ON CERTAIN SUBCLASS OF MEROMORPHIC CLOSE-TO-CONVEX FUNCTIONS

S.P. GOYAL, ONKAR SINGH, AND RAKESH KUMAR

Abstract
In this paper we introduce and investigate a certain subclass of functions which are analytic in the punctured unit disk and meromorphically close-to-convex. The subordination property, inclusion relationship, coefficient inequalities, distortion theorem and a sufficient condition for our subclass of functions are derived. The results presented here would provide extensions of those given in earlier works.

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1. INTRODUCTION
Let \( \Sigma \) denote the class of functions of the form
\[
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n
\] (1.1)
which are analytic in the punctured open unit disk
\[
\mathcal{U}^* = \{ z : z \in \mathbb{C}; 0 < |z| < 1 \} = \mathcal{U}\setminus\{0\}.
\]
where \( \mathcal{U} \) is an open unit disk.
Let \( P \) denote the class of functions \( p \) given by
\[
p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n \quad (z \in \mathcal{U})
\] (1.2)
which are analytic and convex in \( \mathcal{U} \) and satisfy the condition
\[
p(z) \prec \frac{1 + Az}{1 + Bz} \quad (z \in \mathcal{U}; -1 \leq B < A \leq 1)
\] (1.3)
Let \( f, g \in \Sigma \), where \( f \) is given by (1.1) and \( g \) is defined by
\[
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n
\] (1.4)
A function \( f \in \Sigma \) is said to be in the class \( \mathcal{MS}^*(\alpha) \) of meromorphic starlike of order \( \alpha \) if it satisfies the inequality
\[
R \left( \frac{-zf'(z)}{f(z)} \right) > \alpha \quad (z \in \mathcal{U}; 0 \leq \alpha < 1)
\] (1.5)
Moreover, a function \( f \in \Sigma \) is said to be in the class \( \mathcal{MC} \) of meromorphic close-to-convex functions if it satisfies the condition
\[
R \left( \frac{zf'(z)}{g(z)} \right) < 0 \quad (z \in \mathcal{U}; g \in \mathcal{MS}^*(0) \equiv \mathcal{MS}^*)
\] (1.6)

Further let
\[
f(z) = z + a_2 z^2 + \ldots
\] (1.7)
be analytic in \( \mathcal{U} \). If there exists a function \( g(z) \in S^*(\frac{1}{2}) \) such that
\[
R \left( \frac{zf'(z)}{g(z)g(-z)} \right) < 0 \quad (z \in \mathcal{U})
\]
then we say that \( f \in \mathcal{K}_s \), where \( S^*(\frac{1}{2}) \) denotes the usual class of starlike functions of order \( \frac{1}{2} \). The function class \( \mathcal{K}_s \) was introduced and studied by Gao and Zhou [3]. Also Srivastava et al. [10] considered the class \( \mathcal{MS}^*_s \) of meromorphic starlike functions with respect to symmetric points which satisfy the condition
\[
R \left( \frac{zf'(z)}{f(z) - f(-z)} \right) < 0 \quad (z \in \mathcal{U})
\] (1.8)

Again Kowalczyk and Bomba [5] discussed \( \mathcal{K}_s(\gamma) \) of analytic functions related to starlike functions. Let \( f \) is given by (1.3). Then \( f \in \mathcal{K}_s(\gamma) \) if it satisfies the inequality
\[
R \left( \frac{-zf'(z)}{g(z)g(-z)} \right) > \gamma \quad (z \in \mathcal{U}; 0 \leq \gamma < 1)
\] (1.9)
where \( g(z) \in S^*(\frac{1}{2}) \).

