J. Group Theory **16** (2013), 35–49
DOI 10.1515/jgt-2012-0027

Local fusion graphs for symmetric groups

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Communicated by Christopher W. Parker

Abstract. For a group G, π a set of odd positive integers and X a set of involutions of G we define a graph $\mathcal{F}_\pi(G, X)$. This graph, called a π-local fusion graph, has vertex set X with $x, y \in X$ joined by an edge provided $x \neq y$ and the order of xy is in π. In this paper we investigate $\mathcal{F}_\pi(G, X)$ when G is a finite symmetric group for various choices of X and π.

1 Introduction

There is a long and rich history of conjuring up various types of important combinatorial structures from a group. For example Cayley graphs, constructed from a group together with a generating set for that group, have had a considerable presence in such areas as geometric group theory and the study of expander graphs (see [22]). While groups with a BN-pair (such as groups of Lie-type) via their parabolic subgroups give rise to buildings (see [16, Chapter 15]). For a group G and X a subset of G we have the commuting graph, $\mathcal{C}(G, X)$, whose vertices are the elements of X with two distinct elements of X adjacent whenever they commute (for recent work on commuting graphs see [7–12, 14, 18, 19]). Such graphs have had an important impact in the study of finite simple groups, the commuting graphs associated with the Fischer groups [20], which featured in their construction, being a prime example. Variations on this theme have also played a role – see for example the so-called root 4-subgroups of the Held group, on page 230 of [1]. For yet another variety of graph consult [15].

We now discuss a recent combinatorial structure of this genre. Suppose that G is a group, π is a set of positive integers and X is a subset of G. The graph $\mathcal{C}_\pi(G, X)$ is defined to be the graph with X as its vertex set and for $x, y \in X$, x and y are adjacent if $x \neq y$ and the order of xy is in π. We observe, as xy and yx are conjugate elements of G, that the graph $\mathcal{C}_\pi(G, X)$ is undirected. Further, we observe that $\mathcal{C}_{\{2\}}(G, X)$ when X is a set of involutions in G is exactly the commuting involution graph $\mathcal{C}(G, X)$. When the orders of the elements in X are coprime to all the integers in π, we shall call $\mathcal{C}_\pi(G, X)$ a π-*coprimality graph* (or just *coprimality graph* if π is understood).

An important type of coprimality graph arises when X is a set of involutions. For π a set of odd positive integers, we write $\mathcal{F}_\pi(G, X)$ instead of $\mathcal{C}_\pi(G, X)$, and refer to $\mathcal{F}_\pi(G, X)$ as the *π-local fusion graph* for X. In the case when π consists of all odd positive integers, we just write $\mathcal{F}(G, X)$ instead of $\mathcal{F}_\pi(G, X)$, and call $\mathcal{F}(G, X)$ the *local fusion graph* for X. The name 'local fusion' comes from the fact that if $x = x_0, x_1, x_2, \ldots, x_m = y$ is a path in the graph $\mathcal{F}(G, X)$, then $g = g_1 g_2 \ldots g_m$ conjugates x to y where each g_i, $1 \leqslant i \leqslant m$, is an element of the dihedral group $\langle g_{i-1}, g_i \rangle$. In [17], $\{3\}$-local fusion graphs, $\mathcal{F}_{\{3\}}(G, X)$ are investigated for X a G-conjugacy class of involutions. There issues, such as connectedness and what kind of triangles the graph contains, are examined. Further, the case when $G \cong \mathrm{PSL}_2(q)$ (q a prime power) is analysed in detail, the work in [17] being prompted by a tower of graphs associated with a subgroup chain $\mathrm{Alt}(5) \leq PSL_2(11) \leq M_{11} \leq M_{12}$. Each of the graphs in this tower may be viewed as being a restricted type of $\{3\}$-local fusion graph.

The famous Baer–Suzuki Theorem (see [2, (39.6)] or [21, Theorem 3.8.2]), when X is a G-conjugacy class of involutions, may be rephrased using the local fusion graph in the following way. The graph $\mathcal{F}(G, X)$ is totally disconnected if and only if $\langle X \rangle$ is a 2-subgroup of G. For suppose $\mathcal{F}(G, X)$ is totally disconnected, and let $x, y \in X$, with $x \neq y$. Assume that the order of xy is $2^k m$, where m is odd. If $m > 1$, then $(xy)^{2^k} = x(yx \cdots xy) = xx^g$ has odd order m and $x \neq x^g$. Hence x and x^g are adjacent in $\mathcal{F}(G, X)$, a contradiction. Therefore xy has order 2^k. Since, as is well known, $\langle x, y \rangle$ is a dihedral group of order twice that of xy, $\langle x, y \rangle$ is a 2-group, and so $\langle X \rangle$ is a 2-group by the Baer–Suzuki Theorem.

