J. Group Theory **16** (2013), 159–220
DOI 10.1515/jgt-2012-0032

A reduction theorem for the blockwise Alperin weight conjecture

Britta Späth

Communicated by Robert M. Guralnick

Abstract. We show that the blockwise version of the Alperin weight conjecture is true if for every finite non-abelian simple group a set of conditions holds. Furthermore we prove that several series of simple groups satisfy these assumptions. This refines recent work of Navarro and Tiep, who proved an analogous reduction theorem for the non-blockwise version of the Alperin weight conjecture.

1 Introduction

One of the most important conjectures in the modular representation theory of finite groups is the Alperin weight conjecture. Alperin already formulated two versions of this conjecture, one without blocks and one about p-blocks of finite groups, usually called the blockwise version. The blockwise version can be seen as a refinement as it immediately implies the non-blockwise version.

For a finite group G it predicts the number of irreducible Brauer characters in a p-block in terms of associated p-weights of G. Given a p-block B of G, a *p-weight of B* is a pair (Q, ψ) where Q is a radical p-subgroup of G and ψ an irreducible defect zero character of $\mathrm{N}_G(Q)/Q$, belonging to a block b of $\mathrm{N}_G(Q)$ which induces B. The conjecture states the following relation between global and local data associated with the block.

Conjecture 1.1 (Alperin [1]). Let G be a finite group, p be a prime and B be a p-block of G. Then the number of irreducible Brauer characters belonging to B equals the number of G-conjugacy classes of p-weights of B.

There are some general results towards proving the conjecture: In [18], Knörr and Robinson gave a reformulation of this conjecture using alternating sums and descending chains of p-subgroups of G. Puig has refined this conjecture and, in [29, 30], given a theorem reducing it to a statement about non-abelian groups, closely related to simple groups.

The author has been supported by the Deutsche Forschungsgemeinschaft, SPP 1388.

On the other hand, Navarro and Tiep presented in [27] another approach and showed that the non-blockwise version of the Alperin weight conjecture holds whenever a certain set of conditions holds for every non-abelian simple group. They called groups with this property AWC-good. In [6] their inductive condition was reformulated in terms of projective representations. Furthermore, by work of An–Dietrich [2] and Malle [21], it has been proven that the sporadic simple groups and the simple alternating groups are AWC-good. Additionally, Navarro and Tiep showed that all simple groups of Lie type satisfy this inductive condition whenever the prime coincides with the characteristic of the underlying field or the group has an abelian Sylow 2-subgroup.

In this paper we refine the strategy and methods of Navarro–Tiep in order to prove the following statement.

Theorem A. *Let G be a finite group and p be a prime. Assume that every non-abelian simple group S involved in G satisfies the inductive BAW condition from Definition* 4.1. *Then Conjecture* 1.1 *holds for every p-block of G.*

Our strategy can also be adapted such that one considers only blocks with specific defect groups, and this leads to the following statement.

Theorem B. *Let G be a finite group, p be a prime and $\mathcal{R}$ be a set of p-groups which is closed under taking quotients and subgroups. Assume that every non-abelian simple group S involved in G satisfies the inductive BAW condition from Definition* 5.17 *with respect to $\mathcal{R}$. Then Conjecture* 1.1 *holds for every p-block of G, whose defect group is contained in $\mathcal{R}$.*

Recall that a group S is *involved* in G if there exist subgroups

$$H_1 \lhd H_2 \leq G$$

such that the quotient H_2/H_1 is isomorphic to S. Examples for possible sets $\mathcal{R}$ are the set of all abelian p-groups and the set of all p-groups, whose order is at most a given integer. The inductive BAW condition from Definition 4.1 consists of the conditions presented in [27, Section 3] together with some additional compatibility properties that are required in the proof for controlling the blocks, see Remark 4.2. Its relative version is given in Definition 5.17 and considers only specific Brauer and defect zero characters.

Applying this result for appropriate sets $\mathcal{R}$ one might be able to verify the Alperin weight conjecture for blocks with a given defect groups, like in [16] or [9]. The blocks with abelian defect groups of simple groups seem to be understood by [17]. Hence we hope that in a first step the assumption on the simple groups can be verified in the case where $\mathcal{R}$ is chosen to be the set of all abelian p-groups.

While this paper was being finalized, Puig, in [31,32], published a result similar to Theorem A depending on methods and results from [29, 30]. Our approach is completely independent. We give a detailed proof using different methods, which are closely related to character theory. Furthermore we show that several infinite series of simple groups satisfy the inductive BAW condition, namely groups having an abelian Sylow 2-subgroup and the finite groups of Lie type for their defining characteristic. Moreover, for all simple groups with a cyclic outer automorphism group the inductive BAW condition holds with respect to $\mathcal{R}_c$, the set of all cyclic p-groups. The alternating groups and fourteen sporadic simple groups satisfy the inductive BAW condition according to [5, 21].

Theorem C. *Let S be a simple group of Lie type, a Suzuki or a Ree group and p be the characteristic of the underlying field. Then the inductive BAW condition from Definition* 4.1 *holds for S and p.*

As an application of our results we prove the following statement.

Theorem D. *Let G be a group such that every composition factor has an abelian Sylow* 2*-subgroup. Then the blockwise Alperin weight conjecture is true for G and every prime p.*

Main ideas. The proof presented here adapts and extends the strategy of [27] to blocks. Their approach depends on results of [15], which were refined and adapted to controlling blocks in [34]. These methods are used to construct the bijection from Proposition 4.6. A similar bijection has been constructed for height zero characters and now the proof is transferred to Brauer characters. This statement is one of the key steps as it shows how the validity of the inductive BAW condition for a simple group S predicts the representation theory of an arbitrary finite group, see Section 4.

A second main ingredient is the character correspondence from Section 5.12 in [24], sometimes called DGN-correspondence. Using a result from [20], we obtain a bijection for characters lying above two characters related by this correspondence, see Theorem 3.8. Some properties of this bijection were already mentioned in [8].

Conjecture 5.1, a relative and generalized version of the blockwise Alperin weight conjecture, is better suited to reductions and helpful during the inductive proof. Assuming the inductive BAW condition from Definition 4.1 and its implication from Corollary 4.7, we prove Theorem A in Section 5 and sketch how these considerations imply Theorem B. In Section 6 we show that several series of simple groups satisfy the inductive BAW condition.

2 Preliminaries

In this section we fix some notation, for example around defect zero characters. Furthermore, we introduce the concept of a block quadruple, an extension of an ordinary-modular character triple. This will be relevant in the proof of Theorem A.

Let p be a fixed prime, $\mathbf{R}$ be the ring of algebraic integers in $\mathbb{C}$ and I be a maximal ideal of $\mathbf{R}$ with $p\mathbf{R} \subseteq I$. For a given finite group G this defines $\mathrm{IBr}(G)$, the set of irreducible p-Brauer characters of G, see [25, Chapter 2]. The quotient $\mathbb{F} = \mathbf{R}/I$ is a field of characteristic p, and is the algebraic closure of its prime field $\mathbb{F}_p$. We denote by $^* : \mathbf{R} \to \mathbb{F}$ the canonical epimorphism.

For blocks and characters in general we use the notation as introduced in [13] and [25]. For a finite group G and any block $b \in \mathrm{Bl}(G)$ let

$$\lambda_b : \mathrm{Z}(\mathbb{F}G) \longrightarrow \mathbb{F}$$

be the associated central function. For characters $\theta \in \mathrm{Irr}(b) \cup \mathrm{IBr}(b)$ we sometimes write λ_θ instead of λ_b. Furthermore, for an element $g \in G$ we denote by $\mathfrak{Cl}_G(g)$ the G-conjugacy class containing g and by U^+ the sum over the elements in U for any subset $U \subseteq G$. We recall the concept of an induced block.

Definition 2.1 (Induced blocks). Let $H \leq G$ and $b \in \mathrm{Bl}(H)$. We say, $b^G \in \mathrm{Bl}(G)$ *is defined* if the map

$$\lambda_b^G : \mathrm{Z}(\mathbb{F}G) \longrightarrow \mathbb{F} \text{ with } \lambda_b^G(\mathfrak{Cl}_G(g)^+) := \lambda_b\Bigg(\sum_{y \in \mathfrak{Cl}_G(g) \cap H} y\Bigg)$$

is an algebra homomorphism. We denote by $b^G \in \mathrm{Bl}(G)$ the block with $\lambda_{b^G} = \lambda_b^G$ and this block is called the *induced block*.

