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On $\hat{\theta}$-pairs for maximal subgroups of a finite group

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Abstract. For a maximal subgroup M of a finite group G, a $\hat{\theta}$-pair for M is any pair of subgroups (C, D) of G such that (i) $D \trianglelefteq G$, $D \leq C$, (ii) $\langle M, C\rangle = G$, $\langle M, D\rangle = M$, (iii) no non-trivial proper subgroup of C/D is normal in G/D and (iv) either $C = G$ or there exists a subgroup E of G such that C is a maximal subgroup of E and $(EM_G)_G$ is not contained in M. By using this notion, we exhibit some new characterizations on the structure of finite groups.

1 Introduction

Throughout this paper, all groups considered are finite and G always denotes a group. All unexplained notation and terminology are standard, as in [8, 15, 19].

In [6], Deskins defined the concept of completions associated to maximal subgroups of a finite group G as follows: for a maximal subgroup M of G, a subgroup C of G is said to be a *completion* for M if C is not contained in M, while every proper subgroup of C that is normal in G is contained in M. In [29], Zhao introduced the concept of θ-completions for maximal subgroups: for a maximal subgroup M of G, a subgroup C of G is said to be a θ-*completion* for M if C is not contained in M while M_G, the core of M in G, is contained in C and no non-trivial proper subgroup of C/M_G is normal in G/M_G. In [24], Mukherjee and Bhattacharya gave the concept of θ-pairs for maximal subgroups, which has a close relationship with the concepts of completions and θ-completions. Recall that a pair of subgroups (C, D) of G is called a θ-*pair* for a maximal subgroup M of G if the following hold: (i) $D \trianglelefteq G$, $D \leq C$, (ii) $\langle M, C\rangle = G$, $\langle M, D\rangle = M$, (iii) no non-trivial proper subgroup of C/D is normal in G/D.

We denote the set of completions, θ-completions, and θ-pairs for a maximal subgroup M of G by $I(M)$, $\theta I(M)$, and $\theta(M)$, respectively. It is clear that $I(M)$, $\theta I(M)$, and $\theta(M)$ can be partially ordered by set-theoretic inclusion. The max-

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imal elements of $I(M)$ ($\theta I(M)$, and $\theta(M)$) with respect to this partial ordering are called *maximal* completions (*maximal* θ-completions, and *maximal* θ-pairs, respectively). Also, an element C of $I(M)$ ($\theta I(M)$) or an element (C, D) of $\theta(M)$ is called *normal* if C is a normal subgroup of G.

Obviously, if $C \in I(M)$ ($C \in \theta I(M)$), then $(C, (C \cap M)_G)$ ((C, M_G), respectively) is a θ-pair for M. But the converse does not hold in general (cf. [1, Example]). Moreover, a normal completion (a normal θ-completion) C is maximal in $I(M)$ ($\theta I(M)$, respectively). However, a normal θ-pair (C, D) is not necessarily maximal in $\theta(M)$. For example, let $G = Z_6 = Z_2 \times Z_3$ and $M = Z_3$. Clearly $(Z_2, 1)$ is a normal θ-pair for M. But since (G, M) is a θ-pair for M, $(Z_2, 1)$ is not maximal in $\theta(M)$.

There has been a lot of interest in using the notions of completions, θ-completions, and θ-pairs, to investigate the structure of finite groups (see, for example, [1, 2, 4, 7, 12, 13, 17, 21, 22, 24, 26–31]). For the purpose of weakening or dispensing with the maximality imposed on completions and θ-completions, the concepts of s-completions and s-θ-completions (strong θ-completions) for maximal subgroups were introduced by Li et al. in [23] and [9].

Recall that for a maximal subgroup M of G, a completion (θ-completion) C for M is called an *s-completion* (*s-θ-completion*, respectively) if either $C = G$ or there exists a subgroup E of G which is not a completion (θ-completion, respectively) for M such that E contains C as a maximal subgroup. Actually, a maximal completion (a maximal θ-completion) is an s-completion (s-θ-completion, respectively), but the converse is not true in general (cf. [23, Example 1.2]).

As a continuation of the above mentioned ideas, we now introduce the concept of $\hat{\theta}$-pairs for maximal subgroups as follows:

Definition 1.1. For a maximal subgroup M of G, a θ-pair (C, D) for M is called a *$\hat{\theta}$-pair* if either $C = G$ or there exists a subgroup E of G such that C is a maximal subgroup of E and $(EM_G)_G$ is not contained in M.

In fact, many interesting results concerning s-completions, s-θ-completions, and maximal θ-pairs may be nontrivially generalized based on the concept of $\hat{\theta}$-pairs (see below Lemma 2.2). The aim of the present paper is to exhibit some new characterizations for a finite group to be solvable, supersolvable, nilpotent, etc.

2 Preliminaries

In this paper, we use $\mathfrak{X}$ to denote a class of groups. Let $\pi(G)$ denote the set of prime divisors of $|G|$; $|G|_p$ denotes the order of a Sylow p-subgroup of G. If

$H \leq G$, the core of H in G is denoted by H_G. In addition, $\mathfrak{N}$, $\mathfrak{U}$, $\mathfrak{N}_\mathfrak{p}$ and $\mathfrak{S}_\mathfrak{p}$ denote the classes of finite nilpotent groups, finite supersolvable groups, finite p-nilpotent groups and finite p-groups, respectively.

It is convenient to introduce the notion of $\hat{\theta}$-completions for maximal subgroups.

Definition 2.1. For a maximal subgroup M of G, a θ-completion C for M is called a $\hat{\theta}$-*completion* if either $C = G$ or there exists a subgroup E of G such that C is a maximal subgroup of E and E_G is not contained in M.

It is clear that if C is a $\hat{\theta}$-completion for a maximal subgroup M of G, then (C, M_G) is a $\hat{\theta}$-pair for M.

Lemma 2.2. *Suppose that M is a maximal subgroup of G.*

(1) *If C is an s-completion for M, then $(C, (C \cap M)_G)$ is a $\hat{\theta}$-pair for M.*

(2) *If C is an s-θ-completion for M, then C is a $\hat{\theta}$-completion for M and (C, M_G) is a $\hat{\theta}$-pair for M.*

(3) *If (C, D) is a maximal θ-pair for M, then (C, D) is a $\hat{\theta}$-pair for M.*

(4) *If C is a normal θ-completion for M, then C is a $\hat{\theta}$-completion for M.*

(5) *If (C, D) is a normal θ-pair for M, then (C, D) is a $\hat{\theta}$-pair for M.*

Proof. (1) Obviously, $(C, (C \cap M)_G)$ is a θ-pair for M. If $C = G$, then it is trivial. Now assume that $C < G$. Then there exists a subgroup E such that C is maximal in E and E is not a completion for M. Since E is not a completion, we may take a normal subgroup K of G such that $1 < K < E$ and $K \not\leq M$. Then $(EM_G)_G \not\leq M$. Therefore, $(C, (C \cap M)_G)$ is a $\hat{\theta}$-pair for M.