The aim of the present paper is to begin the investigation of π-local fusion graphs for finite symmetric groups.

Theorem 1.1. *Suppose that $G = \mathrm{Sym}(n)$ with $n \geq 5$ and X is a G-conjugacy class of involutions. Then $\mathcal{F}(G, X)$ is connected with* $\mathrm{Diam}(\mathcal{F}(G, X)) = 2$.

For $n = 2$, $\mathcal{F}(G, X)$ consists of just one vertex and for $n = 3$, $\mathcal{F}(G, X)$ is the complete graph on three vertices. While for $n = 4$ and X the conjugacy class of $(1, 2)(3, 4)$ in Sym(4), $\mathcal{F}(G, X)$ consists of three vertices with no edges – if X is the conjugacy class of transpositions in Sym(4), then $\mathcal{F}(G, X)$ is connected of diameter 2. There are π-local fusion graphs where we do encounter larger diameters. For example with $G = \mathrm{Sym}(9)$ and X the G-conjugacy class of $(1, 2)(3, 4)(5, 6)$ we have

$$\mathrm{Diam}(\mathcal{F}_{\{3\}}(G, X)) = \mathrm{Diam}(\mathcal{F}_{\{5\}}(G, X)) = \mathrm{Diam}(\mathcal{F}_{\{7\}}(G, X)) = 3.$$

This all prompts the question as to whether there are groups in which the diameter of local fusion graphs can be arbitrarily large – the answer is yes, and we direct the reader to [4]. For further work on coprimality graphs and symmetric groups see [5], and for more recent developments on local fusion graphs see [3] and [6].

The question of connectivity for π-local fusion graphs is the subject of our second theorem.

Theorem 1.2. *Suppose that $G = \mathrm{Sym}(n)$, X is a G-conjugacy class of involutions and π is a set of odd positive integers. Then $\mathcal{F}_\pi(G, X)$ is either totally disconnected or connected.*

This paper is arranged as follows. Section 2 is mostly concerned with the notion of an 'x-graph' which, for $G \cong \mathrm{Sym}(n)$, encodes the $C_G(t)$-orbits on the conjugacy class of t where t is an involution. Then in Section 3 the x-graphs are put to work in establishing the diameter of local fusion graphs thereby proving Theorem 1.1. The proof of Theorem 1.2, which also employs x-graphs, is to be found in Section 4. Our group theoretic notation is standard as given, for example, in [2].

2 Background results

Throughout this paper t will denote a fixed involution of X, a conjugacy class of $\mathrm{Sym}(n)$. We will sometimes denote $\mathrm{Sym}(m)$ ($m \in \mathbb{N}$) by $\mathrm{Sym}(\Omega)$ where Ω is an m-element set upon which the permutations act. For $g \in \mathrm{Sym}(\Omega)$, the *support* of g, $\mathrm{supp}(g)$, is $\Omega \setminus \mathrm{fix}(g)$, where $\mathrm{fix}(g) = \{\alpha \in \Omega \mid \alpha^g = \alpha\}$. We use $d(\cdot,\cdot)$ to denote the standard graph theoretic distance on $\mathcal{F}_\pi(G, X)$.

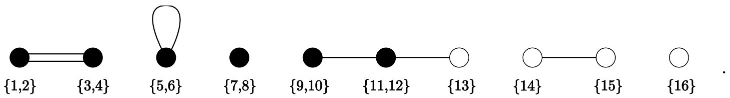
The proofs of Theorems 1.1 and 1.2 feature another graph $\mathcal{G}_x$ referred to as the *x-graph*. Assuming that $G = \mathrm{Sym}(n)$ acts upon $\Omega = \{1, 2, \dots, n\}$ and that $t = (1,2)(3,4)\dots(2b-1, 2b)$, we set

$$\mathcal{V} = \{\{1,2\}, \{3,4\}, \dots, \{2b-1, 2b\}, \{2b+1\}, \dots, \{n\}\}.$$

Thus the elements of $\mathcal{V}$ are just the orbits of $\langle t \rangle$ upon Ω. For each $x \in X$, we define the x-graph $\mathcal{G}_x$ to be the graph with $\mathcal{V}$ as vertex set, and $v_1, v_2 \in \mathcal{V}$ are joined by an edge whenever there exist $\alpha \in v_1$ and $\beta \in v_2$ with $\alpha \neq \beta$ for which $\{\alpha, \beta\}$ is a $\langle x \rangle$-orbit. Additionally the vertices of $\mathcal{G}_x$ corresponding to 2-cycles of t will be coloured black (●) and the other vertices white (○). Therefore $\mathcal{G}_x$ has b black vertices and $n - 2b$ white vertices. Note that the edges in $\mathcal{G}_x$ are in one-to-one correspondence with the 2-cycles of x. So the number of edges in $\mathcal{G}_x$ is the same as the number of black vertices. As an example, taking

