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On 2-Engel groups and Bruck loops

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Abstract. A classical construction associates a Bruck loop with a Moufang or Bol loop on which the squaring map is a permutation. The idea of that construction is now extended to a *quadratically closed* Bol loop or group (where the squaring operation is surjective) that satisfies the 2-Engel condition (for groups: elements commute with their conjugates). The Bruck loop is built on a space of sequences of elements from the original Bol loop. More generally, given a quadratically closed loop with the 2-Engel condition from the join of the varieties of diassociative and (right) Bol loops, a power-associative loop with the right inverse and automorphic inverse properties is constructed in similar fashion.

1 Introduction

On the real line, the reflection $a \circ b$ of a point a in a point b is given as $2b - a$. More generally, in a group, Moufang, (right) Bol or diassociative loop M, one may define the *core* operation as $(ba^{-1})b$.[1] Bruck's original intention with this definition was to obtain an isotopy invariant for Moufang loops [3, Section VII.5]. The core is a right quasigroup, a "symmetric space" in the algebraic sense of Loos [16, 17]. In a Moufang loop on which squaring is bijective, Bruck showed that the core is a two-sided quasigroup, isotopic to a Bol loop with the automorphic inverse property, or what is nowadays known as a Bruck loop [3, Theorem VII.5.2]. Glauberman used this Bruck loop structure to obtain many results about finite Moufang loops of odd order, including their solvability and the validity of Sylow's and Hall's Theorems [7, 8]. Since then, Bruck loops have gained additional importance through their natural occurrence in many areas, such as geometry, matrix decompositions, and special relativity theory – compare [13], [23, Exercises 2.9, 17, 18], and [26], for example.

The aim of the current paper is to investigate the extension of Bruck's construction to loops, not necessarily Moufang, in which the squaring map is not necessarily bijective. Surjectivity is required, so a loop is said to be *quadratically closed* if the squaring map is surjective. For a quadratically closed loop, a process known

[1] This concept is not to be confused with the usage of the word "core" to denote the largest normal subgroup of odd order in a finite group.

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as *bijectivization* is applied to obtain a bijective version of the squaring map on the so-called *history space*, a space of doubly-infinite sequences of loop elements such that each element in the sequence is succeeded by its square. Forward shifts of sequences become proxies for the square, while backward shifts play the role of the square root, a two-sided inverse for the squaring realized as the forward shift.

In order for the history space to stay closed under pointwise multiplication in the original loop, that loop is required to lie in the varieties of diassociative or (right) Bol loops (or the join of these two varieties), and to satisfy a 2-Engel condition generalizing the group condition that elements commute with their conjugates. In these circumstances, it is shown that the history space carries a loop structure with the right inverse and automorphic inverse properties (Proposition 8.5), and forms a Bruck loop if the original loop is Moufang or (right) Bol (Theorem 8.7). With a quadratically closed field K as ground field, matrix groups $A(K)$, $B(K)$, $C(K)$ of respective degrees 1, 2, and 3 satisfy the properties needed for the Bruck loop construction to work (Example 5.5, Theorems 5.7, 5.8). Over the ground field of complex numbers, the construction with $A(\mathbb{C})$ offers a new setting for complex reflections and midpoints on the unit circle, and the Riemann surface of the square root function (Example 6.2, Remark 8.8).

The plan of the paper is as follows. Basic quasigroup, loop, and group properties are summarized in Section 2. Section 3 presents the core construction in the very broad context of right quasigroups, and uses it to formulate a 2-Engel condition that generalizes the familiar group condition. Section 4 describes Bruck's original construction, extended where possible from Moufang loops both to (right) Bol loops and general diassociative loops, and makes a connection with the "twisted" terminology: twisted subsets and twisted subgroups. Quadratic closure forms the topic of Section 5, where the groups $A(K)$, $B(K)$, and $C(K)$ are presented. Section 6 outlines the process of bijectivization. Section 7 then establishes the closure properties of the history space, ready for the main results in Section 8.

2 Engel groups and Bol loops

This section records some basic definitions. Readers may consult [23, 24] for further explanation, or for other basic notation or algebraic conventions used here without explicit clarification.

A *magma* $(M, *)$ is a set M that is equipped with a binary operation $M^2 \to M$; $(x, y) \mapsto x * y$, often called *multiplication*, and occasionally denoted merely by the juxtaposition xy of its arguments. The *square* x^2 of an element x of a magma $(M, *)$ is defined as $x * x$. The element x is said to be *idempotent* if $x^2 = x$. The magma $(M, *)$ itself is said to be *idempotent* if each element x of M is idempotent.

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If m is an element of a magma $(M, *)$, the *right multiplication* $R_*(m)$ is the map $M \to M$; $x \mapsto xm$, while the *left multiplication* $L_*(m)$ is the map $M \to M$; $x \mapsto mx$. If the multiplication is denoted by juxtaposition, then no suffix is placed on the R or L.

Definition 2.1. Let $(M, \cdot)$ be a magma.

(1) The magma is a (*combinatorial*) *right quasigroup* if, for all y and z in M, there is a unique element x of M such that $x \cdot y = z$.

(2) The magma is a(n *equational*) *right quasigroup* $(M, \cdot, /)$ if there is a binary operation $(x, y) \mapsto x/y$ of *right division* such that the identities $(x \cdot y)/y = x = (x/y) \cdot y$ hold in M.

(3) The magma is a(n *equational*) *left quasigroup* $(M, \cdot, \backslash)$ if there is a binary operation $(x, y) \mapsto x\backslash y$ of *left division* such that the identities $y\backslash(y \cdot x) = x = y \cdot (y\backslash x)$ hold in M.

(4) The magma is a (*two-sided*) *quasigroup* $(M, \cdot, /, \backslash)$ if it forms both a right quasigroup $(M, \cdot, /)$ and a left quasigroup $(M, \cdot, \backslash)$. In this case one often says that $(M, \cdot)$ is a quasigroup.

(5) A quasigroup $(M, \cdot)$ is a *loop* $(M, \cdot, 1)$ if there is an element 1 of M, the so-called *identity element*, such that the identities $1 \cdot x = x = x \cdot 1$ hold in M.

(6) An element x of a loop $(M, \cdot, 1)$ is said to be *invertible* if there is an element x^{-1} of M (the *inverse* of x), such that $x \cdot x^{-1} = 1 = x^{-1} \cdot x$.