Statement (2) can be handled similarly.

Statements (3) and (4) are evident.

(5) If $C = G$, then it is trivial. Now suppose that $C < G$. Let E be an arbitrary subgroup of G such that C is maximal in E. As $C \leq (EM_G)_G$ and $C \not\leq M$, we have that $(EM_G)_G \not\leq M$. Hence (C, D) is a $\hat{\theta}$-pair for M. □

Remark. (1) For any maximal subgroup M of G, the $\hat{\theta}$-completions and the $\hat{\theta}$-pairs for M always exist. For example, let A/M_G be a minimal normal subgroup of G/M_G, then it is clear that A is a normal θ-completion for M, and so it is a $\hat{\theta}$-completion for M.

(2) For a maximal subgroup M of G, a $\hat{\theta}$-completion for M is not necessarily an s-θ-completion. For example, let $G = A_5$ be the alternating group of degree 5, $M = A_4$, and $C \in \mathrm{Syl}_5(G)$. Then $M_G = 1$ and C is a θ-completion for M. Since

C is maximal in G and G is also a θ-completion for M, C is a $\hat{\theta}$-completion, but not an s-θ-completion for M.

(3) For a maximal subgroup M of G, a $\hat{\theta}$-pair for M is not necessarily a maximal θ-pair. For example, let $G = S_4$ be the symmetric group of degree 4, $M = A_4$, and $C = S_3$. Obviously, $(C \cap M)_G = 1$ and $(C, 1)$ is a θ-pair for M. Since C is maximal in G and (G, M) is also a θ-pair for M, we see that $(C, 1)$ is a $\hat{\theta}$-pair, but not a maximal θ-pair for M.

Lemma 2.3. *Suppose that M is a maximal subgroup of G, $N \trianglelefteq G$ and $N \le M$. Then C is a $\hat{\theta}$-completion for M if and only if C/N is a $\hat{\theta}$-completion for M/N.*

Proof. It is obvious that C is a θ-completion for M if and only if C/N is a θ-completion for M/N. Suppose that C is a $\hat{\theta}$-completion for M. If $C = G$, then it is trivial. We may, therefore, assume that there exists a subgroup E of G such that C is maximal in E and $E_G \not\le M$. Then C/N is maximal in E/N, and $(E/N)_{(G/N)} = E_G/N \not\le M/N$. Hence C/N is a $\hat{\theta}$-completion for M/N. A similar argument to the one above shows that the converse also holds. □

We say that a class of groups $\mathfrak{X}$ is section-closed if whenever G is an $\mathfrak{X}$-group (that is, $G \in \mathfrak{X}$), every section A/B of G is also an $\mathfrak{X}$-group.

Lemma 2.4. *Let M be a maximal subgroup of G and $\mathfrak{X}$ be a class of groups which is section-closed. Then the following statements are equivalent:*

(a) *There exists a $\hat{\theta}$-completion C for M such that C/M_G is an $\mathfrak{X}$-group.*

(b) *There exists a $\hat{\theta}$-pair (C, D) for M such that C/D is an $\mathfrak{X}$-group.*

Proof. It is trivial that statement (a) implies (b). Now we prove that (b) implies (a). If $C = G$, then it is obvious. We may, therefore, assume that there exists a subgroup E of G such that C is maximal in E and $(EM_G)_G \not\le M$. Since clearly, $D = (C \cap M)_G \le C \cap M_G$ and C/D is an $\mathfrak{X}$-group, $CM_G/M_G \cong C/(C \cap M_G)$ is also an $\mathfrak{X}$-group. Suppose that CM_G is a θ-completion for M; then we have either $(CM_G)_G \le M$ or $CM_G \trianglelefteq G$. If the identity $CM_G = EM_G$ holds, then since $(CM_G)_G = (EM_G)_G \not\le M$, we have that $CM_G \trianglelefteq G$. Hence CM_G is a normal θ-completion for M, and consequently, CM_G is a $\hat{\theta}$-completion for M by Lemma 2.2 (4). If $CM_G < EM_G$, then it is easy to see that CM_G is maximal in EM_G. As $(EM_G)_G \not\le M$, CM_G is a $\hat{\theta}$-completion for M by Definition 2.1. Now consider that CM_G is not a θ-completion for M. Then there exists a minimal normal subgroup L/M_G of G/M_G, which is properly contained in CM_G/M_G. It is clear that L is a $\hat{\theta}$-completion for M such that L/M_G is an $\mathfrak{X}$-group. The lemma is thus proved. □

With an argument like the proof of Lemma 2.4, we can easily obtain the next lemma.

Lemma 2.5. *Let M be a maximal subgroup of G and $\mathfrak{X}$ be a class of groups which is section-closed. Then the following statements are equivalent:*

(a) *There exists a $\hat{\theta}$-completion C for M such that $C^g \not\leq M$ for every $g \in G$ and C/M_G is an $\mathfrak{X}$-group.*

(b) *There exists a $\hat{\theta}$-pair (C, D) for M such that $C^g \not\leq M$ for every $g \in G$ and C/D is an $\mathfrak{X}$-group.*

Recall that a maximal subgroup M of G is said to be c-maximal if M has composite index in G.