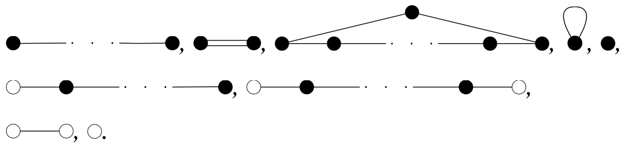
$$n = 16,$$
$$t = (1,2)(3,4)(5,6)(7,8)(9,10)(11,12),$$
$$x = (1,3)(2,4)(5,6)(9,11)(12,13)(14,15),$$

$\mathcal{G}_x$ looks like



Lemma 2.1. *For $x \in X$, the possible connected components of $\mathcal{G}_x$ are*

(i), (ii), (iii)



Proof. This follows from observing that a black vertex can have valency at most two while a white vertex has valency at most one. □

Lemma 2.2. *The following statements hold:*

(i) *Every graph with b black vertices of valency at most two, $n - 2b$ white vertices of valency at most one, and exactly b edges is the x-graph for some $x \in X$.*

(ii) *If $x, y \in X$, then x and y are in the same $C_G(t)$-orbit if and only if $\mathcal{G}_x$ and $\mathcal{G}_y$ are isomorphic graphs (where isomorphisms preserve the colour of vertices).*

(iii) *Let $C_1, C_2, \ldots, C_\ell$ be the connected components of $\mathcal{G}_x$. Assume that x_i and t_i are the corresponding parts of x and t, and let b_i, w_i and c_i be, respectively, the number of black vertices, white vertices and cycles in C_i. Then*

(a) *the order of tx is the least common multiple of the orders of $t_i x_i$, where $i = 1, \ldots, \ell$,*

(b) *for $i = 1, \ldots, \ell$, the order of $t_i x_i$ is $(2b_i + w_i)/(c_i + 1)$.*

Proof. See [8, Lemma 2.1 and Proposition 2.2]. □

Suppose for $x \in X$ the connected components of $\mathcal{G}_x$ are $C_1, C_2, \ldots, C_\ell$, and for each such component let x_i and t_i be the corresponding parts of x and t. Observe that for $i \neq j$ both t_i and x_i commute with both t_j and x_j. So in the above example, $\ell = 6$ with

$$t_1 = (1,2)(3,4), \quad t_2 = (5,6), \quad t_3 = (7,8),$$
$$t_4 = (9,10)(11,12)(13), \quad t_5 = (14)(15), \quad t_6 = (16),$$

and

$$x_1 = (1,3)(2,4), \quad x_2 = (5,6), \quad x_3 = (7)(8),$$
$$x_4 = (9,11)(12,13)(10), \quad x_5 = (14,15), \quad x_6 = (16).$$

We remark that, as $\mathcal{G}_x$ has b edges, the number of connected components of type ●—· · ·—● and of type ○—●—· · ·—●—○ must be equal (including ○—○ in the latter type). This is an important observation for part of the proof of Theorem 1.1. Consider the following simple situation:

$$t = (1,2)(3,4)(5,6)(7)(8,9)(10) \quad \text{and} \quad x = (1)(2,3)(4,5)(6)(7,8)(9,10),$$

with $n = 10$. Then $\mathcal{G}_x$ is

with

$$t_1 = (1,2)(3,4)(5,6), \quad t_2 = (7)(8,9)(10),$$
$$x_1 = (1)(2,3)(4,5)(6), \quad x_2 = (7,8)(9,10)$$

being the parts of t and x corresponding to the two connected components. In our proof of Theorem 1.1 we argue by induction on n, and seek to exploit the symmetric subgroups $\mathrm{Sym}(\Lambda)$, where Λ is the support of a connected part of t. But as we see in this small example, t_1 and x_1 are not conjugate in $\mathrm{Sym}(\{1,2,3,4,5,6\})$, nor are t_2 and x_2 in $\mathrm{Sym}(\{7,8,9,10\})$, and so our inductive strategy will fail. However, this obstacle may be overcome by pairing up connected components ●—· · ·—● and ○—●—· · ·—●—○ of $\mathcal{G}_x$ and applying induction to $\mathrm{Sym}(\Lambda)$ where Λ is the union of the support of t on these two connected components. This kind of issue does not arise with any of the other types of connected components of $\mathcal{G}_x$. While on the subject of potential pitfalls in the proof of Theorem 1.1 we mention the connected components ●═● of $\mathcal{G}_x$. Let t_i and x_i be the parts of t and x corresponding to this connected component, and set $\Lambda = \mathrm{supp}(t_i)$. Then $\mathrm{Sym}(\Lambda) \cong \mathrm{Sym}(4)$ with t_i and x_i having cycle type 2^2 in $\mathrm{Sym}(\Lambda)$, and there is no path between t_i and x_i in the $\mathrm{Sym}(\Lambda)$ local fusion graph. To deal with such connected components of $\mathcal{G}_x$, we are forced to bring all of $\mathcal{G}_x$ into play – this turns out to be a substantial part of the proof of Theorem 1.1.