Motivated by the class \( \mathcal{K}_s(\gamma) \), Seker [8] introduced a new class \( \mathcal{K}_s^{(k)}(\gamma) \) of analytic functions related to starlike functions as follows
Let \( f \) be an analytic function defined by (1.3). Then \( f \in \mathcal{K}_s^{(k)}(\gamma) \), if it satisfies the condition
\[
R \left( \frac{zf'(z)}{g_k(z)} \right) > \gamma \quad (z \in \mathcal{U}; 0 \leq \gamma < 1)
\] (1.10)
where \( g \in S^*(\frac{k-1}{k}) \), \( k \geq 1 \) is a fixed positive integer and \( g_k(z) \) is defined by the following equality
\[
g_k(z) = \prod_{\nu=0}^{k-1} g(\nu, z) \quad (\nu = e^{\frac{2\pi\nu}{k}})
\] (1.11)

Recently Wang et al.[11] considered and investigated the class \( \mathcal{MK} \) of meromorphic close-to-convex function, if \( f \in \Sigma \) it satisfies the inequality
\[
R \left( \frac{f'(z)}{g(z)g(-z)} \right) > 0 \quad (z \in \mathcal{U})
\] (1.12)
where \( g \in \mathcal{MS}^*(\frac{1}{2}) \).

Motivated essentially by the aforementioned function classes \( \mathcal{MK} \) and \( \mathcal{K}_s^{(k)}(\gamma) \), in this paper we introduce and investigate a new class \( \mathcal{MK}^{(k)}[A, B] \) of meromorphic functions.
Definition. A function $f \in \Sigma$ is said to be in the class $\mathcal{MK}^{(k)}[A, B]$ if it satisfies the inequality
\[
-\frac{f'(z)}{z^{k-2}g_k(z)} < \frac{1 + Az}{1 + Bz} \quad (z \in U; -1 \leq B < A \leq 1)
\] (1.13)
where $g \in \mathcal{MS}^*(\frac{k-1}{k})$, $k \geq 1$ is fixed positive integer and $g_k(z)$ is defined by the following equality
\[
g_k(z) = \prod_{\nu=0}^{k-1} \rho^\nu g(\rho^\nu z) \quad (\rho = e^{\frac{2\pi i}{k}})
\] (1.14)

Remark 1. $\mathcal{MK}^{(2)}[1, -1] = \mathcal{MK}$, where $\mathcal{MK}$ were studied by Wang et al.[11]. By simple calculations we see that the inequality (1.13) is equivalent to
\[
\left| \frac{f'(z)}{z^{k-2}g_k(z)} + 1 \right| < \left| \frac{Bf'(z)}{z^{k-2}g_k(z)} + A \right| \quad (z \in U; -1 \leq B < A \leq 1)
\]
\[ k - 1 \sum_{i=0}^{k-1} \alpha_i - (k - 1) \] (2.4)

Thus, if
\[ 0 \leq \sum_{i=0}^{k-1} \alpha_i - (k - 1) < 1 \]
that is,
\[ (k - 1) \leq \sum_{i=0}^{k-1} \alpha_i < k \]

Then
\[ F_k(z) = z^{k-1} \prod_{i=0}^{k-1} \varphi_i(z) \in \mathcal{MS}^* \left( \sum_{i=0}^{k-1} \alpha_i - (k - 1) \right). \] (2.5)

**Lemma 2** (see [2]). Suppose that
\[ h(z) = \frac{1}{z} + \sum_{n=1}^{\infty} c_n z^n \in \mathcal{MS}^* \] (2.6)
Then
\[ |c_n| \leq \frac{2}{n+1} \quad (n \in \mathbb{N}) \] (2.7)
Each of these inequality is sharp, with the extremal function given by
\[ h(z) = z^{-1} (1 + z^{n+1})^{\frac{2}{n+1}} \] (2.8)

**Lemma 3** (see [1]). Let \( p \in P[A,B] \) and \( p(z) = 1 + \sum_{n=1}^{\infty} c_n z^n \)
Then
\[ |c_n| \leq A - B \]
This result is sharp.

**Lemma 4** (see [4]). Let \( p \in P[A,B] \), then for \(|z| = r < 1\)
\[ \frac{1 - Ar}{1 - Br} \leq \text{Re}p(z) \leq |p(z)| \leq \frac{1 + Ar}{1 + Br} \] (2.9)
These bounds are sharp.