Lemma 2.6. *Let $N \trianglelefteq G$ and p be the largest prime divisor of $|N|$. If $P \in \mathrm{Syl}_p(N)$, then either $P \trianglelefteq G$ or any maximal subgroup of G containing $N_G(P)$ is c-maximal.*

Proof. Suppose that $P \ntrianglelefteq G$. Let M be an arbitrary maximal subgroup of G containing $N_G(P)$. Then by the Frattini argument, we have $G = N_G(P)N = MN$. If $|G : M| = |N : M \cap N| = r$ for some prime r, then $N/(M \cap N)_N$ is isomorphic to some subgroup of S_r, where S_r denotes the symmetric group of degree r. Hence r is the largest prime divisor of $|N : (M \cap N)_N|$. Since $P \leq M \cap N$, we have that $r < p$, and so $P \leq (M \cap N)_N$. By the Frattini argument again, we get $N = N_N(P)(M \cap N)_N \leq M \cap N$. It follows that $N \leq M$, a contradiction. Thus M is c-maximal. □

Lemma 2.7 ([25, Corollary 1]). *Let G be a finite group containing a nilpotent maximal subgroup S, and P denote the Sylow 2-subgroup of S. If every subgroup H of P such that $C_P(H) \leq H$ is a normal subgroup of P, then G is solvable.*

Lemma 2.8. *Let M be a solvable maximal subgroup of G. If $N \trianglelefteq G$ such that $G = MN$ and $M \cap N = 1$, then N is solvable.*

Proof. We may assume that $M > 1$ and $M_G = 1$. Since M is solvable, M has a nontrivial normal p-group P for some prime p. By [20, Lemma 8.1.3], there exists a P-invariant Sylow p-subgroup H of N. If $H > 1$, then $N_H(P) > 1$. Since $M_G = 1$, we have that $N_G(P) = M$, and so

$$M \cap N \geq N_G(P) \cap H = N_H(P) > 1,$$

a contradiction. This shows that N is a p'-group. Since

$$C_N(P) \leq N_N(P) = M \cap N = 1,$$

N has a unique P-invariant Sylow q-subgroup Q for any prime divisor q of $|N|$ by [20, Lemma 8.2.3]. Note that for any $m \in M$ and for any $x \in P$, we have that $(Q^m)^x = (Q^{mxm^{-1}})^m = Q^m$. This implies that Q^m is also P-invariant. Consequently, $Q^m = Q$, and so Q is M-invariant. Therefore, $G = MQ$, and thereby $N = Q$ is solvable. □

Lemma 2.9 ([14, Chapter 8, Theorem 3.1]). *Let P be a Sylow p-subgroup of G and p be an odd prime. If $N_G(Z(J(P))$ has a normal p-complement, then so also does G.*

Lemma 2.10 ([3, Lemma 3.1]). *Let $\mathfrak{F}$ be a saturated formation of characteristic π and H be a subnormal subgroup of G containing $O_\pi(\Phi(G))$ such that $H/O_\pi(\Phi(G))$ belongs to $\mathfrak{F}$. Then H belongs to $\mathfrak{F}$.*

Lemma 2.11 ([5, Theorem 3]). *Let $L(G)$ denote the intersection of c-maximal subgroups M of a group G (if no such M exists, then $L(G) = G$). Then $L(G)$ is supersolvable.*

Lemma 2.12 ([16, Lemma 2.9]). *Let $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$ and G be a group with a normal subgroup E such that $G/E \in \mathfrak{F}$. If E is cyclic, then $G \in \mathfrak{F}$.*

Lemma 2.13 ([18, Theorem 12.5.1]). *The groups G of order p^n which contain a cyclic subgroup of index p are of the following types:*

(1) $G = \langle a \mid a^{p^n} = 1, n \geq 1\rangle$,

(2) $G = \langle a, b \mid a^{p^{n-1}} = 1, b^p = 1, bab^{-1} = a, n \geq 2\rangle$,

(3) $G = \langle a, b \mid a^{p^{n-1}} = 1, b^p = 1, bab^{-1} = a^{1+p^{n-2}}, p \text{ is odd}, n \geq 3\rangle$,

(4) $G = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = a^{2^{n-2}}, bab^{-1} = a^{-1}, n \geq 3\rangle$,

(5) $G = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, bab^{-1} = a^{-1}, n \geq 3\rangle$,

(6) $G = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, bab^{-1} = a^{1+2^{n-2}}, n \geq 4\rangle$,

(7) $G = \langle a, b \mid a^{2^{n-1}} = 1, b^2 = 1, bab^{-1} = a^{-1+2^{n-2}}, n \geq 4\rangle$.

Let $\mathfrak{F} = \mathrm{LF}(f)$ be a saturated formation, where f is a formation function. Then we give the following definition of f-central that generalizes the classical notion.

Definition 2.14. Let A and B be subgroups of a group G such that $B \trianglelefteq G$ and $B \leq A$. We say that A/B is f-central in G if $(G/B)^{f(p)} \leq C_{G/B}(A/B)$ for each prime p dividing $|A/B|$. Here $(G/B)^{f(p)}$ denotes the $f(p)$-residual of G/B.

3 Main results

Theorem 3.1. *Let G be a group and let $\mathfrak{X}$ be the class of nilpotent groups X such that for the Sylow 2-subgroup P of X, every subgroup H of P with $C_P(H) \leq H$ is normal in P. For every normal subgroup S of G, we denote by $\Phi_{\mathfrak{X},S}(G)$ the intersection of all c-maximal subgroups M of G such that*

(i) $S \not\leq M$,

(ii) *M fails to have a $\hat{\theta}$-pair (C, D) with C/D is an $\mathfrak{X}$-group ($\Phi_{\mathfrak{X},S}(G) = G$ if no such M exists).*

Then $\Phi_{\mathfrak{X},S}(G) \cap S$ is the maximal normal solvable subgroup of G contained in S.

Proof. First, we claim that the class of groups $\mathfrak{X}$ is section-closed. Suppose that X is an $\mathfrak{X}$-group, that is, X is nilpotent, and for $P \in \mathrm{Syl}_2(X)$, every subgroup H of P with $C_P(H) \leq H$ is normal in P. Let A/B be a section of the group X. Then $(P \cap A)B/B \in \mathrm{Syl}_2(A/B)$. Let K/B be a subgroup of $(P \cap A)B/B$ such that $C_{(P\cap A)B/B}(K/B) \leq K/B$. Then, clearly,

$$K = (P \cap A \cap K)B \quad \text{and} \quad C_{(P\cap A)B}(K/B) \leq K.$$

It follows that $C_{P\cap A}(P \cap A \cap K) \leq P \cap A \cap K$. Since

$$C_P(C_P(P \cap A \cap K)(P \cap A \cap K)) \leq C_P(P \cap A \cap K)(P \cap A \cap K),$$

we have

$$C_P(P \cap A \cap K)(P \cap A \cap K) \trianglelefteq P.$$

This implies that $P \cap A \cap K \trianglelefteq P \cap A$, and so

$$K/B = (P \cap A \cap K)B/B \trianglelefteq (P \cap A)B/B.$$

This shows that A/B is an $\mathfrak{X}$-group. Hence the claim holds.