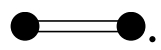
Suppose $x, y \in X$. We shall use $\mathcal{G}_x^y$ to denote the x-graph where y plays the role of t – so the vertices of $\mathcal{G}_x^y$ are the orbits of $\langle y \rangle$ on Ω with vertices w_1, w_2 joined if there exists α in w_1 and β in w_2 with $\alpha \neq \beta$ and $\{\alpha, \beta\}$ an $\langle x \rangle$-orbit. So $\mathcal{G}_x^t$ is just $\mathcal{G}_x$. For more on x-graphs, see [8, Section 2.1].

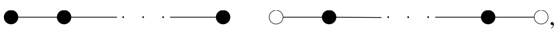
3 The diameter of $\mathcal{F}(G, X)$

In this section we prove Theorem 1.1. So we have $G = \mathrm{Sym}(n)$ with $n \geq 5$, X a G-conjugacy class of involutions and t a fixed element of X. For $n \leq 16$, Magma [13] makes relatively short work of checking that $\mathcal{F}(G, X)$ is connected and has diameter 2. So we may assume $n > 16$.

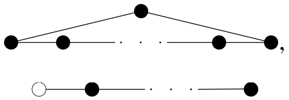
We proceed by induction on n. Let $x \in X$. We aim to show that $d(t, x) \leq 2$. Since there are plainly $x \in X$ for which $d(t, x) > 1$, this would prove that

$$\mathrm{Diam}(\mathcal{F}(G, X)) = 2.$$

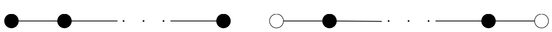
Suppose for the moment that $\mathcal{G}_x$ contains no connected components of type . If $\mathcal{G}_x$ is not connected and not of type

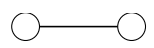


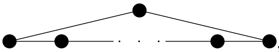
then, by induction, $d(t, x) \leq 2$. Thus, using Lemma 2.1, we may assume $\mathcal{G}_x$ is one of



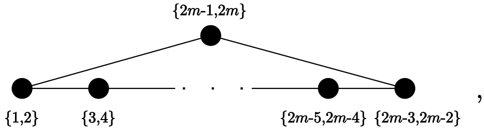
or



(allowing  as a possibility in the latter component). In (3.1), (3.2) and (3.3) we deal with each of these possibilities in turn.

(3.1) If $\mathcal{G}_x$ is , then $d(t, x) \leq 2$.

Assume, without loss of generality, that $t = (1, 2)(3, 4), \ldots, (2m - 1, 2m)$. So $\mathcal{G}_x$ has m black vertices. If m is odd, then tx has odd order by Lemma 2.2 (iii) (b), and so $d(t, x) \leq 1$. While if m is even, we assume that $\mathcal{G}_x$ is labelled like so



and that

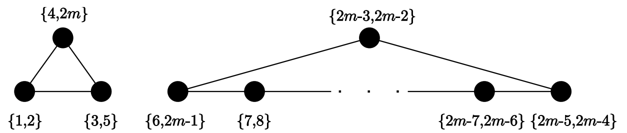
$$x = (1, 2m)(2, 3)(4, 5) \ldots (2m - 4, 2m - 3)(2m - 2, 2m - 1).$$

Note that we have $m \geq 4$. We select

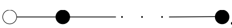
$$y = (1, 2)(3, 5)(4, 2m)(6, 2m - 1)(7, 8)(9, 10) \ldots (2m - 3, 2m - 2).$$

Then $y \in X$ and $ty = (3, 2m, 6)(4, 5, 2m - 1)$, and hence $d(t, y) \leq 1$. Now $\mathcal{G}_x^y$

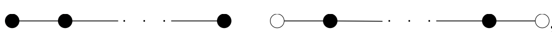
is seen to be



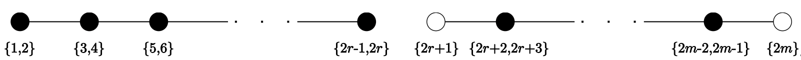
Since the two connected components of $\mathcal{G}_x^y$ have sizes 3 and $m-3$, both of which are odd, Lemma 2.2 (iii) implies that yx has odd order. Therefore $d(x, y) \leq 1$ and so (3.1) holds.

(3.2) If $\mathcal{G}_x$ is , then $d(t, x) = 1$.

Since $\mathcal{G}_x$ is a connected component with one white vertex, (3.2) follows from Lemma 2.2 (iii).