(7) If the subloop of a loop M generated by each element of M (under all the operations $\cdot, /, \backslash$) is associative, and thus forms a cyclic group, then the loop is said to be *power-associative*.

(8) A loop $(M, \cdot, /, \backslash, 1)$ is *diassociative* if for each at most 2-element subset S of M, the subloop generated by the elements of the subset S (under all the operations $\cdot, /, \backslash, 1$) is associative, and thus forms a group.

(9) A loop $(M, \cdot)$ is a (*right*) *Bol loop* if it satisfies the identity $((x \cdot y) \cdot z) \cdot y = x \cdot ((y \cdot z) \cdot y)$, so that $R_.(y)R_.(z)R_.(y) = R_.((y \cdot z) \cdot y)$.

(10) A loop $(M, \cdot)$ is said to be a *Moufang loop* if it satisfies the identity $((x \cdot y) \cdot z) \cdot y = x \cdot (y \cdot (z \cdot y))$.

Note that the identity for right Bol loops becomes trivial if $x = 1$, while the identity for Moufang loops does not. Commutative Bol loops are Moufang. Each element of a power-associative loop is invertible, while diassociative loops are certainly power-associative. By Moufang's theorem, Moufang loops are diassociative

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[3, Section VII.4]. A loop is said to have the *automorphic inverse property* if each element is invertible, and if inversion is an automorphism of the loop. Right Bol loops are power-associative [24, Theorem I.4.2.8].

If a magma $(M, \cdot)$ is a combinatorial right quasigroup, then the unique solution x to the equation $x \cdot y = z$ may be taken as z/y. Conversely, if $(M, \cdot, /)$ is an equational right quasigroup, then the unique solution x to the equation $x \cdot y = z$ may be given as z/y. Thus the concepts of Definition 2.1 (1) and (2) are equivalent: one simply uses the term *right quasigroup*. Similar equivalences exist for left and two-sided quasigroups, justifying the unqualified terms *left quasigroup* and *quasigroup*. Note that the further identities

$$y/(x\backslash y) = x = (y/x)\backslash y \tag{2.1}$$

hold in a (two-sided) quasigroup [23, Section 1.3].

If a magma $(M, \cdot)$ is a right quasigroup, each right multiplication $R_{\cdot}(y)$ by an element y of M is invertible, with $z/y = zR(y)^{-1}$ for $z \in M$. Similarly, left multiplications of left quasigroups are invertible. Now let $(M, \cdot)$ be a quasigroup. The set $\{R_{\cdot}(m), L_{\cdot}(m) \mid m \in M\}$ of all right and left multiplications is a subset of $M!$, the group of all permutations of M. Then the *multiplication group* of $(M, \cdot)$ is the subgroup $\operatorname{Mlt} M$ of $M!$ generated by $\{R_{\cdot}(m), L_{\cdot}(m) \mid m \in M\}$.

The 2-*Engel* condition in a group or diassociative loop M is the law

$$[[x, y], y] = 1. \tag{2.2}$$

The condition has an equivalent formulation as

$$[x^y, x] = 1. \tag{2.3}$$

Compare [10, III.6.4] for the group case of the equivalence. Note that the equivalence still holds in diassociative loops, since it takes place in the subgroup generated by the two variables x and y. In group theory, the 2-Engel conditions originated from Burnside's proof of the local finiteness of groups of exponent 3: such groups are 2-Engel (see [4]). Subsequently, Burnside studied general 2-Engel groups (see [5]).

Diassociative loops of nilpotence class at most 2 satisfy the 2-Engel condition. Conversely, Hopkins [9] showed that 2-Engel groups are nilpotent of class at most 3.[2] (Theorem 5.8 provides an example of a 2-Engel group of nilpotence class 3.) However, while commutative Moufang loops satisfy the 2-Engel condition, there is no bound on their nilpotence class [3, Section VIII.1], [18], [22].

[2] This result is often attributed to Levi [15] – compare [10, III.6.5], for example. Further references, and other results on 2-Engel groups, may be found in [3, Section VII.5], [20, 25].

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3 The core of a quasigroup

Let $(M, \cdot, /)$ be a right quasigroup. The (*right*) *core* of M is defined as the magma $(M, \circ)$ with the operation $x \circ y = (y/x)y$. (Note that this is the opposite of Bruck's original concept, which was defined in the context of Moufang loops (see [3, VII (5.1)]). The current choice gives a better match to other notational conventions.)

Proposition 3.1. *Suppose that $(M, \cdot, /, \backslash)$ is a quasigroup. Then the core of M is an idempotent right quasigroup $(M, \circ, //)$, with $x//y = (x/y)\backslash y$ for all x, y in M.*

Proof. For x, y in M, the identities

$$(x \circ y)//y = \big([(y/x)y]/y\big)\backslash y = (y/x)\backslash y = x$$

and

$$(x//y) \circ y = \big(y/[(x/y)\backslash y]\big)y = (x/y)y = x$$

are verified using (2.1). Also note that $x \circ x = (x/x)x = x$. □

Corollary 3.2. *Let M be a quasigroup. Then the core of M is a quasigroup if and only if the following, equivalent conditions hold:*

(a) *For $x, z \in M$, there is a unique element y of M such that $(y/x)y = z$.*

(b) *For $x, z \in M$, there is a unique element y of M such that $y/x = z/y$.*

(c) *The set $\{L_/(y)R_\cdot(y) \mid y \in M\}$ of permutations of M is sharply transitive, see* [23, Section 8.1].

The core may be used to formulate a general 2-Engel condition within the context of right quasigroups.

Definition 3.3. A right quasigroup $(M, \cdot, /)$ is said to satisfy the (*right*) 2-*Engel condition* if

$$(x \circ y)^2 = x^2 \circ y^2$$

for x, y in M.

Example 3.4. Suppose that $(M, \cdot, /)$ is a trivial right quasigroup with

$$x \cdot y = x/y = x$$

for all x, y in M. Then $(M, \circ, \backslash\backslash)$ is a trivial left quasigroup with

$$x \circ y = x\backslash\backslash y = y$$

for all x, y in M. In particular, trivial right quasigroups satisfy the right 2-Engel condition.

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If M is a diassociative loop, then the consistency of the new right 2-Engel condition with the earlier equivalent 2-Engel conditions (2.2) or (2.3) is established as follows.