Some of the fundamental properties of induced blocks are the following: If $P\mathrm{C}_G(P) \leq H \leq \mathrm{N}_G(P)$ for some p-subgroup P of G, then b^G is defined for every $b \in \mathrm{Bl}(H)$, see [25, Theorem (4.14)].

Let $B \in \mathrm{Bl}(G)$, $P \leq G$ such that P is contained in some defect group of B and H with $P\mathrm{C}_G(P) \leq H \leq \mathrm{N}_G(P)$. Then there exists a block $b \in \mathrm{Bl}(H)$ with $b^G = B$, see [25, Theorem (4.14)]. For any block $b \in \mathrm{Bl}(H)$ for which b^G is defined, every defect group of b is contained in some defect group of b^G, see [25, Lemma (4.13)].

In the following, for any integer n we denote by n_p the maximal power of p dividing n. Our approach is centred around irreducible ordinary and Brauer characters. Hence we will use the following shorthand notations.

Notation 2.2. For a finite group G and $\theta \in \mathrm{Irr}(G) \cup \mathrm{IBr}(G)$ we denote by $\mathrm{bl}(\theta)$ the unique block $b \in \mathrm{Bl}(G)$ with $\theta \in \mathrm{Irr}(b) \cup \mathrm{IBr}(b)$.

For a finite group G we call $\theta \in \mathrm{Irr}(G)$ a *defect zero character of* G if

$$\theta(1)_p = |G|_p.$$

The set of defect zero characters of G is denoted by $\mathrm{dz}(G)$. For $\chi \in \mathrm{Irr}(G)$ we denote by $\chi^0 \in \mathrm{IBr}(G)$ the Brauer character that coincides with χ on all p-regular elements. For $\chi \in \mathrm{dz}(G)$, χ^0 is irreducible, see [25, Theorem (3.18)].

If a group A acts on G, we denote by $\mathrm{dz}_A(G)$ the set of A-invariant characters in $\mathrm{dz}(G)$. For $Z \lhd G$ and $\theta \in \mathrm{Irr}(Z)$ we write $\mathrm{dz}(G \mid \theta)$ for $\mathrm{dz}(G) \cap \mathrm{Irr}(G \mid \theta)$ and

$$\mathrm{rdz}(G \mid \theta) := \left\{ \chi \in \mathrm{Irr}(G \mid \theta) \;\middle|\; \left(\frac{\chi(1)}{\theta(1)}\right)_p = |G/Z|_p \right\}.$$

For a block $B \in \mathrm{Bl}(G)$ let

$$\mathrm{rdz}(B \mid \theta) := \mathrm{Irr}(B) \cap \mathrm{rdz}(G \mid \theta).$$

We denote by $\mathrm{dz}(\mathrm{N}_G(Q)/Q, B)$ the set of $\overline{\chi} \in \mathrm{dz}(\mathrm{N}_G(Q)/Q)$ with $\mathrm{bl}(\chi)^G = B$, where $\chi \in \mathrm{Irr}(\mathrm{N}_G(Q))$ is the lift of $\overline{\chi}$. For $\chi \in \mathbb{Z}\mathrm{Irr}(G)$ and $\phi \in \mathbb{Z}\mathrm{IBr}(G)$ we denote by $\mathrm{Irr}(\chi)$ and $\mathrm{IBr}(\phi)$, respectively the set of irreducible constituents of χ and ϕ, respectively.

Later, radical subgroups will be important.

Notation 2.3. Let G be a finite group. Then a group $P \leq G$ is *p-radical* if it satisfies $\mathrm{O}_p(\mathrm{N}_G(P)) = P$. We denote by $\mathrm{Rad}(G)$ the set of p-radical subgroups of G. If a group A acts on a set $\mathbb{T}$, we denote by $\mathbb{T}/\sim_A$ an arbitrary A-transversal in $\mathbb{T}$. For example $\mathrm{Rad}(G)/\sim_G$ denotes a G-transversal of radical p-subgroup of G.

Let G be a finite group and Z be a normal p-subgroup of G. Then every block $\overline{B} \in \mathrm{Bl}(G/Z)$ is dominated by or contained in a unique block $B \in \mathrm{Bl}(G)$, see [24, Section 5.8] and [25, p. 199]. We write then $\overline{B} \subseteq B$.

Lemma 2.4. *Let G be a finite group, $Z \lhd G$ be a p-group, $\overline{G} := G/Z$, and let $B \in \mathrm{Bl}(G)$.*

(a) *Let $\overline{B}_1, \dots, \overline{B}_s \in \mathrm{Bl}(\overline{G})$ be the blocks with $\overline{B}_i \subseteq B$. Then*

$$\bigcup_{i=1}^{s} \mathrm{IBr}(\overline{B}_i) = \mathrm{IBr}(B).$$

Let D be a defect group of B. Every $\overline{B}_i$ has a defect group, that is contained in D/Z.

(b) *Let Q be a p-group with $Z \lhd Q \leq G$, $b \in \mathrm{Bl}(\mathrm{N}_G(Q))$ and*
$$\overline{b}_1, \dots, \overline{b}_t \in \mathrm{Bl}(\mathrm{N}_G(Q)/Z)$$
be the blocks dominated by b. Then the blocks $\overline{b}_i^G$ $(1 \leq i \leq t)$ are defined and dominated by b^G.

Proof. According to [24, Theorem 5.8.10] the block B dominates blocks of $\overline{G}$, we call them $\overline{B}_1, \dots, \overline{B}_s$. By [24, Lemma 5.8.6 (b)] this implies
$$\bigcup_{i=1}^{s} \mathrm{Irr}(\overline{B}_i) \subseteq \mathrm{Irr}(B) \quad \text{and} \quad \bigcup_{i=1}^{s} \mathrm{IBr}(\overline{B}_i) \subseteq \mathrm{IBr}(B).$$
The normal p-subgroup Z has to be contained in the kernel of every Brauer character $\theta \in \mathrm{IBr}(B)$, see [24, Theorem 5.8.2]. This implies that every Brauer character of G is also a Brauer character of $\overline{G}$, namely $\mathrm{IBr}(G) = \mathrm{IBr}(\overline{G})$. As every block of $\overline{G}$ is dominated by a block of G this implies $\bigcup_{i=1}^{s} \mathrm{IBr}(\overline{B}_i) = \mathrm{IBr}(B)$. The statement about the defect group is known from [25, Theorem (9.9) (a)].

The statement in [23, Proposition 2.4] proves part (b). □

In the following statement we compute the value of the central function associated with the product of two characters.

Lemma 2.5. *Let $N \lhd G$.*

(a) *Furthermore, let $\widetilde{\theta} \in \mathrm{IBr}(G)$ such that $\theta := \widetilde{\theta}_N$ is irreducible, $\overline{\eta} \in \mathrm{IBr}(G/N)$ and $\eta \in \mathrm{IBr}(G)$ its lift. If $g \in G$, then*
$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(g)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(\overline{g})^+),$$
where $\overline{g} = gN \in G/N$ and $L/N = \mathrm{C}_{G/N}(\overline{g})$.

(b) *Let $\widetilde{\theta} \in \mathrm{Irr}(G)$ such that $\theta := \widetilde{\theta}_N$ is irreducible, $\overline{\eta} \in \mathrm{Irr}(G/N)$ and $\eta \in \mathrm{Irr}(G)$ its lift. If $g \in G$, then*
$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = \lambda_{\widetilde{\theta}_L}(\mathfrak{Cl}_L(g)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{G/N}(\overline{g})^+),$$
where $\overline{g} = gN \in G/N$ and $L/N = \mathrm{C}_{G/N}(\overline{g})$.

Proof. We present here only the proof of (a). With little adaption, the proof also implies (b). Let $\overline{G} := G/N$, $\epsilon : G \to \overline{G}$ the canonical epimorphism, $g \in G$ and $\overline{g} := \epsilon(g)$. The set $\epsilon(\mathfrak{Cl}_G(g))$ coincides with $\mathfrak{Cl}_{\overline{G}}(\overline{g})$.