It is clear that $\Phi_{\mathfrak{X},S}(G)$ is a normal subgroup of G. Now let N be an arbitrary normal solvable subgroup of G contained in S. If G has no c-maximal subgroup, then $G = \Phi_{\mathfrak{X},S}(G)$, and G is solvable by [19, Chapter VI, Satz 9.4]. In this case, the theorem holds trivially. We may, therefore, assume that G contains at least a c-maximal subgroup M. If $N \not\leq M$, then $S \not\leq M$. Now take a minimal normal subgroup L/M_G of G/M_G contained in NM_G/M_G. It is easy to see that (L, M_G) is a normal θ-pair for M. Hence (L, M_G) is a $\hat{\theta}$-pair for M by Lemma 2.2 (5). Clearly, L/M_G is elementary abelian, and so L/M_G is an $\mathfrak{X}$-group. This shows that $N \leq \Phi_{\mathfrak{X},S}(G)$.

By Lemma 2.4, a maximal subgroup M of G has no $\hat{\theta}$-pair (C, D) with C/D is an $\mathfrak{X}$-group if and only if M has no $\hat{\theta}$-completion C with C/M_G is an $\mathfrak{X}$-group. (i) Let T be a c-maximal subgroup of G which has no $\hat{\theta}$-completion C such that $S \not\leq T$ and C/T_G is an $\mathfrak{X}$-group. Then $N \leq T$ and T/N is a c-maximal subgroup of G/N. Suppose that T/N has a $\hat{\theta}$-completion A/N such that $(A/N)/(T_G/N)$ is an $\mathfrak{X}$-group. By Lemma 2.3, T has a $\hat{\theta}$-completion A, and evidently, A/T_G is an $\mathfrak{X}$-group. This contradiction shows that $\Phi_{\mathfrak{X},S/N}(G/N) \leq \Phi_{\mathfrak{X},S}(G)/N$. (ii) Conversely, let T/N be a c-maximal subgroup of G/N which has no $\hat{\theta}$-completion C/N such that $S/N \nleq T/N$ and $(C/N)/(T_G/N)$ is an $\mathfrak{X}$-group. With a similar discussion as in (i), we can see that T is also a c-maximal subgroup of G which has no $\hat{\theta}$-completion C such that $S \nleq T$ and C/T_G is an $\mathfrak{X}$-group. This implies that $\Phi_{\mathfrak{X},S}(G)/N \leq \Phi_{\mathfrak{X},S/N}(G/N)$. Therefore, $\Phi_{\mathfrak{X},S/N}(G/N) = \Phi_{\mathfrak{X},S}(G)/N$.

If $N \neq 1$, then $(G/N, S/N)$ satisfies the hypothesis of the theorem by Lemma 2.3. Hence by induction, $(\Phi_{\mathfrak{X},S}(G) \cap S)/N = \Phi_{\mathfrak{X},S/N}(G/N) \cap S/N$ is the maximal normal solvable subgroup of G/N contained in S/N. It follows that $\Phi_{\mathfrak{X},S}(G) \cap S$ is the maximal normal solvable subgroup of G contained in S. So we may assume that S contains no nontrivial normal solvable subgroup of G. In this case, we shall show that $\Phi_{\mathfrak{X},S}(G) \cap S = 1$.

Suppose that $\Phi_{\mathfrak{X},S}(G) \cap S > 1$, and let L be a minimal normal subgroup of G contained in $\Phi_{\mathfrak{X},S}(G) \cap S$. Then L is non-solvable. Let p be the largest prime divisor of $|L|$ and $P \in \mathrm{Syl}_p(L)$. Since $P \ntrianglelefteq G$, there exists a c-maximal subgroup M of G containing $N_G(P)$ by Lemma 2.6. By the Frattini argument,

$$G = N_G(P)L = ML.$$

Hence $\Phi_{\mathfrak{X},S}(G) \cap S \nleq M$. This means that M has a $\hat{\theta}$-pair (C, D) such that C/D is an $\mathfrak{X}$-group. It follows from Lemma 2.4 that M has a $\hat{\theta}$-completion C such that C/M_G is an $\mathfrak{X}$-group. If $C = G$, then G/M_G is nilpotent, and consequently, $L \cong LM_G/M_G$ is solvable, which is absurd. We may, therefore, assume that there exists a subgroup E such that C is maximal in E and $E_G \nleq M$. Note that G/M_G is primitive and LM_G/M_G is a non-solvable minimal normal subgroup of G/M_G. By [8, Chapter A, Theorem 15.2], either G/M_G has a unique non-abelian minimal normal subgroup LM_G/M_G or G/M_G has exactly two non-abelian minimal normal subgroups LM_G/M_G and L^*/M_G. In the former case, since $E_G/M_G > 1$, we have $LM_G/M_G \leq E/M_G$. If $LM_G/M_G \leq C/M_G$, then LM_G/M_G is nilpotent, which is impossible. Hence $LM_G/M_G \nleq C/M_G$. Then, since C/M_G is maximal in E/M_G, we have that $E/M_G = (C/M_G)(LM_G/M_G)$. As C/M_G is an $\mathfrak{X}$-group, by Lemma 2.7, E/M_G is solvable, and so is LM_G/M_G, a contradiction. In the latter case, either $LM_G/M_G \leq E/M_G$ or $L^*/M_G \leq E/M_G$. With an argument similar to the one above, we see that at least one of LM_G/M_G and L^*/M_G is solvable. The final contradiction completes the proof. □

Corollary 3.2. *A group G is solvable if and only if for any c-maximal subgroup M of G, there exists a $\hat{\theta}$-pair (C, D) such that C/D is nilpotent in which the nilpotent class of the Sylow 2-subgroup is at most 2.*

Proof. For the sufficiency part, let $P/D \in \mathrm{Syl}_2(C/D)$ with nilpotent class at most 2. Then $(P/D)/Z(P/D)$ is abelian. Therefore, for any subgroup H/D of P/D with $C_{P/D}(H/D) \le H/D$, we have that $Z(P/D) \le H/D$, and so we get $H/D \trianglelefteq P/D$. Take $S = G$. Then G is solvable by Theorem 3.1. Conversely, assume that G is solvable. Let A/M_G be a minimal normal subgroup of G/M_G. Then clearly, (A, M_G) is a normal θ-pair for M, and so it is a $\hat{\theta}$-pair for M by Lemma 2.2 (5). Since G is solvable, A/M_G is elementary abelian. This shows that the necessity part also holds. □

Remark. Note that [7, Theorem 1], [1, Proposition 1], and [29, Theorem 3.1] can be directly generalized using Corollary 3.2. Furthermore, with similar technology and a slight improvement, [23, Theorem 3.1], [11, Theorem 3.1], and [31, Theorem 3.1] can also be extended by using the notion of $\hat{\theta}$-pairs.