Proposition 3.5. *Let M be a diassociative loop. Then the following conditions are equivalent:*

(a) $\forall\, x, y \in M,\ (x \circ y)^2 = x^2 \circ y^2.$

(b) $\forall\, x, y \in M,\ yxxy = xyyx.$

(c) *Each element of M commutes with its conjugates.*

Proof. Note

$$\begin{aligned} & \forall\, x, y \in M,\ (x \circ y)^2 = x^2 \circ y^2 \\ \Leftrightarrow\quad & \forall\, x, y \in M,\ y(x^{-1}y)(yx^{-1})y = y(yx^{-1})(x^{-1}y)y \\ \Leftrightarrow\quad & \forall\, x, y \in M,\ (x^{-1}y)(yx^{-1}) = (yx^{-1})(x^{-1}y) \\ \Leftrightarrow\quad & \forall\, x, z \in M,\ (x^{-1}zx)z = z(x^{-1}zx), \end{aligned}$$

the latter implication being obtained using the substitution $y = zx$. □

4 Bruck loops

If M is a diassociative loop, then one may extend a group-theoretical definition [21] by calling substructures of the pointed magma $(M, \circ, 1)$ *twisted subsets* of M. Similarly, one may define a *twisted subloop* of a general loop M to be a substructure of the pointed magma $(M, \Delta, 1)$ with $x \Delta y = y(xy)$, extending the *twisted subgroup* terminology of [1]. Note that the set $\mathbb{N}$ of natural numbers (0 included!) is a twisted subgroup of the additive group $\mathbb{Z}$, but not a twisted subset.

Proposition 4.1. *Let M be a diassociative loop.*

(a) *The structure $(M, \Delta, 1)$ is derived from $(M, \circ, 1)$.*

(b) *Each twisted subset of M is a twisted subloop of M.*

(c) *If M is finite, then each twisted subloop of M is a twisted subset of M.*

Proof. For (a), consider elements x, y of M. Then

$$x \Delta y = y(xy) = x^{-1} \circ y = (x \circ 1) \circ y.$$

The statement (b) is an immediate consequence of (a). Now for (c), suppose that N is a twisted subloop of M, with M finite. Let x and y be elements of N. If x has even order $2r$ with a positive integer r, then $xR_\Delta(x)^{r-1} = x^{2r-1} = x^{-1}$, so

$$x \circ y = yx^{-1}y = (xR_\Delta(x)^{r-1}) \Delta y \in N.$$

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On the other hand, if x has odd order $2s + 1$ with a positive integer s, then

$$x^{-1} = x^{2s} = xL_{\triangle}(1)^s,$$

so again $x \circ y = yx^{-1}y = (xL_{\triangle}(1)^s) \triangle y \in N$. □

The following result summarizes properties of the core in a diassociative or right Bol loop. (The statements of the proposition apply equally to a loop lying in the join of the varieties of diassociative and right Bol loops).

Proposition 4.2. *Let M be a diassociative or right Bol loop.*

(a) *Inversion in M is an automorphism of* $(M, \circ)$.

(b) *The core* $(M, \circ)$ *of M forms a right quasigroup* $(M, \circ, \circ)$.

(c) *For $x \in M$, one has $xR_\circ(1) = x^{-1}$ and $xL_\circ(1) = x^2$.*

(d) *If M is both a diassociative and a right Bol loop, so a Moufang loop, then the multiplication group* Mlt M *of M is a transitive group of automorphisms of* $(M, \circ)$.

Proof. For (a), consider x, y in M. Then

$$\begin{aligned}(x^{-1} \circ y^{-1})(x \circ y) &= [(y^{-1}x)y^{-1}][(yx^{-1})y] \\ &= (y^{-1}x)R(y^{-1})R((yx^{-1})y) \\ &= (y^{-1}x)R(y^{-1})R(y)R(x^{-1})R(y) = 1,\end{aligned}$$

so $(x \circ y)^{-1} = x^{-1} \circ y^{-1}$. For (b), one then has

$$xR_\circ(y)^2 = [y(x \circ y)^{-1}]y = [y(x^{-1} \circ y^{-1})]y = \{y[(y^{-1}x)y^{-1}]\}y = x,$$

so $x//y = x \circ y$ in the right quasigroup $(M, \circ, //)$ of Proposition 3.1. For (c), note $x \circ 1 = (1x^{-1})1 = x^{-1}$ and $1 \circ x = (x1)x = x^2$. For statement (d), one may consult [3, p. 120], recalling that the multiplication group of any quasigroup acts transitively [23, p. 36], [24, p. 55]. □

The main result here dates back to Bruck for the Moufang loop case [3, Theorem VII.5.2], and was extended to Bol loops by Belousov [2]. The current proof is designed to prepare for the subsequent work of this paper.

Theorem 4.3. *Let M be a (right) Bol loop where the squaring map*

$$L_\circ(1)\colon M \to M; x \mapsto x^2$$

has an inverse

$$L_\circ(1)^{-1}\colon M \to M; x \mapsto x^{1/2}.$$

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Then the following hold:

(a) *The core of M is a quasigroup.*

(b) *The operation $x \cdot y = xR_\circ(1)^{-1} \circ yL_\circ(1)^{-1} = (y^{1/2}x)y^{1/2}$ yields a right Bol loop $(M, \cdot, 1)$.*

(c) *Inverses in the Bol loop $(M, \cdot, 1)$ coincide with inverses in the original Bol loop M.*

(d) *Inversion is an automorphism of the loop $(M, \cdot, 1)$.*

Proof. For (a), define $x \backslash\backslash y = (y \circ x^{1/2})^{1/2} \circ x^{1/2}$. Now

$$\begin{aligned}\big[(x^{-1/2}y)x^{-1/2}\big]^2 &= \big[(x^{-1/2}y)x^{-1/2}\big]\big[(x^{-1/2}y)x^{-1/2}\big]\\ &= (x^{-1/2}y)R(x^{-1/2})R\big((x^{-1/2}y)x^{-1/2}\big)\\ &= x^{-1/2}R(y)R(x^{-1})R(y)R(x^{-1/2})\\ &= x^{-1/2}R(x \circ y)R(x^{-1/2})\\ &= (x \circ y)^{-1} \circ x^{-1/2}\\ &= \big[(x \circ y) \circ x^{1/2}\big]^{-1},\end{aligned}$$

the final equality following by Proposition 4.2 (a). Thus

$$\big[(x \circ y) \circ x^{1/2}\big]^{-1/2} = (x^{-1/2}y)x^{-1/2}.$$

Then

$$\begin{aligned}x \backslash\backslash (x \circ y) &= \big\{x^{1/2}\big[(x \circ y) \circ x^{1/2}\big]^{-1/2}\big\}x^{1/2}\\ &= \big\{x^{1/2}\big[(x^{-1/2}y)x^{-1/2}\big]\big\}x^{1/2}\\ &= 1R(x^{1/2})R(x^{-1/2})R(y)R(x^{-1/2})R(x^{1/2}) = y.\end{aligned}$$