**Lemma 5** (see [7]). Suppose that \( g \in \mathcal{MS}^* \), then
\[ \frac{(1-r)^2}{r} \leq |g(z)| \leq \frac{(1+r)^2}{r} \quad (|z| = r; 0 < r < 1) \] (2.10)

**Lemma 6** (see [6]). Let \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\). Then
\[ \frac{1 + A_1 z}{1 + B_1 z} < \frac{1 + A_2 z}{1 + B_2 z} \] (2.11)
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3. Main Results

Theorem 1 Let \( g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{M}S^* \left( \frac{k-1}{k} \right) \), then

\[
G_k(z) = z^{k-1} g_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_n z^n \in \mathcal{M}S^* \quad (3.1)
\]

Proof From (1.14), we know that

\[
z^{k-1} g_k(z) = z^{k-1} \prod_{\nu=0}^{k-1} \rho^\nu g(\rho^\nu z)
= z^{k-1} \left( \prod_{\nu=0}^{k-1} \left( \frac{1}{z} + \sum_{n=2}^{\infty} b_n \rho^{\nu(n+1)} z^n \right) \right) \quad (3.2)
\]

Now since

\[ g(z) = \frac{1}{z} + \sum_{n=1}^{\infty} b_n z^n \in \mathcal{M}S^* \left( \frac{k-1}{k} \right) \]

Then by above Lemma 1 and equality (3.2), we can get \( G_k(z) \in \mathcal{M}S^* \).

Corollary For \( k = 2 \) in Theorem 1, we get the result of Theorem 2 obtained by Wang et al.[11].

Remark 2 By Theorem 1 we see that \( G_k(z) \) given by (3.1) belongs to \( \mathcal{M}S^* \). Thus by (1.13), we find that our class \( \mathcal{MK}^{(k)}[A,B] \) is a subclass of the class \( \mathcal{MK} \) of meromorphic close-to-convex functions.

Theorem 2. Let \( f(z) \) given by (1.1) and \( -1 \leq B < A \leq 1 \). if

\[
\sum_{n=1}^{\infty} \left\{ (1 + |B| n |a_n| + (1 + |A|) \frac{2}{n+1} \right\} \leq A - B \quad (3.3)
\]

then \( f \in \mathcal{MK}^{(k)}[A,B] \).

Proof. Let the function \( f(z) \) and \( g_k(z) \) be given by (1.1) and (1.14) respectively. Furthermore, let \( g(z) \in \mathcal{M}S^* \left( \frac{k-1}{k} \right) \).

Then by Theorem 1 and Lemma 2, we have

\[ G_k(z) = z^{k-1} g_k(z) = \frac{1}{z} + \sum_{n=1}^{\infty} B_n z^n \in \mathcal{M}S^* , \]

where \( |B_n| \leq \frac{2}{n+1} \).

Now we obtain

\[
\Delta = |zf'(z) + G_k(z)| - |Bzf'(z) + AG_k(z)|
= \left| \sum_{n=1}^{\infty} n a_n z^n + \sum_{n=1}^{\infty} B_n z^n \right| - \left| (A - B) \frac{1}{z} + A \sum_{n=1}^{\infty} B_n z^n + B \sum_{n=1}^{\infty} n a_n z^n \right|
\]

Thus for \( |z| = r \ (0 \leq r < 1) \), we have

\[
\Delta \leq \sum_{n=1}^{\infty} n |a_n| r^n + \sum_{n=1}^{\infty} |B_n| r^n - \left( (A - B) \frac{1}{r} - |A| \sum_{n=1}^{\infty} |B_n| r^n - |B| \sum_{n=1}^{\infty} n |a_n| r^n \right)
\]
\[ -\frac{1}{r} (A - B) = \sum_{n=1}^{\infty} |B_n| r^n + (1 + |B|) \sum_{n=1}^{\infty} n |a_n| r^n \]

\[ \leq -\frac{1}{r} (A - B) + (1 + |A|) \sum_{n=1}^{\infty} \frac{2}{n+1} r^n + (1 + |B|) \sum_{n=1}^{\infty} n |a_n| r^n \]

\[ \leq 0. \quad \text{(By the given condition)} \]

Thus we have

\[ |zf(z) + G_k(z)| < |Bzf(z) + AG_k(z)| \]

which is equivalent to

\[ \left| \frac{f(z)}{z^{k-2}g_k(z)} + 1 \right| < \left| \frac{Bf(z)}{z^{k-2}g_k(z)} + A \right| \quad (z \in U) \]

which implies that \( f \in \mathcal{M}^{(k)}[A, B] \).