For every $y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})$ we choose an element $g_y \in \mathfrak{Cl}_G(g) \cap \epsilon^{-1}(y)$. We have $\mathfrak{Cl}_G(g) \cap g_y N = \mathfrak{Cl}_{L_y}(g_y)$, where $L_y := \epsilon^{-1}(\mathrm{C}_{\overline{G}}(y))$, and
$$\mathfrak{Cl}_G(g) = \dot{\bigcup_{y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})}} \mathfrak{Cl}_{L_y}(g_y).$$
Note that for $y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})$ the sets $\mathfrak{Cl}_{L_y}(g_y)$ are conjugate in G.

Let $\mathcal{D}$ be an $\mathbb{F}$-representation of G affording $\widetilde{\theta}$ and $\mathcal{Q}$ be an $\mathbb{F}$-representation of G affording η. We compute $(\mathcal{Q} \otimes \mathcal{D})(\mathfrak{Cl}_G(g)^+)$. For $y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})$ we obtain

$$(\mathcal{Q} \otimes \mathcal{D})(\mathfrak{Cl}_{L_y}(g_y)^+) = \mathcal{Q}(g_y) \otimes \lambda_{\widetilde{\theta}_{L_y}}(\mathfrak{Cl}_{L_y}(g_y)^+)\, \mathrm{id}_{\theta(1)},$$

as the representation $\mathcal{Q}$ is constant on $g_y N$, and

$$\mathcal{D}(\mathfrak{Cl}_{L_y}(g_y)^+) = \lambda_{\widetilde{\theta}_{L_y}}(\mathfrak{Cl}_{L_y}(g_y)^+)\, \mathrm{id}_{\widetilde{\theta}(1)}$$

by the definition of $\lambda_{\widetilde{\theta}_{L_y}}$. For every $y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})$ the set $\mathfrak{Cl}_{L_y}(g_y)$ is G-conjugate to $\mathfrak{Cl}_{L_{\overline{g}}}(g)$ and hence

$$\lambda_{\widetilde{\theta}_{L_y}}(\mathfrak{Cl}_{L_y}(g_y)^+) = \lambda_{\widetilde{\theta}_{L_{\overline{g}}}}(\mathfrak{Cl}_{L_{\overline{g}}}(g)^+).$$

This implies

$$\begin{aligned}(\mathcal{Q} \otimes \mathcal{D})(\mathfrak{Cl}_G(g)^+) &= \sum_{y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})} (\mathcal{Q} \otimes \mathcal{D})(\mathfrak{Cl}_{L_y}(g_y)^+) \\ &= \sum_{y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})} \mathcal{Q}(g_y) \otimes \lambda_{\widetilde{\theta}_{L_{\overline{g}}}}(\mathfrak{Cl}_{L_{\overline{g}}}(g)^+)\mathrm{id}_{\theta(1)}.\end{aligned}$$

Because $\eta = \overline{\eta} \circ \epsilon$, we have

$$\sum_{y \in \mathfrak{Cl}_{\overline{G}}(\overline{g})} \mathcal{Q}(g_y) = \lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+)\, \mathrm{id}_{\eta(1)},$$

and hence

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = \lambda_{\widetilde{\theta}_{L_{\overline{g}}}}(\mathfrak{Cl}_{L_{\overline{g}}}(g)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+). \qquad \square$$

A further tool is given by block quadruples which is an ordinary-modular character triple together with a distinguished block. In order to clarify the notation we sketch the definition of an ordinary-modular character triple from [25, Exercise (8.10)].

An ordinary character triple (G, N, θ) is a pair of groups $N \lhd G$ with a G-invariant character $\theta \in \mathrm{Irr}(N)$. If in addition $\theta^0 \in \mathrm{IBr}(N)$, we call (G, N, θ) *ordinary-modular character triple*.

Recall that an *isomorphism* $(\tau, \sigma) : (G, N, \theta) \to (G_1, N_1, \theta_1)$ is given by

- an isomorphism $\tau : G/N \to G_1/N_1$,
- a map

  $$\sigma_J : \mathbb{Z}\mathrm{Irr}(J \mid \theta) \cup \mathbb{Z}\mathrm{IBr}(J \mid \theta^0) \longrightarrow \mathbb{Z}\mathrm{Irr}(J_1 \mid \theta_1) \cup \mathbb{Z}\mathrm{IBr}(J_1 \mid \theta_1^0)$$

  for every J with $N \leq J \leq G$ and $\tau(J/N) = J_1/N_1$,

such that these maps satisfy a list of additional properties. In particular the map σ_J satisfies $\sigma_J(\chi)^0 = \sigma_J(\chi^0)$ for every $N \leq J \leq G$ and $\chi \in \mathrm{Irr}(J \mid \theta)$.

An isomorphism of ordinary-modular character triples with defect zero characters also preserves the partitioning of the characters into blocks. For the proof we recall the definition of decomposition numbers. For $B \in \mathrm{Bl}(G)$ and $\chi \in \mathrm{Irr}(B)$ we can write χ^0 as

$$\chi^0 = \sum_{\psi \in \mathrm{IBr}(B)} d_{\chi,\psi}\psi.$$

The integers $d_{\chi\psi}$ are called the *decomposition numbers* and determine the blocks, since

$$\mathrm{IBr}(B) = \left\{\psi \in \mathrm{IBr}(G) \mid d_{\chi,\psi} \neq 0 \text{ for some } \chi \in \mathrm{Irr}(B)\right\}$$

for $B \in \mathrm{Bl}(G)$, see [25, Theorem (3.3)]. For $Z \lhd G$ and $b \in \mathrm{Bl}(Z)$ we denote by $\mathrm{Bl}(G \mid b)$ the set of blocks of G that cover b.

Lemma 2.6. *Let* (G, N, θ), (G_1, N_1, θ_1) *be isomorphic ordinary-modular character triples with* $\theta \in \mathrm{dz}(N)$ *and* $\theta_1 \in \mathrm{dz}(N_1)$ *and* (τ, σ) *an isomorphism between them. Let* $b := \mathrm{bl}(\theta)$, $b_1 := \mathrm{bl}(\theta_1)$, $N \leq J \leq G$ *and* $N_1 \leq J_1 \leq G_1$, *such that*

$$\tau(J/N) = J_1/N_1.$$

For every block $B \in \mathrm{Bl}(J \mid b)$ *there exists a block* $B_1 \in \mathrm{Bl}(J_1)$ *such that*

$$\sigma_J(\mathrm{IBr}(B)) = \mathrm{IBr}(B_1).$$

Hence σ_J *determines a bijection*

$$\sigma_J : \mathrm{Bl}(J \mid b) \longrightarrow \mathrm{Bl}(J_1 \mid b_1) \quad \text{with } \sigma_J(\mathrm{IBr}(B)) = \mathrm{IBr}(\sigma_J(B)).$$

Furthermore, $\mathrm{Irr}(\sigma_J(B)) = \sigma_J(\mathrm{Irr}(B))$ *for every* $B \in \mathrm{Bl}(J \mid b)$.

Proof. According to the definition of isomorphic ordinary-modular character triples, the decomposition numbers are preserved, more precisely the equation

$$d_{\chi,\psi} = d_{\sigma_J(\chi),\sigma_J(\psi)}$$

holds for every $N \leq J \leq G$, $\chi \in \mathrm{Irr}(J \mid \theta)$ and $\psi \in \mathrm{IBr}(J \mid \theta^0)$.