On the other hand,

$$\begin{aligned}x \circ (x \backslash\backslash y) &= \big[(x \backslash\backslash y)x^{-1}\big](x \backslash\backslash y)\\ &= \big\{\big(\big[x^{1/2}(y \circ x^{1/2})^{-1/2}\big]x^{1/2}\big)x^{-1}\big\}\big\{\big[x^{1/2}(y \circ x^{1/2})^{-1/2}\big]x^{1/2}\big\}\\ &= \big\{\big[x^{1/2}(y \circ x^{1/2})^{-1/2}\big]x^{-1/2}\big\}\big\{\big[x^{1/2}(y \circ x^{1/2})^{-1/2}\big]x^{1/2}\big\}\\ &= x^{1/2}R\big((y \circ x^{1/2})^{-1/2}\big)R\big(x^{-1/2}\big)\\ &\qquad \times R\big(x^{1/2}\big)R\big((y \circ x^{1/2})^{-1/2}\big)R\big(x^{1/2}\big)\\ &= x^{1/2}R\big(y \circ x^{1/2}\big)^{-1}R\big(x^{1/2}\big)\\ &= x^{1/2}R\big(x^{1/2}\big)^{-1}R(y)R\big(x^{1/2}\big)^{-1}R\big(x^{1/2}\big) = y.\end{aligned}$$

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Thus $(M, \circ, \backslash\backslash)$ is a left quasigroup. The proof of (a) is completed by Proposition 4.2 (b).

For (b), note that

$$\begin{aligned}
\big[(x \cdot y) \cdot z\big] \cdot y &= \{y^{1/2}\big[\big(z^{1/2}\{(y^{1/2}x)y^{1/2}\}\big)z^{1/2}\big]\}y^{1/2} \\
&= 1R\big(y^{1/2}\big)R\big(z^{1/2}\big)R\big(y^{1/2}\big)R\big(x\big)R\big(y^{1/2}\big)R\big(z^{1/2}\big)R\big(y^{1/2}\big) \\
&= 1R\big((y^{1/2}z^{1/2})y^{1/2}\big)R(x)R\big((y^{1/2}z^{1/2})y^{1/2}\big) \\
&= (tx)t
\end{aligned}$$

with $t = (y^{1/2}z^{1/2})y^{1/2}$. Setting $x = 1$ yields $t^2 = (y \cdot z) \cdot y$. Thus

$$[(x \cdot y) \cdot z] \cdot y = \big\{[(y \cdot z) \cdot y]^{1/2}x\big\}[(y \cdot z) \cdot y]^{1/2} = x \cdot [(y \cdot z) \cdot y],$$

the right Bol identity in $(M, \cdot, 1)$.

For (c), note that $x^{-1} \cdot x = (x^{1/2}x^{-1})x^{1/2} = 1$. Finally, (d) follows from Proposition 4.2 (a). □

In acknowledgment of this theorem, Bol loops that have the automorphic inverse property of Theorem 4.3 (d) are now described as *Bruck loops*.[3]

Remark 4.4. Let M be a Moufang loop satisfying the condition of Theorem 4.3.

(1) If M is commutative, then the Bol loop $(M, \cdot, 1)$ reduces to M.

(2) If M is not commutative, then M does not have the automorphic inverse property. Thus in this case, Theorem 4.3 may be seen as endowing M with the automorphic inverse property, at the cost of weakening the Moufang property to the (right) Bol property.

(3) If M satisfies the 2-Engel condition, then Proposition 3.5 (b) implies

$$x \cdot y = y^{1/2}x^{1/2}x^{1/2}y^{1/2} = x^{1/2}y^{1/2}y^{1/2}x^{1/2} = y \cdot x,$$

so that $(M, \cdot, 1)$ is a commutative Moufang loop.

5 Quadratic closure

Recall that a field K is said to be 2-*closed* or *quadratically closed* if each element of K is a square; i.e., the squaring map $x \mapsto x^2$ is surjective. Algebraically closed

[3] Other names, e.g., “B-loop” [7], “K-loop” [13] or “gyrocommutative gyrogroup” [26], have been used in various contexts by various authors. The latter structures were axiomatized differently, until their equivalence to Bruck loops was established by Kreuzer [14].

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fields are quadratically closed, as are finite fields of characteristic 2 – compare, say, [24, p. 189]. Any field that is not of characteristic 2 may be closed quadratically by successive adjunction of square roots of elements [19, Proposition 18.3]. For current purposes, it is convenient to extend the terminology of quadratic closure to magmas.

Definition 5.1. A magma is said to be *quadratically closed* if the squaring map $x \mapsto x^2$ is surjective.

Example 5.2. An idempotent magma is quadratically closed.

Example 5.3. A finite group of odd order is quadratically closed; indeed its squaring map is bijective.

Remark 5.4. If the squaring map of a Bol loop M is bijective, then the Bruck loop construction of Theorem 4.3 applies. Thus in the current paper, interest is focused on quadratically closed loops whose squaring map is not injective.

Example 5.5. Let K be a quadratically closed field of characteristic distinct from 2. Then the multiplicative group A or $A(K)$ of non-zero elements of K is quadratically closed. Squaring is not injective, since the distinct elements 1 and -1 square to 1. Thinking of $A(K)$ as a 2-Engel group of invertible (1×1)-matrices, Theorems 5.7 and 5.8 below provide analogous examples of quadratically closed 2-Engel groups $B(K)$ and $C(K)$ of (2×2)- and (3×3)-matrices respectively.