Theorem 3.3. *Let G be a group and $\mathfrak{X}$ be the class of all groups X for which there exists at most one odd prime q such that X is not q-closed but X is $\{2, q\}$-closed. For any given normal subgroup S of G, we denote by $\widetilde{\Phi}_{\mathfrak{X},S}(G)$ the intersection of all maximal subgroups M of G such that*

(i) $S \not\le M$,

(ii) *M fails to have a $\hat{\theta}$-pair (C, D) such that $C^g \not\le M$ for every $g \in G$ and C/D is an $\mathfrak{X}$-group ($\widetilde{\Phi}_{\mathfrak{X},S}(G) = G$ if no such M exists).*

Then $\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S$ is the maximal normal solvable subgroup of G contained in S.

Proof. Obviously, the class of groups $\mathfrak{X}$ is section-closed and $\widetilde{\Phi}_{\mathfrak{X},S}(G)$ is a normal subgroup of G. Let N be an arbitrary normal solvable subgroup of G contained in S and M be a maximal subgroup of G. If $N \not\le M$, then $S \not\le M$ and there exists a $\hat{\theta}$-pair (L, M_G) for M such that L/M_G is a minimal normal solvable subgroup of G/M_G by Lemma 2.2 (5). Hence L/M_G is an $\mathfrak{X}$-group, which implies that $N \le \widetilde{\Phi}_{\mathfrak{X},S}(G)$. By using Lemma 2.3 and 2.5, in the same way as in the proof of Theorem 3.1, we have that $\widetilde{\Phi}_{\mathfrak{X},S/N}(G/N) = \widetilde{\Phi}_{\mathfrak{X},S}(G)/N$. If $N \ne 1$, then the theorem holds by induction. We may, therefore, assume that S contains no nontrivial normal solvable subgroup of G. We shall show that $\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S = 1$.

Suppose that $\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S > 1$, and let L be a minimal normal subgroup of G contained in $\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S$. Evidently L is non-solvable. If $L \le \Phi(G)$, then L is solvable, a contradiction. Thus $L \not\le \Phi(G)$, and thereby we may choose a maximal subgroup M of G such that $G = ML$. It is easy to see that for any nontrivial normal subgroup K of G,

$$\widetilde{\Phi}_{\mathfrak{X},S}(G)K/K \le \widetilde{\Phi}_{\mathfrak{X},SK/K}(G/K).$$

Using Lemma 2.3, the theorem holds for $(G/K, SK/K)$ by induction. Hence

$\widetilde{\Phi}_{\mathfrak{X},SK/K}(G/K) \cap SK/K$ is solvable. As

$$(\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S)K/K \leq (\widetilde{\Phi}_{\mathfrak{X},S}(G)K \cap SK)/K \leq \widetilde{\Phi}_{\mathfrak{X},SK/K}(G/K) \cap SK/K,$$

$(\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S)K/K$ is also solvable. It follows that $(\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S)/L$ is solvable. If $L \cap K = 1$, then $\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S$ is solvable, and so $\widetilde{\Phi}_{\mathfrak{X},S}(G) \cap S = 1$. We may, therefore, assume that L is contained in any nontrivial normal subgroup of G. This shows that L is the unique minimal normal subgroup of G, and so $M_G = 1$.

Since $\widetilde{\Phi}_{\mathfrak{X},S}(G) \nleq M$, by hypothesis and Lemma 2.5, M has a $\hat{\theta}$-completion C such that $C^g \nleq M$ for every $g \in G$ and there exists at most one odd prime q such that C is not q-closed but C is $\{2,q\}$-closed. If $C = G$, then G is 2-closed, and so G is solvable by the Feit–Thompson Theorem, which is impossible. Hence G has a subgroup E such that C is maximal in E and $E_G \nleq M$. This implies that $L \leq E$. If $L \leq C$, then L is 2-closed. Therefore, L is solvable, a contradiction. It follows that $E = CL$. Note that C is solvable. If $C \cap L = 1$, then L is solvable by Lemma 2.8, a contradiction again. Thus $C \cap L > 1$.

Suppose that $C \cap L$ has a nontrivial Sylow p-subgroup P for a prime $p \neq q$ (where the prime p may be 2). Then since $C \cap L$ is p-closed, $P \trianglelefteq C \cap L$, and consequently $P \trianglelefteq C$. As C is maximal in E, $N_E(P) = E$ or C. If $P \trianglelefteq E$, then $P \trianglelefteq L$, which contradicts the fact that L is non-solvable. Therefore, $N_E(P) = C$, and so $N_L(P) = C \cap L$. If $P \notin \mathrm{Syl}_p(L)$, then L has a Sylow p-subgroup P^* properly containing P. It follows that $P < N_{P^*}(P) = C \cap P^* = P$, a contradiction. Hence $P \in \mathrm{Syl}_p(L)$. Assume that $p > 2$. Denote by $J(P)$ the Thompson subgroup of P. Since

$$C = N_E(P) \leq N_E(Z(J(P))) < E,$$

we have that $N_E(Z(J(P))) = C$. As $N_L(Z(J(P))) = C \cap L$ is $\{2,q\}$-closed and r-closed for every prime $r \notin \{2,q\}$, it follows that $N_L(Z(J(P)))$ is p-nilpotent. Then by Lemma 2.9, L is p-nilpotent, and so $p \nmid |L|$, a contradiction. This implies that $C \cap L$ contains no nontrivial Sylow p-subgroup when $p \notin \{2,q\}$. It follows that $\pi(C \cap L) \subseteq \{2,q\}$ and if $2 \mid |C \cap L|$, then $|C \cap L|_2 = |L|_2$.

If $2 \nmid |C \cap L|$, then $C \cap L$ is a q-group. Since q is odd, L is q-nilpotent as above, which is absurd. Hence $2 \mid |C \cap L|$, and thereby L has a nontrivial Sylow 2-subgroup P contained in $C \cap L$. Let T be a maximal subgroup of G containing $N_G(P)$. By the Frattini argument, $G = N_G(P)L = TL$. As $\widetilde{\Phi}_{\mathfrak{X},S}(G) \nleq T$, by hypothesis and Lemma 2.5, there exists a $\hat{\theta}$-completion C' for T such that $(C')^g \nleq T$ for every $g \in G$ and there exists at most one odd prime r such that C' is not r-closed but C' is $\{2,r\}$-closed. With a similar argument to the one above, we see that $C' \cap L > 1$ and L has a nontrivial Sylow 2-subgroup P' contained in $C' \cap L$ such that $C' \leq N_G(P')$. By Sylow's theorem, $P = (P')^l$ for an element $l \in L$. Hence $(C')^l \leq (N_G(P'))^l = N_G((P')^l) = N_G(P) \leq T$. The final contradiction completes the proof. □

Remark. Theorem 3.3 above does not hold in general if we replace 'maximal' by 'c-maximal'. For example, let $S = G = A_5$ and $C = A_4$. Then for every c-maximal subgroup M of G, $(C, 1)$ is a $\hat{\theta}$-pair for M such that C is an $\mathfrak{X}$-group. However, G is not solvable.