(3.3) If $\mathcal{G}_x$ is , then $d(t, x) \leq 2$.

Without loss we may label $\mathcal{G}_x$ as follows:

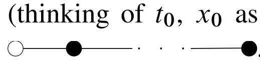


where

$$t = (1,2)(3,4)(5,6)\dots(2r-1,2r)(2r+1)\dots$$
$$\dots(2r+2,2r+3)\dots(2m-2,2m-1)(2m).$$

We may assume that

$$x = (2,3)(4,5)\dots(2r-2,2r-1)(2r+1,2r+2)\dots(2m-1,2m).$$

Set $t_0 = (1,2)t$ and $x_0 = x(2m-1,2m)$. Then t_0 and x_0 are H-conjugate, where $H = \mathrm{Sym}(\Omega \setminus \{1, 2m\})$. Observing that $\mathcal{G}_{x_0}^{t_0}$ (thinking of t_0, x_0 as involutions in H) has two connected components of type , we deduce from Lemma 2.2 (iii) that t_0x_0 has odd order. Let $y = (1,2m)t_0$. Then $y \in X$ and

$$ty = (1,2)t_0(1,2m)t_0 = (1,2)(1,2m) = (1,2,2m),$$

whence $d(t, y) \leq 1$. Also, as t_0 and x_0 fix 1 and $2m$,

$$\begin{aligned} yx &= (1,2m)t_0x_0(2m-1,2m) \\ &= t_0x_0(1,2m)(2m-1,2m) \\ &= t_0x_0(1,2m-1,2m). \end{aligned}$$

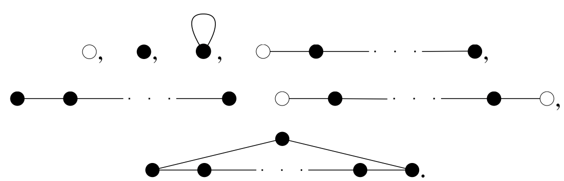
Now $t_0x_0 \in H$ is a product of two disjoint odd cycles of lengths, say, m_1, m_2. If $2m-1$ is in say the latter cycle of t_0x_0, then tx is a disjoint product of an m_1-cycle and an (m_2+2)-cycle. Thus yx has odd order and so $d(x, y) \leq 1$. Therefore $d(t, x) \leq 2$, which proves (3.3).

Taken together (3.1), (3.2) and (3.3) prove Theorem 1.1 when $\mathcal{G}_x$ contains no connected components of type ●═●. It therefore remains to analyse $\mathcal{G}_x$ when it has connected components of type ●═●. If there are an even number of ●═● connected components, then, as the local fusion graphs for Sym(8) have diameter two, by pairing them up and using induction we obtain our result. Thus we may assume $\mathcal{G}_x$ contains exactly one ●═● connected component. Let $\mathcal{H}_x$ denote the union of all the other connected components of $\mathcal{G}_x$. Also we may assume

$$t = (1, 2)(3, 4)t_0, \quad x = (1, 3)(2, 4)x_0$$

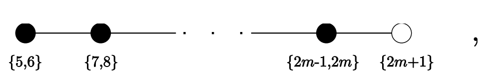
where t_0 and x_0 are involutions in $H = \mathrm{Sym}(\Omega \setminus \{1, 2, 3, 4\})$.

Let $\mathcal{C}_x$ be a subgraph of $\mathcal{H}_x$, where $\mathcal{C}_x$ is one of



Then $t_0 = t_1t_2$, $x_0 = x_1x_2$ where t_1, x_1 are the parts in $\mathcal{C}_x$ and t_2, x_2 the parts in $\mathcal{H}_x \setminus \mathcal{C}_x$. Then t_2 and x_2 are conjugate involutions in some symmetric subgroup of G and the x_2-graph (with respect to t_2) is $\mathcal{H}_x \setminus \mathcal{C}_x$. Since $\mathcal{H}_x$ contains no subgraph ●═●, we can find y_2 in this conjugacy class such that t_2y_2 and y_2x_2 have odd order. Since y_2 commutes with both $(1, 2)(3, 4)t_1$ and $(1, 3)(2, 4)x_1$, without loss we may assume that $\mathcal{H}_x = \mathcal{C}_x$. We now work through the possibilities for $\mathcal{H}_x$ making repeated use of Lemma 2.2 (iii) to show $d(t, x) \leq 2$. The first three possibilities listed above do not need attention as $n \geq 16$.