Proposition 5.6. *The Moufang loop of non-zero real octonions is quadratically closed.*

Proof. Each non-zero octonion l sits in a copy of the complex numbers contained within the octonions. There, two solutions x to the equation $x^2 - l = 0$ may be found. □

Theorem 5.7. *Let K be a quadratically closed field of characteristic distinct from* 2*. Let B or $B(K)$ denote the group of invertible, upper triangular* (2×2)*-matrices over K. Then B is a quadratically closed,* 2*-Engel group for which squaring is not injective.*

Proof. First note that B, as a group of nilpotence class 2, is a 2-Engel group. Now take an element

$$\begin{bmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{bmatrix}$$

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of B. There are two cases to consider:

(1) If the non-zero elements l_{11} and l_{22} are equal, choose a square root r_{11} of the l_{11} in the quadratically closed field K, and take $r_{22} = r_{11}$. Since K is not of characteristic 2, the sum $r_{11} + r_{22} = 2r_{11}$ is non-zero.

(2) If the non-zero elements l_{11} and l_{22} are distinct, choose respective square roots r_{11} of l_{11} and r_{22} of l_{22} in the quadratically closed field K. Since $r_{11} = -r_{22}$ would imply the contradiction $l_{11} = r_{11}^2 = r_{22}^2 = l_{22}$, the sum $r_{11} + r_{22}$ is again non-zero.

Noting that $r_{11} + r_{22}$ is invertible in each case, one may now take

$$r_{12} = (r_{11} + r_{22})^{-1} l_{12}.$$

Then

$$\begin{bmatrix} r_{11} & r_{12} \\ 0 & r_{22} \end{bmatrix}^2 = \begin{bmatrix} r_{11}^2 & (r_{11} + r_{22})r_{12} \\ 0 & r_{22}^2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} \\ 0 & l_{22} \end{bmatrix},$$

so B is quadratically closed. Finally, note that the scalar matrices I_2 and $-I_2$ in B both square to I_2. They are distinct, since K is not of characteristic 2, so squaring is not injective in B. □

Theorem 5.8. *Let K be a quadratically closed field of characteristic distinct from* 2*. Let C or $C(K)$ denote the group of invertible, upper triangular* (3×3)*-matrices over K with equal diagonal entries. Then C is a quadratically closed,* 2*-Engel group for which squaring is not injective.*

Proof. Consider elements

$$l = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ 0 & l_{11} & l_{23} \\ 0 & 0 & l_{11} \end{bmatrix} \quad \text{and} \quad x = \begin{bmatrix} x_{11} & x_{12} & x_{13} \\ 0 & x_{11} & x_{23} \\ 0 & 0 & x_{11} \end{bmatrix}$$

of C. Then

$$l^x = \begin{bmatrix} l_{11} & l_{12} & l_{13} + \begin{vmatrix} l_{12} & x_{11}^{-1}x_{12} \\ l_{23} & x_{11}^{-1}x_{23} \end{vmatrix} \\ 0 & l_{11} & l_{23} \\ 0 & 0 & l_{11} \end{bmatrix},$$

while both ll^x and $l^x l$ reduce to

$$\begin{bmatrix} l_{11}^2 & 2l_{11}l_{12} & 2l_{11}l_{13} + l_{12}l_{23} + l_{11} \begin{vmatrix} l_{12} & x_{11}^{-1}x_{12} \\ l_{23} & x_{11}^{-1}x_{23} \end{vmatrix} \\ 0 & l_{11}^2 & 2l_{11}l_{23} \\ 0 & 0 & l_{11}^2 \end{bmatrix}.$$

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Download Date | 10/17/19 5:29 AM

Thus C is a 2-Engel group (of nilpotence class 3). Since both I_3 and $-I_3$ square to I_3, and K is not of characteristic 2, squaring in C is not injective. To see that it is surjective, however, consider the element l of C, and the choice of an element

$$r = \begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{11} & r_{23} \\ 0 & 0 & r_{11} \end{bmatrix}$$

of C with

$$r^2 = \begin{bmatrix} r_{11}^2 & 2r_{11}r_{12} & 2r_{11}r_{13} + r_{12}r_{23} \\ 0 & r_{11}^2 & 2r_{11}r_{23} \\ 0 & 0 & r_{11}^2 \end{bmatrix} = \begin{bmatrix} l_{11} & l_{12} & l_{13} \\ 0 & l_{11} & l_{23} \\ 0 & 0 & l_{11} \end{bmatrix} = l.$$

Since K is quadratically closed, l_{11} has a square root r_{11} in K. Take

$$r_{12} = (2r_{11})^{-1}l_{12} \quad \text{and} \quad r_{23} = (2r_{11})^{-1}l_{23},$$

so that $r_{12}r_{23} = (2r_{11})^{-2}l_{12}l_{23}$. Setting

$$r_{13} = (2r_{11})^{-1}l_{13} - (2r_{11})^{-3}l_{12}l_{23}$$

completes the specification of an element r of C with $r^2 = l$. □

6 Bijectivization

Let $T\colon X \to X$ be a surjective function, for example the squaring map of a quadratically closed right quasigroup. This function may fail to be bijective. The process of bijectivization, as described in this section, is designed to correct that defect. The construction does not appear to be as well known as it deserves, although related ideas may be found in the literature (compare, say, [6, Section 10.5], [11,12], [24, I.2.2.1]).

Consider the *sequence space* $X^{\mathbb{Z}}$ of functions $a\colon \mathbb{Z} \to X;\ r \mapsto a_r$. Within the sequence space, consider the *history space*

$$\widetilde{X} = \{a \in X^{\mathbb{Z}} \mid \forall\, r \in \mathbb{Z},\ a_{r+1} = a_r T\}.$$

Interpreting $T\colon X \to X$ as the evolution operator of a dynamical system in the sense of [24, O§4.2], elements of the history space represent *histories* of states that change over discrete time. The *bijectivization* of $T\colon X \to X$ is defined as the function

$$\widetilde{T}\colon \widetilde{X} \to \widetilde{X}; a \mapsto a^{\widetilde{T}},$$

the *shift operator* $\widetilde{T}$ on $\widetilde{X}$ with

$$(a^{\widetilde{T}})_r = a_{r+1} = a_r T$$

Unauthenticated
Download Date | 10/17/19 5:29 AM

for each integer r. For each integer r, the *snapshot* at time r is the restriction $\beta_r\colon \widetilde{X} \to X;\ a \mapsto a_r$ of the evaluation function $X^{\mathbb{Z}} \to X;\ a \mapsto a_r$. More generally, for a *time interval* $I = \{r, r+1, \dots, r+k\}$ in $\mathbb{Z}$, the *video clip* over I is the restriction β_I to $\widetilde{X}$ of the restriction projection $X^{\mathbb{Z}} \to X^I;\ a \mapsto a|_I$.