Let $B \in \mathrm{Bl}(J \mid b)$. Because $\mathrm{IBr}(b) = \left\{\theta^0\right\}$ and $\mathrm{Irr}(b) = \{\theta\}$, we have

$$\mathrm{IBr}(B) \subseteq \mathrm{IBr}(J \mid \theta^0) \quad \text{and} \quad \mathrm{Irr}(B) \subseteq \mathrm{Irr}(J \mid \theta).$$

Analogously

$$\mathrm{IBr}(B_1) \subseteq \mathrm{IBr}(J_1 \mid {\theta_1}^0) \quad \text{and} \quad \mathrm{Irr}(B_1) \subseteq \mathrm{Irr}(J_1 \mid \theta_1)$$

for every $B_1 \in \mathrm{Bl}(J_1 \mid b_1)$. This implies that for every block $B \in \mathrm{Bl}(J \mid b)$ the characters $\sigma_J(\mathrm{Irr}(B))$ form a connected component in the Brauer graph. By [25, Theorem (3.19)] we have $\sigma_J(\mathrm{Irr}(B)) = \mathrm{Irr}(B_1)$ for some $B_1 \in \mathrm{Bl}(J_1 \mid b_1)$. Analogously, the statement holds also for Brauer characters. □

For our reduction purpose the following statement plays a key role. Instead of an arbitrary ordinary-modular character triple with a defect zero character we can consider one whose character is linear and faithful.

Lemma 2.7 ([25, Exercise (8.13)]). *Let (G, N, θ) be an ordinary-modular character triple with $\theta \in \mathrm{dz}(N)$. Then there exists an isomorphic ordinary-modular character triple (G_1, N_1, θ_1) such that $N_1 \leq \mathrm{Z}(G_1)$, N_1 is a p'-group and θ_1 a faithful character of N_1.*

We construct a candidate for (G_1, N_1, θ_1) in the following, such that there are additional relations between (G, N, θ) and (G_1, N_1, θ_1).

Recall that according to [13, Theorem (11.2)] there exists a projective $\mathbb{C}$-representation $\mathcal{P}$ of G for any character triple (G, N, θ), such that $\mathcal{P}_N$ is a representation affording θ and it is multiplicative on $(G \times N) \cup (N \times G)$. According to the proof of [25, Theorem (8.28)] the projective representation $\mathcal{P}$ can be chosen such that $Z := \langle \alpha(g, g') \mid g, g' \in G \rangle$ is a finite group, where $\alpha : G \times G \to \mathbb{C}^*$ is the factor set of $\mathcal{P}$. In such a situation we call $\mathcal{P}$ *projective representation of G associated with θ*.

For $N \lhd G$ and a G-invariant character $\phi \in \mathrm{IBr}(N)$ one can analogously define the notion of a projective representations of G associated with ϕ.

2.8 "Favorite" isomorphic ordinary-modular character triples

Let (G, N, θ) be a character triple with $\theta \in \mathrm{dz}(N)$, $\mathcal{P}$ be a projective $\mathbb{C}$-representation of G associated with θ, $\alpha : G \times G \to \mathbb{C}^\times$ be the factor set of $\mathcal{P}$, and $Z := \langle \alpha(g, g') \mid g, g' \in G \rangle$. Since θ is a defect zero character, the projective representation $\mathcal{P}$ can be chosen such that Z is a p'-group, see [25, Exercises (3.10) and (8.13)].

Like in the proof of [25, Theorem (8.28)], the factor set of $\mathcal{P}$ determines a finite group $\widehat{G}$, where $\widehat{G} := \{(g, \zeta) \mid g \in G, \zeta \in Z\}$ with multiplication

$$(g_1, \zeta_1)(g_2, \zeta_2) = (g_1 g_2, \zeta_1 \zeta_2 \alpha(g_1, g_2)).$$

As α is a 2-cocycle of G, it follows that $\widehat{G}$ is a finite group. Via $\epsilon : \widehat{G} \to G$ with $(g, \zeta) \mapsto g$ we see that $\widehat{G}$ is a central extension of G by Z. As $\epsilon_{N_0} : N_0 \to N$ with $N_0 := \{(n, 1) \mid n \in N\}$ is an isomorphism, $\theta_0 := \theta \circ \epsilon_{N_0}$ is an irreducible character of N_0.

By the construction of $\widehat{G}$ the map $\widetilde{\mathcal{D}} : \widehat{G} \to \mathrm{GL}_{\theta(1)}(\mathbb{C})$ defined by

$$\widetilde{\mathcal{D}}(g,\zeta) = \zeta\mathcal{P}(g) \quad \text{for every } (g,\zeta) \in \widehat{G}$$

is a representation of $\widehat{G}$. The character $\widetilde{\theta_0}$ afforded by $\widetilde{\mathcal{D}}$ is an extension of θ_0.

For any group J with $N \leq J \leq G$ we denote by $\widehat{J}$ the group $\epsilon^{-1}(J)$ and let $\overline{J} := \widehat{J}/N_0$, in particular $\overline{G} := \widehat{G}/N_0$, $\widehat{N} := \epsilon^{-1}(N)$ and $\overline{N} := \widehat{N}/N_0$. Every character of $\mathrm{Irr}(J \mid \theta)$ lifts to a character in $\mathrm{Irr}(\widehat{J} \mid \theta_0 \times 1_Z)$. Recall that by [25, Theorem (8.18) and Corollary (8.20)] every character of $\mathrm{Irr}(\widehat{J} \mid \theta_0 \times 1_Z)$ can be written as $(\widetilde{\theta_0})_{\widehat{J}}\eta$ where $\nu \in \mathrm{Irr}((\widetilde{\theta_0})_Z)$ and $\eta \in \mathrm{Irr}(\widehat{J} \mid \nu)$ with $N_0 \leq \ker(\eta)$. Let

$$\overline{(\widetilde{\theta_0})_{\widehat{J}}\eta} \in \mathrm{Irr}(J \mid \theta)$$

be the character that lifts to $(\widetilde{\theta_0})_{\widehat{J}}\eta$. Then

$$\mathrm{Irr}(J \mid \theta) = \left\{ \overline{(\widetilde{\theta_0})_{\widehat{J}}\eta} \;\middle|\; \eta \in \mathrm{Irr}(\overline{J} \mid 1_{N_0} \times \nu) \right\}.$$

An analogous statement holds for $\mathrm{IBr}(J \mid \theta^0)$, namely

$$\mathrm{IBr}(J \mid \theta^0) = \left\{ \overline{(\widetilde{\theta}_0^0)_{\widehat{J}}\eta} \;\middle|\; \eta \in \mathrm{IBr}(\overline{J} \mid 1_{N_0} \times \nu^0) \right\},$$

where $\overline{(\widetilde{\theta_0}^0)_{\widehat{J}}\eta}$ denotes the character that lifts to $(\widetilde{\theta_0}^0)_{\widehat{J}}\eta \in \mathrm{IBr}(\widehat{J})$. Let

$$\tau : G/N \longrightarrow \overline{G}/\overline{N}$$

be the canonical isomorphism and

$$\sigma_J : \mathbb{Z}\mathrm{IBr}(J \mid \theta^0) \cup \mathbb{Z}\mathrm{Irr}(J \mid \theta) \longrightarrow \mathbb{Z}\mathrm{IBr}(\overline{J} \mid \nu^0) \cup \mathbb{Z}\mathrm{Irr}(\overline{J} \mid \nu)$$

be the bijection defined by

$$\mathbb{Z}\mathrm{Irr}(J \mid \theta) \ni \overline{\widetilde{\theta}_{\widehat{J}}\eta} \mapsto \overline{\eta} \in \mathbb{Z}\mathrm{Irr}(\overline{J} \mid \nu)$$

and

$$\mathbb{Z}\mathrm{IBr}(J \mid \theta^0) \ni \overline{\widetilde{\theta}^0_{\widehat{J}}\eta} \mapsto \overline{\eta} \in \mathbb{Z}\mathrm{IBr}(\overline{J} \mid \nu^0),$$

where $\eta \in \mathbb{Z}\mathrm{Irr}(\widehat{J})$ is the lift of $\overline{\eta}$. Then (τ,σ) is an isomorphism between the ordinary-modular character triples (G, N, θ) and $(\overline{G}, \overline{N}, \nu)$.

For the isomorphic character triples (G, N, θ) and $(\overline{G}, \overline{N}, \nu)$ the blocks covering $\mathrm{bl}(\theta)$ and $\mathrm{bl}(\nu)$ are related by the following.