Corollary 3.4. *Let G be a group. Then G is solvable if and only if for any maximal subgroup M of G, there exists a $\hat{\theta}$-pair (C, D) such that $C^g \not\leq M$ for every $g \in G$ and C/D is nilpotent.*

Proof. The sufficiency part directly follows from Theorem 3.3, and the proof of the necessity part is similar to the proof of Corollary 3.2. □

Remark. Note that [26, Theorem] directly follows from Corollary 3.4 (the necessity part is evident). In fact, Corollary 3.4 still holds if 'maximal' is replaced by 'c-maximal'. Therefore, similar results, e.g., [23, Theorem 3.2], [9, Theorem 4.2], and [22, Theorem 1] can be extended by using the notion of $\hat{\theta}$-pairs.

Theorem 3.5. *Let G be a group and $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Suppose that S is a normal subgroup of G. If for any c-maximal subgroup M of G not containing $S^{\mathfrak{F}}$, the $\mathfrak{F}$-residual of S, there exists a $\hat{\theta}$-pair (C, D) for M such that C/D is cyclic, and for any prime p with $|G : M|_p = p^n$ (where $n \geq 2$ is an integer), $C/C \cap M_G$ satisfies*

(i) $|C/C \cap M_G| \nmid p\,(p^{n-1} - 1)$ *when* $p \mid |C/C \cap M_G|$,

(ii) $|C/C \cap M_G| \nmid p^n - 1$ *when* $p \nmid |C/C \cap M_G|$,

then either $S \in \mathfrak{F}$ or G has a homomorphic image isomorphic to S_4 and S has a homomorphic image isomorphic to S_4 or A_4.

Proof. Assume that the result is false and let (G, S) be a counter-example for which $|G| + |S|$ is minimal.

First, we claim that for any c-maximal subgroup M of G not containing $S^{\mathfrak{F}}$, there exists a $\hat{\theta}$-completion for M, which we still denote by C, such that C/M_G is cyclic, and for any prime p with $|G : M|_p = p^n$ $(n \geq 2)$, C/M_G satisfies

(i) $|C/M_G| \nmid p\,(p^{n-1} - 1)$ when $p \mid |C/M_G|$,

(ii) $|C/M_G| \nmid p^n - 1$ when $p \nmid |C/M_G|$.

In fact, since $C/D = C/(C \cap M)_G$ is cyclic, CM_G/M_G is cyclic. With a discussion similar to the proof of Lemma 2.4, either CM_G is a $\hat{\theta}$-completion for M

or CM_G/M_G contains a minimal normal subgroup L/M_G of G/M_G such that L is a $\hat{\theta}$-completion for M. In the former case, for any prime p with $|G:M|_p = p^n$ $(n \geq 2)$, by hypothesis, it is obvious that

(i) $|CM_G/M_G| \nmid p\,(p^{n-1}-1)$ when $p \mid |CM_G/M_G|$,

(ii) $|CM_G/M_G| \nmid p^n - 1$ when $p \nmid |CM_G/M_G|$.

In the latter case, since clearly, $G = ML$ and $(L \cap M)/M_G = 1$, we have that $|G:M| = |L:M_G|$. As L/M_G is cyclic of prime order, for any prime p dividing $|G:M|$, $|G:M| = |L:M_G| = p$. Hence the claim holds.

As $S \notin \mathfrak{F}$, $S^{\mathfrak{F}} > 1$. If G is a simple group, then $S^{\mathfrak{F}} = S = G$. If every maximal subgroup of G has a prime index in G, then $G \in \mathfrak{U}$, and so $S \in \mathfrak{U} \subseteq \mathfrak{F}$, which is impossible. Hence G has at least a c-maximal subgroup M such that there exists a $\hat{\theta}$-completion C for M with C is cyclic. As $C < G$, there exists a subgroup E of G such that C is maximal in E and $E_G \nleq M$. This induces that $E = G$, and so G has a cyclic maximal subgroup C. Then by [19, Chapter IV, Satz 7.4], G is solvable. Therefore, G is cyclic of prime order, and so is S, a contradiction. Thus G is not a simple group.

For any non-trivial normal subgroup N of G, we now show that $(G/N, SN/N)$ satisfies the hypothesis. If M/N is a c-maximal subgroup of G/N not containing $(SN/N)^{\mathfrak{F}}$, then since clearly, $(SN/N)^{\mathfrak{F}} \leq S^{\mathfrak{F}}N/N$, M is a c-maximal subgroup of G not containing $S^{\mathfrak{F}}$. By the above claim, M has a $\hat{\theta}$-completion C such that C/M_G is cyclic, and for any prime p with $|G:M|_p = p^n$ $(n \geq 2)$, we have that

(i) $|C/M_G| \nmid p\,(p^{n-1}-1)$ when $p \mid |C/M_G|$,

(ii) $|C/M_G| \nmid p^n - 1$ when $p \nmid |C/M_G|$.

Now, it follows from Lemma 2.3 that M/N has a $\hat{\theta}$-completion C/N such that $(C/N)/(M_G/N)$ is cyclic, and for any prime p with $|(G/N):(M/N)|_p = p^n$ $(n \geq 2)$, we also have that

(i) $|(C/N)/(M_G/N)| \nmid p\,(p^{n-1}-1)$ when $p \mid |(C/N)/(M_G/N)|$,

(ii) $|(C/N)/(M_G/N)| \nmid p^n - 1$ when $p \nmid |(C/N)/(M_G/N)|$.

By the choice of (G, S), either $SN/N \in \mathfrak{F}$ or G/N has a homomorphic image isomorphic to S_4 and SN/N has a homomorphic image isomorphic to S_4 or A_4. But the latter case is contrary to our assumption. Hence $SN/N \in \mathfrak{F}$. We may, therefore, assume that N is the unique minimal normal subgroup of G. Then clearly, $N \leq S$ and $S/N \in \mathfrak{F}$. Since $1 < S^{\mathfrak{F}} \trianglelefteq G$, we have that $S^{\mathfrak{F}} = N$.