Proposition 6.1. *The function $\widetilde{T}\colon \widetilde{X} \to \widetilde{X}$ is bijective. Its two-sided inverse is given by the function $U\colon \widetilde{X} \to \widetilde{X};\ a \mapsto a^U$ with $(a^U)_r = a_{r-1}$ for each integer r.*

Proof. For each history a in $\widetilde{X}$ and integer r, one has

$$(a^{U\widetilde{T}})_r = (a^U)_{r+1} = a_r \quad \text{and} \quad (a^{\widetilde{T}U})_r = (a^{\widetilde{T}})_{r-1} = a_r. \qquad \square$$

In the context of Proposition 6.1, the function $U\colon \widetilde{X} \to \widetilde{X}$ is known as the *backward shift operator*, and $\widetilde{T}$ may be called the *forward shift operator*.

Example 6.2. The complex squaring function $\sigma\colon \mathbb{C} \to \mathbb{C};\ w \mapsto w^2$ has the bijectivization $\widetilde{\sigma}\colon \widetilde{\mathbb{C}} \to \widetilde{\mathbb{C}}$. The Riemann surface R for the square root function may be realized as

$$R = \{(z_0, z_1) \in \mathbb{C}^2 \mid z_0^2 = z_1\}$$

along with the projection $R \to \mathbb{C};\ (z_0, z_1) \mapsto z_1$. One obtains maps

$$\begin{gathered} \widetilde{\mathbb{C}} \xrightarrow{\beta_{\{0,1\}}} R \xrightarrow{\sqrt{}} \mathbb{C}, \\ z \mapsto (z_0, z_1) \mapsto z_0 \end{gathered}$$

in which $\beta_{\{0,1\}}\colon \mathbb{C} \to R$ is a video clip, and $\sqrt{}\colon R \to \mathbb{C}$ represents the single-valued square root defined on the Riemann surface R. The bijectivization $\widetilde{\mathbb{C}}$ provides a cover of R.

7 The square history core

If M is a quadratically closed right quasigroup, the squaring map $\sigma\colon M \to M;\ x \mapsto x^2$ on M is surjective. The bijectivization procedure of Section 6 yields the history space $\widetilde{M}$ for the squaring map, as a subspace of the function space $M^{\mathbb{Z}}$. This section is concerned with the closure properties of the history space under various pointwise operations on the function space.

Lemma 7.1. *Suppose that M is a quadratically closed, 2-Engel right quasigroup. Let $\widetilde{M}$ be the history space for the surjective squaring map on M. Then $\widetilde{M}$ is a substructure of the pointwise core $(M, \circ)^{\mathbb{Z}}$.*

Proof. Consider x, y in $\widetilde{M}$ and an integer r. Then

$$(x \circ y)_r^2 = (x_r \circ y_r)^2 = x_r^2 \circ y_r^2 = x_{r+1} \circ y_{r+1} = (x \circ y)_{r+1}. \qquad \square$$

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Definition 7.2. Let M be a quadratically closed, 2-Engel right quasigroup. Then the magma $(\widetilde{M}, \circ)$ is called the *square history core* of M.

Corollary 7.3. *Suppose that M is a quadratically closed, 2-Engel diassociative loop. Then the square history core is a twisted subset of the pointwise loop $M^{\mathbb{Z}}$.*

In particular, Proposition 4.1 (b) implies the following.

Corollary 7.4. *Suppose that M is a quadratically closed, 2-Engel group. Then the square history core is a twisted subgroup of the pointwise group $M^{\mathbb{Z}}$.*

The remainder of this section considers closure of the square history core $\widetilde{M}$ under powers, when M is a power-associative loop. Here, the 2-Engel condition plays no role.

Lemma 7.5. *Let M be a quadratically closed power-associative loop. Then for each integer n, the history space $\widetilde{M}$ for the surjective squaring map on M is closed under the pointwise n-th power map on $M^{\mathbb{Z}}$.*

Proof. Consider a history a in $\widetilde{M}$, so $(a_r)^2 = a_{r+1}$ for each integer r. Then

$$((a^n)_r)^2 = ((a_r)^n)^2 = ((a_r)^2)^n = (a_{r+1})^n = (a^n)_{r+1},$$

so a^n also lies in $\widetilde{M}$. □

Corollary 7.6. *The restriction to $\widetilde{M}$ of the pointwise squaring map on $M^{\mathbb{Z}}$ coincides with the forward shift operator $\widetilde{\sigma}$ on $\widetilde{M}$.*

Proof. For a history a in $\widetilde{M}$ and integer r, one has

$$(a^2)_r = (a_r)^2 = (a_r)^\sigma = (a^{\widetilde{\sigma}})_r. \qquad □$$

Proposition 6.1 now yields the following result, essentially saying that the backward shift operator extracts square roots in $\widetilde{M}$.

Corollary 7.7. *Let ρ denote the backward shift operator on $\widetilde{M}$. Then*

$$(a^\rho)^2 = a = (a^2)^\rho \tag{7.1}$$

for each history a in $\widetilde{M}$.

An extension of the exterior equality of (7.1) is worth noting for use in computations. For a history a in $\widetilde{M}$ and integer n, it allows one to set $a^{n\rho} = (a^n)^\rho = (a^\rho)^n$ unambiguously.

Lemma 7.8. *Let M be a quadratically closed power-associative loop. Then for each integer n, the backward shift operator ρ commutes with the pointwise n-th power map on $\widetilde{M}$.*

Unauthenticated
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Proof. For a history a in $\widetilde{M}$ and integer r, one has

$$((a^\rho)^n)_r = ((a^\rho)_r)^n = (a_{r-1})^n = (a^n)_{r-1} = ((a^n)^\rho)_r. \qquad \square$$

Lemma 7.9. *Let M be a quadratically closed power-associative loop. Then for each history a in $\widetilde{M}$ and integer n, one has $a^\rho a^n = a^n a^\rho$.*

Proof. For each integer r, one has

$$(a^\rho a^n)_r = a_{r-1} a_r^n = a_{r-1} a_{r-1}^{2n} = a_{r-1}^{2n} a_{r-1} = a_r^n a_{r-1} = (a^n a^\rho)_r. \qquad \square$$

8 Engel loops and Bruck loops

Suppose that M is a quadratically closed, 2-Engel, power-associative loop.