Proposition 2.9. *Let* (G, N, θ) *and* $(\overline{G}, \overline{N}, \nu)$ *be the character triples from* 2.8. *Let* $N \leq J \leq G$, $b \in \mathrm{Bl}(J \mid \mathrm{bl}(\theta))$ *and* $B \in \mathrm{Bl}(G \mid \mathrm{bl}(\theta))$. *Let* $\widehat{b} \in \mathrm{Bl}(\widehat{J})$ *be the block dominating* b *and* $\overline{b} := \sigma_J(b) \in \mathrm{Bl}(\overline{J})$. *Furthermore, let* $\widehat{B} \in \mathrm{Bl}(\widehat{G})$ *be the block dominating* B *and* $\overline{B} := \sigma_G(B) \in \mathrm{Bl}(\overline{G})$. *Then the following statements are equivalent:*

(i) b^G *is defined and coincides with* B.

(ii) $\widehat{b}^{\widehat{G}}$ *is defined and coincides with* $\widehat{B}$.

(iii) $\overline{b}^{\overline{G}}$ *is defined and coincides with* $\overline{B}$.

Proof. In order to prove the statement we consider characters of the mentioned blocks. Let $\overline{\eta}_0 \in \mathrm{Irr}(\overline{b})$, $\overline{\eta} \in \mathrm{Irr}(\overline{B})$, and denote their lifts to $\widehat{J}$ and $\widehat{G}$ by η_0 and η. Let $\chi_0 \in \mathrm{Irr}(J \mid \theta)$ and $\chi \in \mathrm{Irr}(G \mid \theta)$ be such that $\sigma_J(\chi_0) = \eta_0$ and $\sigma_J(\chi) = \eta$. Then χ_0 lifts to $\widehat{\chi}_0 := (\widetilde{\theta}_0)_{\widehat{J}}\eta_0 \in \mathrm{Irr}(\widehat{J} \mid \theta_0)$ and χ to $\widehat{\chi} := \widetilde{\theta}_0\eta \in \mathrm{Irr}(\widehat{G} \mid \theta_0)$. This implies $\chi \in \mathrm{Irr}(B)$, $\widehat{\chi} \in \mathrm{Irr}(\widehat{B})$, $\chi_0 \in \mathrm{Irr}(b)$, and $\widehat{\chi}_0 \in \mathrm{Irr}(\widehat{b})$.

In a first step we prove that (i) and (ii) are equivalent. The statement in (i) is equivalent to $\lambda_\chi = \lambda_{\chi_0}^G$. According to [24, Lemma 5.3.1 (i)] we have

$$\lambda_{\chi_0}^G(\mathfrak{Cl}_G(g)^+) = \left(\frac{|\mathfrak{Cl}_G(g)|\chi_0^G(g)}{\chi_0^G(1)}\right)^* \quad \text{for every } g \in G.$$

Hence the statement in (i) holds if and only if

$$\left(\frac{|\mathfrak{Cl}_G(g)|\,\chi(g)}{\chi(1)}\right)^* = \left(\frac{|\mathfrak{Cl}_G(g)|\,\chi_0^G(g)}{\chi_0^G(1)}\right)^* \quad \text{for every } g \in G.$$

Let $\epsilon_1 : \widehat{G} \to \widehat{G}/N_0$ be the canonical epimorphism. Since $\widehat{\chi}$ and $\widehat{\chi}_0$ are the lifts of χ and χ_0, this is equivalent to

$$\left(\frac{|\mathfrak{Cl}_G(\epsilon_1(\dot{g}))|\,\widehat{\chi}(\dot{g})}{\widehat{\chi}(1)}\right)^* = \left(\frac{|\mathfrak{Cl}_G(\epsilon_1(\dot{g}))|\,\widehat{\chi}_0^{\widehat{G}}(\dot{g})}{\widehat{\chi}_0^{\widehat{G}}(1)}\right)^* \quad \text{for every } \dot{g} \in \widehat{G}. \tag{2.1}$$

Analogously the statement in (ii), namely $\lambda_{\widehat{\chi}} = \lambda_{\widehat{\chi}_0}^{\widehat{G}}$, holds if and only if

$$\left(\frac{|\mathfrak{Cl}_{\widehat{G}}(\dot{g})|\,\widehat{\chi}(\dot{g})}{\widehat{\chi}(1)}\right)^* = \left(\frac{|\mathfrak{Cl}_{\widehat{G}}(\dot{g})|\,\widehat{\chi}_0^{\widehat{G}}(\dot{g})}{\widehat{\chi}_0^{\widehat{G}}(1)}\right)^* \quad \text{for every } \dot{g} \in \widehat{G}. \tag{2.2}$$

In the next step we prove that for $g \in G$ and $\dot{g} \in \widehat{G}$ with $\epsilon_1(\dot{g}) = g$ the conjugacy class lengths satisfy

$$p \nmid \frac{|\mathfrak{Cl}_{\widehat{G}}(\dot{g})|}{|\mathfrak{Cl}_G(g)|}.$$

We have $\mathfrak{Cl}_{\widehat{G}}(\dot{g}) \cap \dot{g}Z = \dot{g}U$ for some set $U \subseteq Z$. Then

$$|\mathfrak{Cl}_{\widehat{G}}(\dot{g})| = |U||\mathfrak{Cl}_G(g)|.$$

Since $Z \leq \mathrm{Z}(\widehat{G})$, the group U is a subgroup of Z and hence $p \nmid |Z|$ implies $p \nmid |U|$. Equation (2.1) for g is equivalent to equation (2.2) for $\dot{g}$. Accordingly the statements in (i) and (ii) are equivalent.

Next we prove that (iii) implies (ii). Let $\epsilon_2 : \widehat{G} \to \overline{G}$ be the canonical epimorphism, $\dot{g} \in \widehat{G}$, $\overline{g} \in \overline{G}$ with $\dot{g}N_0 = \overline{g}$ and $C := \epsilon_2^{-1}(\mathrm{C}_{\overline{G}}(\overline{g}))$. By Lemma 2.5 (a) we have

$$\left(\frac{|\mathfrak{Cl}_{\overline{G}}(\overline{g})|\overline{\eta}(\overline{g})}{\overline{\eta}(1)}\right)^* \lambda_{(\widetilde{\theta}_0)_C}(\mathfrak{Cl}_C(\dot{g})^+) = \lambda_{\widetilde{\theta}_0\eta}(\mathfrak{Cl}_{\widehat{G}}(\dot{g})^+).$$

Since the conjugacy classes satisfy

$$|\mathfrak{Cl}_{\widehat{G}}(\dot{g})| = |\widehat{G} : \mathrm{C}_{\widehat{G}}(\dot{g})| = |\widehat{G} : C||C : \mathrm{C}_{\widehat{G}}(\dot{g})| = |\mathfrak{Cl}_{\overline{G}}(\overline{g})||\mathfrak{Cl}_C(\dot{g})|,$$

and according to [24, Lemma 5.3.1 (i)], we have

$$\begin{aligned}
\lambda_{\overline{\eta}_0^{\overline{G}}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+)\lambda_{(\widetilde{\theta}_0)_C}(\mathfrak{Cl}_C(\dot{g})^+) &= \left(\frac{|\mathfrak{Cl}_{\overline{G}}(\overline{g})|\overline{\eta}_0^{\overline{G}}(\overline{g})}{\overline{\eta}_0^{\overline{G}}(1)}\right)^* \lambda_{(\widetilde{\theta}_0)_C}(\mathfrak{Cl}_C(\dot{g})^+)\\
&= \left(\frac{|\mathfrak{Cl}_{\overline{G}}(\overline{g})|\overline{\eta}_0^{\overline{G}}(\overline{g})}{\overline{\eta}_0^{\overline{G}}(1)}\right)^* \left(\frac{|\mathfrak{Cl}_C(\dot{g})|\widetilde{\theta}_0(\dot{g})}{\widetilde{\theta}_0(1)}\right)^*\\
&= \left(\frac{|\mathfrak{Cl}_{\widehat{G}}(\dot{g})|((\widetilde{\theta}_0)_{\widehat{J}}\eta_0)^{\widehat{G}}(\overline{g})}{((\widetilde{\theta}_0)_{\widehat{J}}\eta_0)^{\widehat{G}}(1)}\right)^*\\
&= \lambda_{(\widetilde{\theta}_0)_{\widehat{J}}\eta_0}^{\widehat{G}}(\mathfrak{Cl}_{\widehat{G}}(\dot{g})^+).
\end{aligned}$$

This proves that we obtain

$$\lambda_{(\widetilde{\theta}_0)_{\widehat{J}}\eta_0}^{\widehat{G}}(\mathfrak{Cl}_{\widehat{G}}(\dot{g})^+) = \lambda_{\widetilde{\theta}_0\eta}(\mathfrak{Cl}_{\widehat{G}}(\dot{g})^+)$$

from

$$\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+) = \lambda_{\overline{\eta}_0^{\overline{G}}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+)$$

by multiplying with $\lambda_{(\widetilde{\theta}_0)_C}(\mathfrak{Cl}_C(\dot{g})^+)$ where $\dot{g} \in \widehat{G}$ and $\overline{g} = \dot{g}N_0 \in \overline{G}$. Accordingly (iii) follows from (ii).