If $N \leq \Phi(G)$, then $S \in \mathfrak{F}$ by Lemma 2.10, a contradiction. Hence G has at least one maximal subgroup not containing N. Suppose that for any maximal subgroup T not containing N, T has a prime index in G. Then $N \leq L(G)$, and so

N is supersolvable by Lemma 2.11. Evidently, $G = TN$ and $N \cap T = 1$. Since $|G : T|$ is a prime, N is cyclic, and thereby $S \in \mathfrak{F}$ by Lemma 2.12, a contradiction. Therefore, there exists a c-maximal subgroup M of G not containing N. Obviously $M_G = 1$. By the above claim, M has a $\hat{\theta}$-completion C such that C is cyclic, and for any prime p with $|G : M|_p = p^n$ $(n \geq 2)$, C satisfies that

(i) $|C| \nmid p\,(p^{n-1} - 1)$ when $p \mid |C|$,

(ii) $|C| \nmid p^n - 1$ when $p \nmid |C|$.

If $C = G$, then it is trivial. So G has a subgroup E such that C is maximal in E and $E_G \not\leq M$. This implies that $N \leq E$ by the uniqueness of N. If $N \leq C$, then N is cyclic. By Lemma 2.12, $S \in \mathfrak{F}$, which contradicts our assumption. Hence $N \not\leq C$, and so $E = CN$ for C is maximal in E. It follows from [19, Chapter IV, Satz 7.4] that E is solvable. Consequently, N is an elementary abelian p-subgroup for some prime p. This implies that $|N| = |G : M| = p^n$ and $C_G(N) = N$ by [8, Chapter A, Theorem 15.2]. As clearly, N is noncyclic, $n \geq 2$.

First, we assume that $C \trianglelefteq E$. Then $|E : C| = |N : C \cap N| = p$ by the maximality of C. Since $C \cap N$ is cyclic and $C \cap N > 1$, we have that $|C \cap N| = p$, and so $|N| = p^2$ and $n = 2$. Therefore, $|C| \nmid p\,(p-1)$ for $p \mid |C|$. If C is not a p-group, then C has a nontrivial Hall p'-subgroup $C_{p'} \trianglelefteq E$. It follows that $C_{p'} \leq C_E(N) = N$, a contradiction. Hence C is a p-group, and so is E. Note that $E/N = E/C_E(N) \lesssim \mathrm{Aut}(Z_p \times Z_p) \cong \mathrm{GL}(2, p)$, we have that $|E| \leq p^3$. If $|E| = p^2$, then $E = N$ and $|C| = p$, a contradiction. Thus $|E| = p^3$. Suppose that $p > 2$. Then E is a group of type (1), (2) or (3) in Lemma 2.13. Since N is noncyclic, E is noncyclic, and thereby E is not a group of type (1). If E is a group of type (2), then E is abelian. As $C_E(N) = N$, we have that $E = N$, which is absurd. Now let $E = \langle a, b\rangle$ be a group of type (3) satisfying that $a^{p^2} = 1$, $b^p = 1$, and $bab^{-1} = a^{1+p}$. In view of that $[a, b] = a^{-1}b^{-1}ab = (a^p)^{-1} \in Z(E)$, it is easy to calculate that

$$\Omega_1(E) = \langle g \in E \mid g^p = 1\rangle = \langle a^p\rangle \times \langle b\rangle = N.$$

Since $E = N(M \cap E)$ and $N \cap M = 1$, we have $|M \cap E| = p$. This implies that $M \cap E \leq \Omega_1(E) = N$, a contradiction. Hence $p = 2$, and E is a group of type (1), (2), (4) or (5) in Lemma 2.13. By arguing similarly as above, we can see that E is not a group of type (1) or (2). Thus E is either a quaternion group or a dihedral group of order 8. Note that $G \cong G/M_G \lesssim S_4$ for $|G : M| = |N| = 4$. Since G has a subgroup of order 8, $G = S_4$. It follows from the fact $S \notin \mathfrak{U}$ that $S = S_4$ or A_4, which contradicts the choice of (G, S).

Now consider that $C \ntrianglelefteq E$. Then we have $N_E(C) = C$. Hence for any element $g \in E \backslash C$, we have that $C \neq C^g$, and so $E = \langle C, C^g\rangle$. Since C and C^g are both cyclic, we get $E \leq C_E(C \cap C^g)$. Therefore, $C \cap C^g \leq Z(E) \leq C_E(N) = N$.

As $E = CN$ and N is abelian, $C \cap N \trianglelefteq E$. Then $(C/C \cap N) \cap (C/C \cap N)^{\bar{g}} = 1$ for any element $\bar{g} \in (E/C \cap N) \backslash (C/C \cap N)$. This shows that $E/C \cap N$ is a Frobenius group with complement $C/C \cap N$. Let $H/C \cap N$ be the Frobenius kernel of $E/C \cap N$ such that

$$E/C \cap N = (C/C \cap N)(H/C \cap N) \quad \text{and} \quad (C/C \cap N) \cap (H/C \cap N) = 1.$$

By [19, Chapter V, Satz 8.16], either $N \leq H$ or $H \leq N$. But since $|H| = |N|$, we have the identity $H = N$. It follows that $|C/C \cap N| \mid (|N/C \cap N| - 1)$ and $N/C \cap N \in \mathrm{Syl}_p(E/C \cap N)$ by [19, Chapter V, satz 8.3]. Hence $N \in \mathrm{Syl}_p(E)$, and thereby $C \cap N \in \mathrm{Syl}_p(C)$. Obviously, we have $|C \cap N| \leq p$. If $p \mid |C|$, then $|C \cap N| = p$. Since $|N| = |G : M| = p^n$, we have that $|C/C \cap N| \mid (p^{n-1} - 1)$, and so $|C| \mid p(p^{n-1} - 1)$, a contradiction. If $p \nmid |C|$, then $|C \cap N| = 1$. It follows that $|C| \mid p^n - 1$, also a contradiction. The theorem is thus proved. □

Recall that for a class of groups $\mathfrak{X}$, a maximal subgroup M of G is said to be $\mathfrak{X}$-normal if $G/M_G \in \mathfrak{X}$; otherwise it is said to be $\mathfrak{X}$-abnormal.