If $\mathcal{H}_x$ is



then

$$t = (1, 2)(3, 4)(5, 6)(7, 8)\ldots(2m-1, 2m)(2m+1)$$

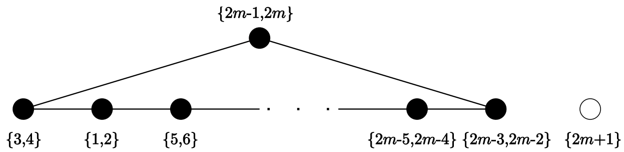
and, without loss of generality,

$$x = (1, 3)(2, 4)(5)(6, 7)(8, 9)\ldots(2m, 2m+1).$$

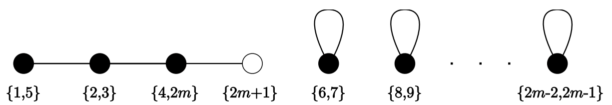
In the case when m is odd, we select

$$y = (1, 5)(2, 3)(4, 2m)(6, 7)(8, 9)\ldots(2m-2, 2m-1)(2m+1),$$

and then $\mathcal{G}_y$ is



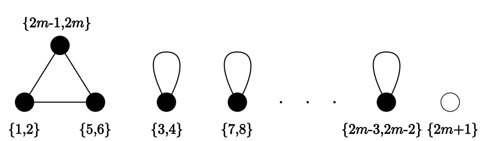
while $\mathcal{G}_x^y$ is



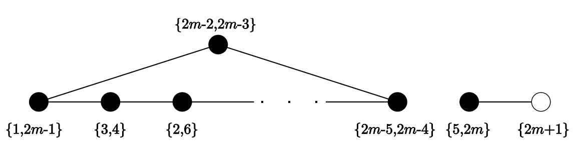
So $\mathcal{G}_y$ consists of a cycle of m black vertices and one white vertex while $\mathcal{G}_x^y$ has one connected component with three black vertices and one white vertex with each of the other components being a cycle with one black vertex. Consequently ty and yx both have odd order by Lemma 2.2 (iii), whence $d(t, x) \leq 2$. If m were to be even, instead we choose

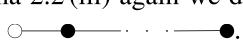
$$y = (1, 2m - 1)(2, 6)(3, 4)(5, 2m)(7, 8)(9, 10) \dots (2m - 3, 2m - 2)$$

which gives $\mathcal{G}_y$ as

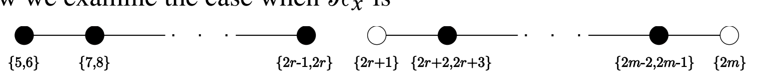


and $\mathcal{G}_x^y$ as



Here the cycle of black vertices in $\mathcal{G}_x^y$ has $m - 1$ black vertices whence using Lemma 2.2 (iii) again we deduce that $d(t, x) \leq 2$, and this settles the case when $\mathcal{H}_x$ is .

Now we examine the case when $\mathcal{H}_x$ is



So

$$t = (1, 2)(3, 4)(5, 6) \dots (2r - 1, 2r)(2r + 1) \dots$$
$$\dots (2r + 2, 2r + 3) \dots (2m - 2, 2m - 1)(2m)$$

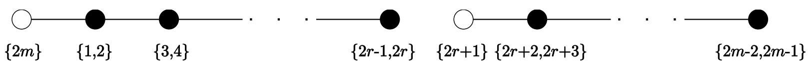
and

$$x = (1,3)(2,4)(5)(6,7)(8,9)\dots(2r-2,2r-1)(2r)\dots$$
$$\dots(2r+1,2r+2)\dots(2m-1,2m).$$

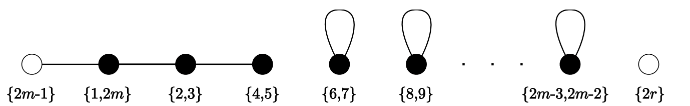
Choosing

$$y = (1,2m)(2,3)(4,5)\dots(2r-2,2r-1)(2r+1,2r+2)\dots(2m-3,2m-2),$$

we observe that $\mathcal{G}_y$ is

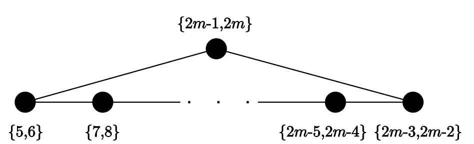


and $\mathcal{G}_x^y$ is



Yet again Lemma 2.2 (iii) shows that $d(t,y) \leq 1 \geq d(y,x)$ so dealing with this possibility for $\mathcal{H}_x$.