Next, we give the coefficient estimates of functions belonging to the class \( \mathcal{M}^{(k)}[A, B] \).

**Theorem 3.** Let \( f \in \mathcal{M}^{(k)}[A, B] \) \((-1 \leq B < A \leq 1)\) and \( g_k(z) \) is given by (1.1) and (1.14) respectively. Then for \( k \geq 1 \), we have

\[ \sum_{k=1}^{n} |ka_k + B_k|^2 - \sum_{k=1}^{n-1} |A.B_k + kBa_k|^2 < (A - B)^2 \quad (3.4) \]

**Proof** Let \( f \in \mathcal{M}^{(k)}[A, B] \). Then we have

\[ -zf(z) - G_k(z) = \frac{1 + Aw(z)}{1 + Bw(z)} \]

where \( w \) is an analytic function in \( U \), \( |w(z)| < 1 \) for \( z \in U \) and \( G_k(z) = z^{k-1}g_k(z) \).

Then,

\[ -zf(z) - G_k(z) = (A.G_k(z) + Bzf(z)) w(z). \]

Thus, putting

\[ w(z) = \sum_{n=1}^{\infty} t_n z^n, \]

we obtain

\[ -\sum_{n=1}^{\infty} na_n z^n - \sum_{n=1}^{\infty} B_n z^n = \left\{ (A - B) \sum_{n=1}^{\infty} A.B_n z^n + \sum_{n=1}^{\infty} nBa_n z^n \right\} \left( \sum_{n=1}^{\infty} t_n z^n \right) \quad (3.5) \]

Now equating the coefficient of \( z^n \), we get

\[ -na_n - B_n = (A - B) t_{n+1} + (A.B_1 + Ba_1) t_{n-1} + \ldots + \{ A.B_{n-1} + (n - 1)Ba_{n-1} \} t_1 \]

and thus the coefficient combination on the R.H.S. of (3.5) depends only upon the coefficient combinations

\( (A.B_1 + Ba_1), \ldots, \{ A.B_{n-1} + (n - 1)Ba_{n-1} \} \)
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Hence for \( n \geq 1 \)

\[
\left[ (A - B) \frac{1}{z} + \sum_{k=1}^{n-1} (A.B_k + kB_k) z^k \right] w(z) = \sum_{k=1}^{n} (-ka_k - B_k) z^k + \sum_{k=n+1}^{\infty} d_k z^k
\]  

(3.6)

Then squaring the modulus of both sides of the above equality and integrating along \( |z| = r \) and using the fact that \( |w(z)| < 1 \), we obtain

\[
\sum_{k=1}^{n} |ka_k + B_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |d_k|^2 r^{2k} < (A - B)^2 \frac{1}{r^2} + \sum_{k=1}^{n-1} |A.B_k + kB_k|^2 r^{2k}
\]  

(3.7)

Letting \( r \to 1 \) on both sides of (3.7), we obtain

\[
\sum_{k=1}^{n} |ka_k + B_k|^2 < (A - B)^2 + \sum_{k=1}^{n-1} |A.B_k + kB_k|^2
\]

Hence we have

\[
\sum_{k=1}^{n} |ka_k + B_k|^2 - \sum_{k=1}^{n-1} |A.B_k + kB_k|^2 < (A - B)^2
\]

which implies the required inequality.

**Theorem 4.** Suppose that

\[
f(z) = \frac{1}{z} + \sum_{n=1}^{\infty} a_n z^n \in M K^{(k)} [A, B]
\]

Then

\[
|a_n| \leq \frac{(A - B)}{n} \left[ -1 + 2 \sum_{m=1}^{n} \frac{1}{m} \right] + \frac{2}{n (n + 1)}
\]  

(3.8)

**Proof** Suppose that \( f \in M K^{(k)}[A, B] \). Then we know that

\[
\frac{-f'(z)}{z^{k-\frac{1}{2}} g_k(z)} < \frac{1 + Az}{1 + Bz}
\]

this implies that

\[
\frac{-zf'(z)}{G_k(z)} < \frac{1 + Az}{1 + Bz}
\]