Definition 8.1. For histories a and b in $\widetilde{M}$, define

$$a\backslash\backslash b = (b \circ a^\rho)^\rho \circ a^\rho$$

in $\widetilde{M}$. Note that the closure of $\widetilde{M}$ under \\ follows from the results of Section 7.

Lemma 8.2. *Let a and b be histories in $\widetilde{M}$. Then*

$$(a\backslash\backslash b)_r = \big(a_{r-1}(b \circ a^\rho)_r^{-\rho}\big)a_{r-1}$$

for each integer r.

The derivation of the next result is modeled on the proof of Theorem 4.3 (a).

Proposition 8.3. *Suppose that M is a quadratically closed, 2-Engel diassociative or right Bol loop. Then the square history core of M forms a quasigroup $(\widetilde{M}, \circ, \circ, \backslash\backslash)$.*

Proof. By Proposition 4.2 (b), the core of M forms a right quasigroup $(M, \circ, \circ)$. Thus so does its power $(M, \circ, \circ)^{\mathbb{Z}}$. By Lemma 7.1, it follows that the square history core forms a right quasigroup $(\widetilde{M}, \circ, \circ)$.

Consider histories a and b in $\widetilde{M}$, and an integer r. Note that the power loop $M^{\mathbb{Z}}$ is diassociative or right Bol, respectively. Then by Lemma 8.2,

$$\begin{aligned}
\big(a \circ (a\backslash\backslash b)\big)_r &= \big((a\backslash\backslash b)_r a_r^{-1}\big)(a\backslash\backslash b)_r \\
&= \big\{\big[\big(a_{r-1}(b \circ a^\rho)_r^{-\rho}\big)a_{r-1}\big]a_r^{-1}\big\}\big[\big(a_{r-1}(b \circ a^\rho)_r^{-\rho}\big)a_{r-1}\big] \\
&= a_{r-1} R\big((b \circ a^\rho)_r^{-\rho}\big) R(a_{r-1}) R(a_r^{-1}) R(a_{r-1}) \\
&\qquad \times R\big((b \circ a^\rho)_r^{-\rho}\big) R(a_{r-1}) \\
&= a_{r-1} R\big((b \circ a^\rho)_r^{-1}\big) R(a_{r-1}) \\
&= a_{r-1} R\big((a_{r-1}^{-1} b_r) a_{r-1}^{-1}\big) R(a_{r-1}) = b_r,
\end{aligned}$$

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so $(\widetilde{M}, \circ, \circ, \backslash\backslash)$ satisfies the identity $a \circ (a\backslash\backslash b) = b$. Now

$$\begin{aligned}\left[(a_{r-1}^{-1}b_r)a_{r-1}^{-1}\right]^2 &= \left[(a_{r-1}^{-1}b_r)a_{r-1}^{-1}\right]\left[(a_{r-1}^{-1}b_r)a_{r-1}^{-1}\right]\\ &= a_{r-1}^{-1}R(b_r)R(a_{r-1}^{-1})R(a_{r-1}^{-1})R(b_r)R(a_{r-1}^{-1})\\ &= a_{r-1}^{-1}R\big((a \circ b)_r\big)R(a_{r-1}^{-1}) = \big((a \circ b) \circ a^{\rho}\big)_r^{-1},\end{aligned}$$

so $((a \circ b) \circ a^{\rho})_r^{-\rho} = (a_{r-1}^{-1}b_r)a_{r-1}^{-1}$. Lemma 8.2 then yields

$$\begin{aligned}\big(a\backslash\backslash(a \circ b)\big)_r &= \Big(a_{r-1}\big((a \circ b) \circ a^{\rho}\big)_r^{-\rho}\Big)a_{r-1}\\ &= a_{r-1}R(a_{r-1}^{-1})R(b_r)R(a_{r-1}^{-1})R(a_{r-1}) = b_r,\end{aligned}$$

so $(\widetilde{M}, \circ, \circ, \backslash\backslash)$ satisfies the identity $a\backslash\backslash(a \circ b) = b$. This completes the proof that $(\widetilde{M}, \circ, \circ, \backslash\backslash)$ is a two-sided quasigroup. □

The pointwise identity element of the power loop $M^{\mathbb{Z}}$, which also lies in $\widetilde{M}$, will be denoted simply by 1.

Lemma 8.4. *Let M be a quadratically closed, 2-Engel power-associative loop. Then $L_\circ(1)$ acts as the forward shift operator on the history space $\widetilde{M}$, while $R_\circ(1)$ acts as inversion.*

Proof. Consider a history a and integer r. Then

$$(aL_\circ(1))_r = (1 \circ a)_r = (a_r 1)a_r = a_r^2 = a_{r+1},$$

while $(aR_\circ(1))_r = (a \circ 1)_r = (1a_r^{-1})1 = a_r^{-1}$. □

Proposition 8.5. *Suppose that M is a quadratically closed, 2-Engel diassociative or right Bol loop. Define an operation*

$$a \cdot b = aR_\circ(1)^{-1} \circ bL_\circ(1)^{-1}$$

on $\widetilde{M}$. Then the following hold:

(a) *$(\widetilde{M}, \cdot, 1)$ is a loop.*

(b) *For $a, b \in \widetilde{M}$, one has $a \cdot b = a^{-1} \circ b^{\rho} = (b^{\rho}a)b^{\rho}$.*

(c) *Pointwise inverses in $M^{\mathbb{Z}}$ provide inverses in the loop $(\widetilde{M}, \cdot, 1)$.*

(d) *The loop $(\widetilde{M}, \cdot, 1)$ has the right inverse property $(a \cdot b) \cdot b^{-1} = a$.*

(e) *In the loop $(\widetilde{M}, \cdot, 1)$, inversion is an automorphism.*

(f) *The loop $(\widetilde{M}, \cdot, 1)$ is power-associative, and integer powers in $(\widetilde{M}, \cdot, 1)$ coincide with the pointwise powers in $M^{\mathbb{Z}}$.*

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Proof. (a) Since $1 \circ 1 = 1$, one has $1 = 1L_\circ(1) = 1R_\circ(1)$. If a and b are histories, then $a \cdot 1 = aR_\circ(1)^{-1} \circ 1L_\circ(1)^{-1} = aR_\circ(1)^{-1} \circ 1 = a$, and dually $1 \cdot b = b$. Thus the isotope $(\widetilde{M}, \cdot, 1)$ of the quasigroup $(\widetilde{M}, \circ, \circ, \backslash\backslash)$ is a loop.