We now consider our final case which is when $\mathcal{H}_x$ is



Thus

$$t = (1,2)(3,4)(5,6)(7,8)\dots(2m-1,2m)$$

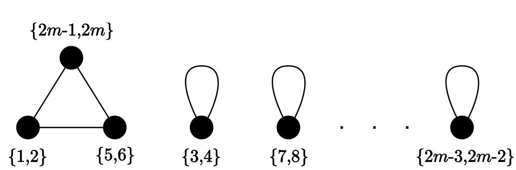
and, without loss,

$$x = (1,3)(2,4)(6,7)(8,9)\dots(2m,5).$$

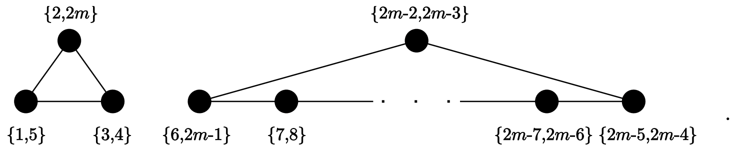
When m is even, we select

$$y = (1,5)(2,2m)(3,4)(6,2m-1)(7,8)(9,10)\dots(2m-3,2m-2)$$

and as a result $\mathcal{G}_y$ is



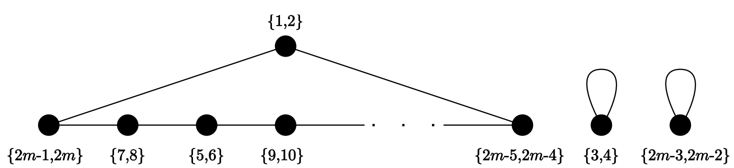
and $\mathcal{G}_x^y$ is



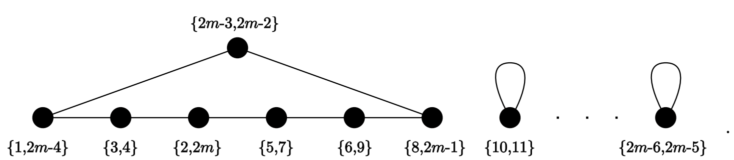
Before dealing with m odd we recall that we are assuming $n(=2m) \geq 16$. So $2m-4 > 10$ and therefore the choice we now make gives us an element of X. Take

$$y = (1, 2m-4)(2, 2m)(3, 4)(5, 7)(6, 9)(8, 2m-1)(10, 11)\dots$$
$$\dots(12, 13)\dots(2m-6, 2m-5)(2m-3, 2m-2).$$

Hence $\mathcal{G}_y$ is



and $\mathcal{G}_x^y$ is



Use of Lemma 2.2 (iii) shows that whether m is even or odd we have $d(t, x) \leq 2$.

Having successfully dealt with all the possibilities for $\mathcal{H}_x$, the proof of Theorem 1.1 is complete.

4 Connectedness of $\mathcal{F}_\pi(G, X)$

As promised here we prove Theorem 1.2 which we restate.

Theorem 4.1. *Suppose that $G = \mathrm{Sym}(n)$, X is a G-conjugacy class of involutions and π is a set of odd positive integers. Then $\mathcal{F}_\pi(G, X)$ is either totally disconnected or connected.*

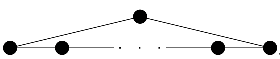
Proof. We argue by induction on n, with $n = 1$ clearly holding. Assume that $\mathcal{F}_\pi(G, X)$ is not totally disconnected, and let $t \in X$ be such that Y, the connected component of t in $\mathcal{F}_\pi(G, X)$, has $|Y| > 1$. Put $K = \mathrm{Stab}_G(Y)$. If $K = G$, then $Y = X$ and hence $\mathcal{F}_\pi(G, X)$ is connected. So we now suppose $K \neq G$, and argue for a contradiction.

Let $x \in Y$ with $d(t, x) = 1$. Then $z = tx$ has order in the set π, and we have

(4.1) $\langle C_G(t), C_G(x)\rangle \leq K$,

(4.2) $\mathrm{supp}(t) \cup \mathrm{supp}(x) = \Omega$.

If (4.2) is false, then t and x both fix some $\alpha \in \Omega$. So $t, x \in G_\alpha \cong \mathrm{Sym}(n-1)$. Since t and x are G_α-conjugate and the order of tx is in π, it follows by induction that $\mathcal{F}_\pi(G_\alpha, X \cap G_\alpha)$ is connected. Therefore $G_\alpha \leq K$, and so, as $K \neq G$ and G_α is a maximal subgroup of G, $K = G_\alpha$. If t fixes a further element of Ω, say β, then, by (4.1), $(\alpha, \beta) \in C_G(t) \leq K$, contrary to $K = G_\alpha$. So t (and hence also x) fixes only α. Thus $\mathcal{G}_x$ has only one white node (namely $\{\alpha\}$) with the remaining connected components being either

 or .

Without loss we assume $\alpha = n$.