Next we verify that (ii) implies (iii). Let $\overline{g} \in \overline{G}$ and $C := \epsilon_2^{-1}(\mathrm{C}_{\overline{G}}(\overline{g}))$. If there exists some $\dot{g} \in \widehat{G}$ with $\epsilon_2(\dot{g})N_0 = \overline{g}$ and $\lambda_{(\widetilde{\theta}_0)_C}(\mathfrak{Cl}_C(\dot{g})^+) \neq 0$, then

$$\lambda_{(\widetilde{\theta}_0)_{\widehat{J}}\eta_0}^{\widehat{G}}(\mathfrak{Cl}_{\widehat{G}}(\dot{g})^+) = \lambda_{\widetilde{\theta}_0\eta}(\mathfrak{Cl}_{\widehat{G}}(\dot{g})^+)$$

implies

$$\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+) = \lambda_{\overline{\eta}_0^{\overline{G}}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+).$$

We show that such an element $\dot g$ exists. The coset $\dot g N_0$ is the disjoint union of C-conjugacy classes, namely

$$\dot g N_0 = \dot{\bigcup_{t\in\mathbb{T}}} \mathfrak{Cl}_C(t)$$

for some set $\mathbb{T} \subseteq \dot g N_0$. By [13, Lemma (8.14) (c)] the equation

$$\sum_{c\in \dot g N_0} \widetilde{\theta}_0(c)\widetilde{\theta}_0(c^{-1}) = |N_0|$$

holds and implies

$$\sum_{c\in\mathbb{T}} \frac{|\mathfrak{Cl}_C(c)|\widetilde{\theta}_0(c)}{\theta_0(1)}\widetilde{\theta}_0(c^{-1}) = \frac{|N_0|}{\theta_0(1)}.$$

Because $\theta \in \mathrm{dz}(N)$ and hence $\theta_0 \in \mathrm{dz}(N_0)$, this implies

$$\sum_{c\in\mathbb{T}} \lambda_{(\widetilde{\theta}_0)_C}(\mathfrak{Cl}_C(c)^+)\widetilde{\theta}_0(c^{-1})^* \neq 0.$$

Hence there exists some $\dot g \in \widehat{G}$ with $\dot g N_0 = \overline{g}$ and $\lambda_{(\widetilde{\theta}_0)_C}(\mathfrak{Cl}_C(\dot g)^+) \neq 0$. For this element

$$\lambda^{\widehat{G}}_{(\widetilde{\theta}_0)_{\hat J}\eta_0}(\mathfrak{Cl}_{\widehat{G}}(\dot g)^+) = \lambda_{\widetilde{\theta}_0\eta}(\mathfrak{Cl}_{\widehat{G}}(\dot g)^+)$$

implies

$$\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+) = \lambda^{\overline{G}}_{\overline{\eta}_0}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+).$$

This proves that (ii) implies (iii). □

The following statement is a consequence of the above proposition.

Corollary 2.10. *Let* (G, N, θ), $(\overline{G}, \overline{N}, \nu)$ *and* (τ, σ) *as in* 2.8. *Let* $N \leq J \leq G$ *and* $b \in \mathrm{Bl}(J \mid \mathrm{bl}(\theta))$. *Let* $\widehat{b} \in \mathrm{Bl}(\widehat{J})$ *be the block dominating* b *and consider the block* $\overline{b} := \sigma_J(b) \in \mathrm{Bl}(\overline{J})$. *Then* b, $\widehat{b}$ *and* $\overline{b}$ *have isomorphic defect groups.*

Proof. As Z is a cyclic, central p'-group, b and $\widehat{b}$ have isomorphic defect groups by [25, Theorem (9.9)]. By [22, Theorem 1.4 (ii)], the blocks $\widehat{b}$ and $\overline{b}$ have isomorphic defect group. □

The proof of the above proposition also shows the following well-known fact about dominating blocks.

Corollary 2.11. *Let* $Z \lhd G$ *be finite groups,* $\overline{G} := G/Z$, $Z \leq J \leq G$, $\overline{J} := J/Z$ *and* $\overline{b} \in \mathrm{Bl}(\overline{J})$. *Assume* $\overline{b}^{\overline{G}}$ *is defined. Then for* $b \in \mathrm{Bl}(J)$ *dominating* $\overline{b}$ *the block* b^G *is defined and dominates* $\overline{b}^{\overline{G}}$.

Proof. This can be shown using the ideas from the proof of the above proposition. Let $\epsilon : G \to \overline{G}$ be the canonical epimorphism. Lemma 5.3.1 (i) of [24] implies

$$\lambda_b^G(\mathfrak{Cl}_G(g)^+) = \left(\frac{|\mathfrak{Cl}_G(g)|}{|\mathfrak{Cl}_{\overline{G}}(\epsilon(g))|}\right)^* \lambda_{\overline{b}}^{\overline{G}}(\mathfrak{Cl}_{\overline{G}}(\epsilon(g))^+) \quad \text{for every } g \in G.$$

Let $\overline{\chi} \in \mathrm{Irr}(\overline{b}^{\overline{G}})$, and $\chi \in \mathrm{Irr}(G)$ its lift. Then for every $g \in G$ and $\overline{g} = \epsilon(g)$ we have

$$\begin{aligned}\lambda_\chi(\mathfrak{Cl}_G(g)^+) &= \left(\frac{|\mathfrak{Cl}_G(g)|}{|\mathfrak{Cl}_{\overline{G}}(\overline{g})|}\right)^* \lambda_{\overline{\chi}}(\mathfrak{Cl}_{\overline{G}}(g)^+)\\ &= \left(\frac{|\mathfrak{Cl}_G(g)|}{|\mathfrak{Cl}_{\overline{G}}(\overline{g})|}\right)^* \lambda_{\overline{b}}^{\overline{G}}(\mathfrak{Cl}_{\overline{G}}(g)^+) = \lambda_b^G(\mathfrak{Cl}_G(g)^+).\end{aligned}$$

Since $\lambda_\chi = \lambda_b^G$, b^G is defined and is the block dominating $\overline{b}^{\overline{G}}$. □

Based on the properties proven in Proposition 2.9 we introduce the following shorthand notation.

Definition 2.12 (Block quadruple). A *block quadruple* is a quadruple (G, Z, θ, B) such that (G, Z, θ) is an ordinary-modular character triple with $\theta \in \mathrm{dz}_G(Z)$ and $B \in \mathrm{Bl}(G)$ such that B covers $\mathrm{bl}(\theta)$. An isomorphism (τ, σ) of the ordinary-modular character triples (G, Z, θ) and (G_1, Z_1, θ_1) with $\sigma_G(B) = B_1$ is called an *isomorphism between the block quadruples* (G, Z, θ, B) *and* $(G_1, Z_1, \theta_1, B_1)$. We call a block quadruple (G, Z, θ, B) *central* if $Z \leq \mathrm{Z}(G)$, and a central block quadruple is called *faithful* if θ is faithful.

The statement of Proposition 2.9 can be reformulated using block quadruples.