Statement (b) follows by Lemma 8.4.

(c) For a history a, consider the pointwise inverse a^{-1} in $M^{\mathbb{Z}}$. Then by Corollary 7.6, Lemmas 7.8 and 7.9, one has

$$a^{-1} \cdot a = \bigl(a^\rho a^{-1}\bigr)a^\rho = a^{\rho 2} a^{-1} = a^{\rho \widetilde{\sigma}} a^{-1} = aa^{-1} = 1.$$

Replacing a by a^{-1} here yields $a \cdot a^{-1} = 1$. Thus the pointwise inverse a^{-1} in $M^{\mathbb{Z}}$ serves as an inverse in the loop $(\widetilde{M}, \cdot, 1)$.

(d) Statements (b) and (c) yield

$$(a \cdot b) \cdot b^{-1} = \bigl((b^\rho a)b^\rho\bigr) \cdot b^{-1} = \bigl\{b^{-\rho}[(b^\rho a)b^\rho]\bigr\}b^{-\rho} = a.$$

(e) By Proposition 4.2 (a), Lemma 7.8, and statement (b) above, one has

$$(a \cdot b)^{-1} = (a^{-1} \circ b^\rho)^{-1} = a \circ b^{-\rho} = a^{-1} \cdot b^{-1}.$$

(f) For a history a and integers m, n, r, Lemma 7.8 and statement (b) above yield $(a^m \cdot a^n)_r = (a^{n\rho} a^m a^{n\rho})_r = a_r^{n\rho} a_r^m a_r^{n\rho} = a_{r-1}^n a_r^m a_{r-1}^n = a_r^m a_r^n = (a^{m+n})_r$, so $a^m \cdot a^n = a^{m+n}$. □

Remark 8.6. Proposition 8.5 extends to the case where the original loop M lies in the join of the varieties of diassociative and right Bol loops. A dual version holds in the join of the varieties of diassociative and left Bol loops.

If the original loop M in Proposition 8.5 is a right Bol loop, then the loop $(\widetilde{M}, \cdot, 1)$ on the history space becomes a right Bruck loop, as summarized in the main theorem below. The final step needed to complete the proof is the same as that used for Theorem 4.3 (b). Examples of Moufang loops giving non-trivial applications of the theorem (in particular, not reducing to Theorem 4.3) are provided by the respective groups $B(K)$ and $C(K)$ of Theorems 5.7 and 5.8, for a quadratically closed field K of characteristic distinct from 2. In these cases, the right Bruck loop is actually a commutative Moufang loop, by the reasoning outlined in Remark 4.4 (3).

Theorem 8.7. *Let M be a quadratically closed,* 2*-Engel right Bol loop. Let $\widetilde{M}$ be the history space $\{a \in M^{\mathbb{Z}} \mid \forall\, r \in \mathbb{Z},\ a_r^2 = a_{r+1}\}$ for the squaring operation on M, with backward shift operator ρ on $\widetilde{M}$ given by $a_r^\rho = a_{r-1}$ for each a in $\widetilde{M}$ and integer r. Define the operation $a \cdot b = (b^\rho a)b^\rho$ on $\widetilde{M}$, and consider the constant sequence* 1. *Then $(\widetilde{M}, \cdot, 1)$ is a right Bruck loop, and integral powers in $\widetilde{M}$ coincide with pointwise powers in the Moufang loop $M^{\mathbb{Z}}$.*

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Proof. In view of Proposition 8.5, it remains to show that $(\widetilde{M},\cdot,1)$ is a right Bol loop when M is a right Bol loop. Consider histories a, b and c. Then

$$\begin{aligned}((a\cdot b)\cdot c)\cdot b &= \{b^\rho[(c^\rho\{(b^\rho a)b^\rho\})c^\rho]\}b^\rho \\ &= 1R(b^\rho)R(c^\rho)R(b^\rho)R(a)R(b^\rho)R(c^\rho)R(b^\rho) \\ &= 1R((b^\rho c^\rho)b^\rho)R(a)R((b^\rho c^\rho)b^\rho) = (da)d\end{aligned}$$

for $d = (b^\rho c^\rho)b^\rho$. Setting $a = 1$ yields $d^2 = (b\cdot c)\cdot b$. One thus obtains the right Bol law $((a\cdot b)\cdot c)\cdot b = (d^\rho a)d^\rho = a\cdot((b\cdot c)\cdot b)$. □

Remark 8.8. Let K be a quadratically closed field of characteristic distinct from 2. Theorem 8.7 certainly applies to the abelian group $A(K)$ of Example 5.5, the multiplicative group of the field K. By the effect noted in Remark 4.4 (1), $(\widetilde{A(K)},\cdot,1)$ just reproduces the pointwise abelian group structure on $\widetilde{A(K)}$, and is therefore trivial from the loop-theoretic viewpoint. However, the Riemann surface cover $\widetilde{\mathbb{C}}$ from Example 6.2 is the union of $\widetilde{A(\mathbb{C})}$ with the singleton containing the constant sequence 0. Furthermore, one may restrict the complex squaring map $\sigma\colon \mathbb{C}\to\mathbb{C}$ to the unit circle $S = \{\exp(2\pi i\theta) \mid \theta\in\mathbb{R}\}$. The core right quasigroup $(S,\circ)$ given by Proposition 4.2 (1) implements complex reflections:

$$\exp(2\pi i\theta)\circ\exp(2\pi i\varphi) = \exp(2\pi i(2\varphi-\theta)).$$

Then its cover $(\widetilde{S},\circ)$ is a subquasigroup of the quasigroup $(\widetilde{A(\mathbb{C})},\circ)$ provided by Proposition 8.3. Left division in the quasigroup $(\widetilde{S},\circ)$ essentially takes midpoints of pairs of points on the unit circle in each snapshot, ambiguities being resolved by the previous history of the points in the snapshot.

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