Suppose that $\mathcal{G}_x$ has

as a component. So, without loss of generality,

$$t = (1,2)(3,4)\ldots(n-2, n-1) = (1,2)t_1$$

and $x = (1,2)x_1$, where $x_1 \in \mathrm{Sym}(\{3, 4, \ldots, n-1\})$. As $K \neq G$, we must have $n > 3$. Thus $t_1, x_1 \in H = \mathrm{Sym}(\{3, 4, \ldots, n-1\})$ with t_1 and x_1 being H-conjugate involutions and the order of t_1x_1, being the same as that of tx, lies in π. Using induction again we infer that $\mathcal{F}_\pi(H, t_1^H)$ is connected. Hence, in $\mathcal{F}_\pi(H, t_1^H)$ there is a path from t_1 to

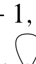
$$s_1 = (3,4)(5,6)\ldots(n-4, n-3)(n-1, n),$$

say $t_1 = y_0, y_1, \ldots, y_m = s_1$ $(y_i \in t_1^H)$. Consequently

$$t = (1,2)t_1 = (1,2)y_0, (1,2)y_1, \ldots (1,2)y_m = (1,2)s_1$$

is a path in $\mathcal{F}_\pi(G, X)$ from t to

$$(1,2)(3,4)(5,6)\ldots(n-4, n-3)(n-1, n).$$

But then $(n-1, n) \in K$, whereas $K = G_\alpha$. This rules out  as being a connected component of $\mathcal{G}_x$.

Let $t = t_1 t_2 \cdots t_k$ and $x = x_1 x_2 \cdots x_k$, where

$$t_1 = (1,2)\dots(\ell_1 - 1, \ell_1),$$
$$t_2 = (\ell_1 + 1, \ell_1 + 2)\dots(\ell_1 + \ell_2 - 1, \ell_1 + \ell_2),$$
$$\vdots$$

and

$$x_1 = (2,3)(4,5)\dots(\ell_1 - 2, \ell_1 - 1)(1, \ell_1),$$
$$x_2 = (\ell_1 + 2, \ell_1 + 3)\dots(\ell_1 + \ell_2 - 2, \ell_1 + \ell_2 - 1)(\ell_1 + 1, \ell_1 + \ell_2),$$
$$\vdots$$

So the elements $x_1, x_2, \dots$ correspond to the connected components of $\mathcal{G}_x$. By Lemma 2.2 (iii) (b), $t_1 x_1$ has order $m = \ell_1/2$. Now the order of $z = tx$ is the least common multiple of the orders of $t_1x_1, t_2x_2, \dots, t_kx_k$, whence m must be odd. Put

$$w = (n, 1, 3, 5, \dots, m-2, m-1, m-3, \dots, 6, 4, 2).$$

Then w is a cycle of length m, and so of order m. Further (by design) $w^{t_1} = w^{-1}$ and hence

$$y_1 = t_1 w = (1,n)(2,3)(4,5)\dots(m-3, m-2)(m, m+1)\dots$$
$$\dots(m+2, m+3)\dots(\ell_1 - 1, \ell_1)$$

is conjugate to t_1. Also, of course, $t_1 y_1 = w$ has order m. So $y = y_1 t_2 \cdots t_k \in X$ and the order of ty is the same as that of tx. Therefore we have $y \in Y$ and hence $(1,n) \in K$. This contradicts the earlier deduction that $K = G_\alpha$, and with this we have proven (4.2).

(4.3) K acts transitively and primitively on Ω.

Since $C_G(t)$ and $C_G(x)$ have shape $2^k \mathrm{Sym}(2k) \times \mathrm{Sym}(n-2k)$, $k = |\mathrm{supp}(t)|/2$, and t and x do not commute, (4.1) and (4.2) imply that K is transitive on Ω. Suppose K does not act primitively on Ω. Then we may choose a nontrivial block Λ for K with $\alpha \in \Lambda \cap \mathrm{supp}(t)$. If $\Lambda \not\subseteq \mathrm{supp}(t)$, then the action of $C_G(t)$ on Ω results in $\Lambda = \Omega$. Thus $\Lambda \subseteq \mathrm{supp}(t)$. Again, using the action of $C_G(t)$ on Ω we deduce that either $\Lambda = \mathrm{supp}(t)$ or $\Lambda = \{\alpha, \beta\}$ where $\beta = \alpha^t$. Since t and x do not commute, we may further assume that $\alpha \in \mathrm{supp}(x)$ is such that $\alpha^x \notin \{\alpha, \beta\}$. So $\alpha \in \mathrm{supp}(x)$ and a similar argument yields that either $\Lambda = \mathrm{supp}(x)$ or $\Lambda = \{\alpha, \alpha^x\}$. In view of (4.2) this then implies that $\Lambda = \Omega$, contrary to Λ being a nontrivial block. Thus (4.3) holds.

Plainly $C_G(t)$, and hence K, contains transpositions. Thus Jordan's theorem [23] and (4.3) force $K = G$. With this contradiction the proof of the theorem is complete. □

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Received November 19, 2010; revised June 28, 2012.

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