Corollary 2.13. *The following hold:*

(a) *For any block quadruple (G, Z, θ, B) the construction of* 2.8 *gives a central faithful block quadruple $(G_1, Z_1, \theta_1, B_1)$, which is isomorphic to (G, Z, θ, B). Furthermore, the defect groups of B and B_1 are isomorphic.*

(b) *Let Q be a group such that $N \leq Q \leq G$ and Q/N is a p-group. Then* $\mathrm{bl}(\psi)^G$ *is defined for every $\psi \in \mathrm{Irr}(\mathrm{N}_G(Q) \mid \theta)$. Furthermore,* $\mathrm{bl}(\psi)^G = B$ *if and only if* $\mathrm{bl}(\sigma_Q(\psi))^{G_1} = B_1$.

Proof. Part (a) is clear from the above.

Observe that $\tau(Q) = Z \times Q_p$ for $Q_p \in \mathrm{Syl}_p(Q)$. Here Q_p is unique. As Q/N is a p-group of G/N, this implies that

$$\tau(\mathrm{N}_G(Q)) = \mathrm{N}_{G_1}(Q_p).$$

Accordingly $\mathrm{bl}(\sigma_{\mathrm{N}_G(Q)}(\psi))^{G_1}$ is defined by [25, Theorem (4.14)]. By Proposition 2.9, $\mathrm{bl}(\psi)^G$ is defined and the equalities

$$\mathrm{bl}(\psi)^G = B \quad \text{and} \quad \mathrm{bl}(\sigma_{\mathrm{N}_G(Q)}(\psi))^{G_1} = B_1$$

are equivalent. □

Hence we can introduce the following set.

Notation 2.14. Let (G, N, θ, B) be a block quadruple, Q/N be a p-group of G/N and $\widetilde{\theta} \in \mathrm{Irr}(Q)$. For $\psi \in \mathrm{Irr}(\mathrm{N}_G(Q) \mid \theta)$ the block $\mathrm{bl}(\psi)^G$ is defined. Hence the set

$$\mathrm{rdz}(\mathrm{N}_G(Q), B \mid \widetilde{\theta}) := \left\{\psi \in \mathrm{rdz}(\mathrm{N}_G(Q) \mid \widetilde{\theta}) \;\middle|\; \mathrm{bl}(\psi)^G = B\right\}$$

is well-defined.

Before our main application of block quadruples we recall a bijection between defect zero and relative defect zero characters from [26].

Theorem 2.15. *Let $O \lhd G$ be a p-group and $\nu \in \mathrm{Irr}(O)$ be G-invariant.*

(a) *There exists a bijection*

$$\Gamma_\nu : \mathrm{dz}(G/O) \longrightarrow \mathrm{rdz}(G \mid \nu).$$

(b) *The block* $\mathrm{bl}(\psi)$ *is dominated by* $\mathrm{bl}(\Gamma_\nu(\psi))$ *for every* $\psi \in \mathrm{dz}(G/O)$.

(c) *Let $Z \leq \mathrm{Z}(G)$ be a p'-group, $\mu \in \mathrm{Irr}(Z)$, $\overline{\mu} \in \mathrm{Irr}(ZO/O)$ be the associated character and $\psi \in \mathrm{Irr}(G/O \mid \overline{\mu}) \cap \mathrm{dz}(G/O)$. Then $\Gamma_\nu(\psi) \in \mathrm{Irr}(G \mid \mu)$.*

Proof. The bijection in (a) is given in [26, Theorem 2.1].

For the proof of the remaining parts we recall here the construction of $\Gamma_\nu(\psi)$. For every element $g \in G$ there exist unique elements $g_p \in G$ and $g_{p'} \in G^0$ such that g_p is a p-element and $g = g_p g_{p'} = g_{p'} g_p$. For every $g \in G$ with $g_p \in O$ there exists a unique extension ν_g to $\langle O, g\rangle$ of ν by [13, Corollary (8.16)]. The character ν_g satisfies $\nu_g(g) = \nu(1)$ whenever $g \in \mathrm{C}_G(O)^0$. For $\psi \in \mathrm{dz}(G/O)$ and its lift $\psi_0 \in \mathrm{Irr}(G)$ the character $\Gamma_\nu(\psi)$ is defined by

$$\Gamma_\nu(\psi)(g) = \begin{cases} \psi_0(g)\nu_g(g) & \text{if } g_p \in O, \\ 0 & \text{otherwise.} \end{cases}$$

By this it is clear that λ_{ψ_0} and $\lambda_{\Gamma_\nu(\psi)}$ satisfy

$$\lambda_{\psi_0}(\mathfrak{Cl}_G(g)^+) = \lambda_{\Gamma_\nu(\psi)}(\mathfrak{Cl}_G(g)^+) \quad \text{for every } g \in \mathrm{C}_G(O)^0.$$

Both blocks $\mathrm{bl}(\psi_0)$ and $\mathrm{bl}(\Gamma_\nu(\psi))$ have defect groups containing O since $O \lhd G$. According to [24, Theorem 5.1.11], $\mathrm{bl}(\psi_0)$ and $\mathrm{bl}(\Gamma_\nu(\psi))$ have the same defect group, and $\mathrm{bl}(\psi_0) = \mathrm{bl}(\Gamma_\nu(\psi))$ by [24, Corollary 5.1.12].

Since Z is a p'-group, (b) implies (c). □

We give a blockwise version of [27, Theorem 2.4].

Theorem 2.16. *Let $N \lhd G$, Q/N be a p-subgroup of G/N and let $M := \mathrm{N}_G(Q)$. Let $\theta \in \mathrm{dz}_M(N)$. Then there exists an M-invariant extension $\widehat{\theta}$ of θ to Q. If $\widetilde{\theta}$ is another M-invariant extension of θ to Q, then there exists a bijection*

$$\Pi : \mathrm{rdz}(M \mid \widetilde{\theta}) \longrightarrow \mathrm{rdz}(M \mid \widehat{\theta})$$

such that

$$\Pi\big(\mathrm{rdz}(M \mid \widehat{\theta}) \cap \mathrm{Irr}(B)\big) = \mathrm{rdz}(M \mid \widetilde{\theta}) \cap \mathrm{Irr}(B)$$

for every $B \in \mathrm{Bl}(M)$. If $\mathrm{rdz}(M \mid \widetilde{\theta}) \neq \emptyset$, then Q/N is p-radical in M/N.

Proof. Without loss of generality we can assume that $G = M$. Furthermore, instead of considering the quadruple (G, N, θ, B) we may analyse a character triple $(G_1, N_1, \theta_1, B_1)$ that is isomorphic to (G, N, θ, B) via some (τ, σ) and is defined as in 2.8. Then N_1 is a central p'-group and θ_1 a faithful character. Let $Q_1 \leq G_1$ such that $\tau(Q/N) = Q_1/N_1$. Because $N_1 \leq \mathrm{Z}(G_1)$ and $p \nmid |N_1|$, Q_1 is the direct product of N_1 and its Sylow p-subgroup Q_p. The character $\widehat{\theta_1} := \theta_1 \times 1_{Q_p}$ is an G_1-invariant extension of θ_1. Then $\widehat{\theta} := \sigma_Q^{-1}(\widehat{\theta_1})$ is a G-invariant extension of θ to Q.

Assume that $\mathrm{rdz}(G \mid \widehat{\theta}) \neq \emptyset$. Via σ_G this implies $\mathrm{rdz}(G_1 \mid \widehat{\theta_1}) \neq \emptyset$. We observe that $Q_p \lhd G_1$ and $Q_p \leq \ker(\widehat{\theta_1})$. Hence every character of $\mathrm{rdz}(G_1 \mid \widehat{\theta_1})$ corresponds to some defect zero characters of G_1/Q_p. As any defect zero character of G_1/Q_p lies in a defect zero block, the group Q_p is radical in G_1 and hence Q/N is radical in G/N.