Corollary 3.6. *Let G be an S_4-free group and $\mathfrak{F}$ be a saturated formation containing $\mathfrak{U}$. Then $G \in \mathfrak{F}$ if and only if for any c-maximal subgroup M of G which is $\mathfrak{F}$-abnormal, there exists a $\hat{\theta}$-pair (C, D) for M such that C/D is cyclic and $|C : C \cap M_G| \geq |G : M|$.*

Proof. The necessity part is obvious. Now assume that $|C : C \cap M_G| \geq |G : M|$. If for some prime p with $|G : M|_p = p^n$ $(n \geq 2)$, either

$$|C/C \cap M_G| \mid p\,(p^{n-1} - 1)$$

or

$$|C/C \cap M_G| \mid p^n - 1,$$

then $|C/C \cap M_G| < p^n \leq |G : M|$, which is impossible. Take $S = G$ and Theorem 3.5 applies. □

Remark. Note that [2, Theorem 1], [23, Theorem 3.5], and [10, Theorem 10] can be directly generalized by Corollary 3.6 (Corollary 3.2 may also be required).

Finally, we focus attention on the case in which assumptions are imposed on θ-pairs without any restriction.

Theorem 3.7. *Let G be a group and $\mathfrak{F} = \mathrm{LF}(f)$ a saturated formation, where f is an integrated formation function. Then $G \in \mathfrak{F}$ if and only if for any $\mathfrak{F}$-abnormal maximal subgroup M of G, there exists a θ-pair (C, D) for M such that C/D is f-central (in sense of Definition* 2.14*) in G.*

Proof. The necessity is clear. We need only prove the sufficiency. Assume that it is false and let G be a counter-example with minimal order. Note that by Definition 2.14, if C/D is f-central in G and A/B is a section of C/D with $B \trianglelefteq G$, then A/B is f-central in G. Using an analogue of the proof of Lemma 2.4, we can obtain that for any $\mathfrak{F}$-abnormal maximal subgroup M of G, M has a θ-completion C such that C/M_G is f-central in G. Let N be a minimal normal subgroup of G. It is easy to check that G/N also satisfies the hypothesis of the theorem. Hence by induction, $G/N \in \mathfrak{F}$. Then since $\mathfrak{F}$ is a saturated formation, N is the unique minimal normal subgroup of G and $N \not\leq \Phi(G)$. Thus there exists a maximal subgroup M of G such that $G = MN$ and $M_G = 1$. If M is $\mathfrak{F}$-normal, then $G \in \mathfrak{F}$, contrary to our assumption. We may, therefore, assume that M is $\mathfrak{F}$-abnormal. It follows that M has a θ-completion C such that $C/1$ is f-central in G. Hence we have $G^{f(p)} \leq C_G(C)$ for each prime divisor p of $|C|$ by definition. If $G^{f(p)} = 1$, then we get $G \in f(p) \subseteq \mathfrak{F}$ for f is integrated, which is absurd. Now assume that $G^{f(p)} > 1$. Then $N \leq G^{f(p)} \leq C_G(C)$, and so $C \leq C_G(N)$. Since $M_G = 1$ and $C > 1$, by [8, Chapter A, Theorem 15.2], it follows that N is elementary abelian and $N = C_G(N)$. This induces that p is the unique prime dividing $|N|$. Since $C \leq N \leq G^{f(p)} \leq C_G(C)$, we have that

$$1 < C \leq Z(G^{f(p)}) \trianglelefteq G.$$

Hence we get $N \leq Z(G^{f(p)})$, and so $G^{f(p)} \leq C_G(N) = N$. This implies that $G^{f(p)} = N$. Therefore, $G/C_G(N) = G/N \in f(p)$. But since $G/N \in \mathfrak{F}$, we obtain that $G \in \mathfrak{F}$. The final contradiction completes the proof. □

Remark. Note that [1, Theorem 1] and [22, Theorem 8] can be generalized by Theorem 3.7.

Corollary 3.8. *Let G be a group. Suppose that $\mathfrak{F} = \mathrm{LF}(f)$ and $\mathfrak{H} = \mathrm{LF}(h)$ are saturated formations such that $\mathfrak{F} \subseteq \mathfrak{H}$, where f and h are the canonical local definitions or the smallest local definitions (cf.* [8, Chapter IV, Definition (3.9)]*). Then $G \in \mathfrak{H}$ if and only if for any $\mathfrak{H}$-abnormal maximal subgroup M of G, there exists a θ-pair (C, D) for M such that C/D is f-central (in sense of Definition* 2.14*) in G.*

Proof. By [8, Chapter IV, Proposition (3.11)], $f(p) \subseteq g(p)$ for all $p \in \mathbb{P}$. We may prove the corollary similarly as Theorem 3.7. □

Corollary 3.9. *Let G be a group and $\mathfrak{F}$ a saturated formation containing $\mathfrak{N}$. Then $G \in \mathfrak{F}$ if and only if for any $\mathfrak{F}$-abnormal maximal subgroup M of G, there exists a θ-pair (C, D) for M such that $C/D \leq Z(G/D)$.*

Proof. In fact, the smallest local definition f of $\mathfrak{N}$ can be defined as follows: $f(p) = 1$ for each prime p. Therefore, the result follows from Corollary 3.8. □

Remark. Note that [29, Theorem 3.9] can be directly generalized by Corollary 3.9.

Corollary 3.10. *Let G be a group and $\mathfrak{F}$ be a saturated formation containing $\mathfrak{N}_{\mathfrak{p}}$. Then $G \in \mathfrak{F}$ if and only if for any $\mathfrak{F}$-abnormal maximal subgroup M of G, there exists a θ-pair (C, D) for M such that $O^p(G/D) \leq C_{G/D}(C/D)$.*

Proof. Note that the canonical local definition f of $\mathfrak{N}_{\mathfrak{p}}$ can be defined as follows: $f(p) = \mathfrak{S}_{\mathfrak{p}}$ and $f(q) = \mathfrak{N}_{\mathfrak{p}}$ for each prime $q \neq p$. As

$$(G/D)/O^p(G/D) \in \mathfrak{S}_{\mathfrak{p}} \subseteq \mathfrak{N}_{\mathfrak{p}},$$

we have that

$$(G/D)^{\mathfrak{N}_{\mathfrak{p}}} \leq (G/D)^{\mathfrak{S}_{\mathfrak{p}}} \leq O^p(G/D).$$

Since $O^p(G/D) \leq C_{G/D}(C/D)$, C/D is f-central in G, and so Corollary 3.8 applies. □

Remark. Note that [24, Theorem 4.7] directly follows from Corollary 3.10.

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