theorem 7 by Wang et al.[11]

**Theorem 6.** Let \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\). Then
\[
\mathcal{MK}^{(k)}(A_1, B_1) \subset \mathcal{MK}^{(k)}(A_2, B_2)
\]

(3.13)
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Proof. Suppose that \( f \in \mathcal{M}^{(k)}[A_1, B_1] \), we have

\[
\frac{-f'(z)}{z^{k-2}g_k(z)} < \frac{1 + A_1 z}{1 + B_1 z}
\]

Since \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\), by lemma 6, we get

\[
\frac{-f'(z)}{z^{k-2}g_k(z)} < \frac{1 + A_1 z}{1 + B_1 z} < \frac{1 + A_2 z}{1 + B_2 z}
\]

Hence \( f \in \mathcal{M}^{(k)}[A_2, B_2] \).

This means that \( \mathcal{M}^{(k)}(A_1, B_1) \subset \mathcal{M}^{(k)}(A_2, B_2) \).

Hence the proof of Theorem 6 is complete.

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