If we set

\[
q(z) = \frac{-zf'(z)}{G_k(z)}
\]  

(3.9)

it follows that

\[
q(z) = 1 + d_1 z + d_2 z^2 + \ldots \in P
\]

In view of Lemma 3, we know that

\[
|d_n| \leq A - B \quad (n \in N)
\]
By substituting the series expressions of functions $f$, $G_k$ and $q$ in (3.9), we obtain

$$
(1 + d_1z + d_2z^2 + \ldots + d_nz^n + \ldots) \left( \frac{1}{z} + B_1z + B_2z^2 + \ldots + B_nz^n + \ldots \right)
$$

(3.10)

$$
= \frac{1}{z} - a_1z - 2a_2z^2 - \ldots - na_nz^n - \ldots
$$

Comparing like coefficients of $z^n$ in (3.10), we get

$$
-na_n = B_1d_{n-1} + B_2d_{n-2} + \ldots + B_{n-2}d_2 + B_{n-1}d_1 + B_n + d_{n+1}
$$

(3.11)

By using Lemma 2 and 3, we get

$$
n|a_n| \leq 2(A-B) \left[ 1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n} \right] + \frac{2}{n+1} - (A-B)
$$

Thus

$$
|a_n| \leq \frac{(A-B)}{n} \left[ -1 + 2 \sum_{m=1}^{n} \frac{1}{m} \right] + \frac{2}{n(n+1)}
$$

which implies the required inequality. The proof of Theorem 4 is completed.

**Theorem 5.** Let $f \in \mathcal{MK}^{(k)}[A, B]$. Then

$$
\frac{(1-r)^2}{r^2} \left( \frac{1-Ar}{1-Br} \right) \leq |f'(z)| \leq \frac{(1+r)^2}{r^2} \left( \frac{1+Ar}{1+Br} \right)
$$

(|z| = r, 0 < r < 1)

(3.12)

**Proof.** Suppose $f \in \mathcal{MK}^{(k)}[A, B]$. By definition we know that

$$
\frac{-zf'(z)}{G_k(z)} \leq \frac{1+Az}{1+Bz}
$$

and since $G_k(z) \in \mathcal{MS}^*$, thus by Lemma 5, we have

$$
\frac{(1-r)^2}{r} \leq |G_k(z)| \leq \frac{(1+r)^2}{r}
$$

and also by Lemma 4, we have

$$
\frac{(1-Ar)}{1-Br} \leq |q(z)| \leq \frac{(1+Ar)}{1+Br}
$$

Thus by virtue of (3.9) and Lemma 5, we obtain

$$
\frac{(1-r)^2}{r^2} \left( \frac{1-Ar}{1-Br} \right) \leq |f'(z)| \leq \frac{(1+r)^2}{r^2} \left( \frac{1+Ar}{1+Br} \right)
$$

Thus the proof is complete.

**Remark** For $A = 1$ and $B = -1$ in Theorem 5, we obtain result of Theorem 7 by Wang et al.[11]

**Theorem 6.** Let $-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1$. Then

$$
\mathcal{MK}^{(k)}(A_1,B_1) \subset \mathcal{MK}^{(k)}(A_2,B_2)
$$

(3.13)
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Proof. Suppose that \( f \in \mathcal{MK}^{(k)}[A_1, B_1] \), we have

\[
\frac{-f'(z)}{z^{k-2}g_k(z)} < \frac{1 + A_1 z}{1 + B_1 z}
\]

Since \(-1 \leq B_2 \leq B_1 < A_1 \leq A_2 \leq 1\), by lemma 6, we get

\[
\frac{-f'(z)}{z^{k-2}g_k(z)} < \frac{1 + A_1 z}{1 + B_1 z} < \frac{1 + A_2 z}{1 + B_2 z}
\]

Hence \( f \in \mathcal{MK}^{(k)}[A_2, B_2] \).

This means that \( \mathcal{MK}^{(k)}(A_1, B_1) \subset \mathcal{MK}^{(k)}(A_2, B_2) \).

Hence the proof of Theorem 6 is complete.

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