It remains to establish the existence of Π. Let $\widetilde{\theta}$ be a further G-invariant extension of θ to Q. Then $\sigma_Q(\widetilde{\theta})$ is of the form $\widetilde{\theta_1} := \theta_1 \times \nu$ for some G_1-invariant linear character $\nu \in \mathrm{Irr}(Q_p)$. Let $\Gamma_\nu : \mathrm{dz}(G_1/Q_p) \to \mathrm{rdz}(G_1 \mid \nu)$ be the bijection from Theorem 2.15. By Theorem 2.15 (c) the map Γ_ν^{-1} defines a bijection

$$\Pi_1 : \mathrm{rdz}(G_1 \mid \widetilde{\theta_1}) \longrightarrow \mathrm{rdz}(G_1 \mid \widehat{\theta_1}).$$

Because of Theorem 2.15 (b) every $\psi \in \mathrm{rdz}(G_1 \mid \widetilde{\theta_1})$ satisfies

$$\mathrm{bl}(\psi) = \mathrm{bl}(\Pi_1(\psi)).$$

Accordingly $\Pi := \sigma_G^{-1} \circ \Pi_1 \circ \sigma_G$ is a bijection with the required properties. □

3 Z-sections and projective $\mathbb{F}$-representations

In this section we present some technical tools. Like in [34] a central role in the reduction is played by so-called Z-sections and projective $\mathbb{F}$-representations, and their interplay with linear representations and central functions associated with these representations. We recall the definition of a Z-section from [34, Section 2], and adapt the constructions from there to Brauer characters and $\mathbb{F}$-representations.

In general we use projective $\mathbb{F}$-representations in the sense of [25, Chapter 8].

Definition 3.1. Let $Z \lhd G$ and let $\epsilon : G \to G/Z$ be the canonical epimorphism. A Z*-section* $\mathrm{rep} : G/Z \to G$ is a map with $\mathrm{rep}(1) = 1$ and $\epsilon \circ \mathrm{rep} = \mathrm{id}_{G/Z}$.

Remark 3.2. Let $Z \lhd G$, $p \nmid |Z|$, and let $\mathrm{rep} : G/Z \to G$ be a Z-section. Let $\chi \in \mathrm{IBr}(G)$ such that $\mathrm{IBr}(\chi_Z) = \{\nu\}$ with $\nu(1) = 1$, for example if $Z \leq \mathrm{Z}(G)$. Let $\mathcal{D}$ be an $\mathbb{F}$-representation affording χ. Then $\mathcal{P} := \mathcal{D} \circ \mathrm{rep}$ is a projective $\mathbb{F}$-representation of G/Z. We say that $\mathcal{P}$ *is obtained from* χ *using* rep. The factor set α of $\mathcal{P}$ satisfies

$$\alpha(\overline{g}, \overline{g}') = \nu(\mathrm{rep}(\overline{g})\,\mathrm{rep}(\overline{g}')\,\mathrm{rep}(\overline{g}\,\overline{g}')^{-1})^* \quad \text{for every } \overline{g}, \overline{g}' \in G/Z.$$

The values of the central function associated with χ are determined by $\mathcal{P}$, ν and rep in the following situation.

Proposition 3.3. *Let* $\chi \in \mathrm{IBr}(G)$ *and* $\mathcal{D}$ *an* $\mathbb{F}$*-representation of* G *affording* χ*. Let* $Z \lhd G$ *with* $p \nmid |Z|$, $\overline{G} := G/Z$*, and* $\mathrm{rep} : \overline{G} \to G$ *a* Z*-section. Assume that* $\mathrm{IBr}(\chi_Z) = \{\nu\}$ *for some faithful linear character* ν*. Let* $\mathcal{P}$ *be the projective representation of* $\overline{G}$ *obtained from* χ *using* rep*. For* $\overline{g} \in \overline{G}$ *let*

$$\mathrm{Stab}_{\overline{G}}^{\mathcal{P}}(\overline{g}) = \left\{y \in \overline{G} \;\middle|\; \mathcal{P}(\overline{g})^{\mathcal{P}(y)} = \mathcal{P}(\overline{g})\right\},$$

and $k_{\mathcal{P}}(\overline{g}) := \frac{|\mathrm{Stab}_{\overline{G}}^{\mathcal{P}}(\overline{g})\mathrm{C}_{\overline{G}}(\overline{g})|}{|\mathrm{C}_{\overline{G}}(\overline{g})|}$.

(a) *For every* $\overline{g} \in \overline{G}$ *there exists a unique element* $\lambda_{\mathcal{P}}(\overline{g}) \in \mathbb{F}$ *with*

$$\lambda_{\mathcal{P}}(\overline{g})\,\mathrm{id}_{\chi(1)} = \sum_{M \in \{\mathcal{P}(\overline{g})^{\mathcal{P}(\overline{y})} \mid \overline{y} \in \overline{G}\}} M.$$

(b) *For every* $z \in Z$ *and* $\overline{g} \in \overline{G}$ *then*

$$\lambda_\chi(\mathfrak{Cl}_G(z\,\mathrm{rep}(\overline{g}))^+) = \nu(z)^* k_{\mathcal{P}}(\overline{g})^* \lambda_{\mathcal{P}}(\overline{g}).$$

Proof. We prove this by a series of intermediate results.

Let $g \in G$. Observe that in $\{\mathcal{D}(g') \mid g' \in \mathfrak{Cl}_G(g)\}$ as a multiset every matrix occurs with multiplicity k, where

$$k = |\{g' \in \mathfrak{Cl}_G(g) \mid \mathcal{D}(g') = \mathcal{D}(g)\}|.$$

By definition this implies $\lambda_\chi(\mathfrak{Cl}_G(g)^+) = k^*\lambda_{\mathcal{D}}(g)$, where $\lambda_{\mathcal{D}}(g) \in \mathbb{F}$ is defined by

$$\lambda_{\mathcal{D}}(g)\,\mathrm{id}_{\chi(1)} = \sum_{M \in \{\mathcal{D}(g)^{\mathcal{D}(y)} \mid y \in G\}} M.$$

Let $\overline{g} \in \overline{G}$ and $z \in Z$ such that $g = z\,\mathrm{rep}(\overline{g})$. As ν is a linear G-invariant character,

$$\sum_{M \in \{\mathcal{D}(\mathrm{rep}(\overline{g}))^{\mathcal{D}(y)} \mid y \in G\}} M = \sum_{M \in \{\mathcal{P}(\overline{g})^{\mathcal{P}(y)} \mid y \in \overline{G}\}} M.$$

Hence $\lambda_{\mathcal{P}}(\overline{g})$ is well-defined as well and $\lambda_{\mathcal{D}}(\mathrm{rep}(\overline{g})) = \lambda_{\mathcal{P}}(\overline{g})$. This proves (a).

As ν is linear and $p \nmid |Z|$, we have

$$\mathcal{D}(z\,\mathrm{rep}(\overline{g})) = \nu(z)^*\mathcal{P}(\overline{g})$$

and $\lambda_\chi(\mathfrak{Cl}_G(z\,\mathrm{rep}(\overline{g}))^+) = \nu(z)^*\lambda_\chi(\mathfrak{Cl}_G(\mathrm{rep}(\overline{g}))^+)$. We easily see

$$\begin{aligned} k &= |\{g' \in \mathfrak{Cl}_G(g) \mid \mathcal{D}(g') = \mathcal{D}(g)\}| \\ &= \frac{|\{y \in G \mid \mathcal{D}(g)^{\mathcal{D}(y)} = \mathcal{D}(g)\}|}{|\mathrm{C}_G(g)|} = \frac{|\mathrm{Stab}_{\overline{G}}^{\mathcal{P}}(\overline{g})||Z|}{|\mathrm{C}_G(g)|}. \end{aligned}$$

This proves $\lambda_\chi(\mathfrak{Cl}_G(g)^+) = k^*\nu(z)^*\lambda_{\mathcal{D}}(\mathrm{rep}(\overline{g}))$ by the definition of λ_χ.

As ν is faithful and G-invariant, we have $Z \leq \mathrm{Z}(G)$ and $Z \leq \mathrm{C}_G(g)$. From $\mathrm{C}_G(g)/Z = \mathrm{Stab}_{\overline{G}}^{\mathcal{P}}(\overline{g}) \cap \mathrm{C}_{\overline{G}}(\overline{g})$ we obtain

$$\{y \in G \mid \mathcal{D}(g)^{\mathcal{D}(y)} = \mathcal{D}(g)\}/\mathrm{C}_G(g) \cong \left(\mathrm{Stab}_{\overline{G}}^{\mathcal{P}}(\overline{g})\mathrm{C}_{\overline{G}}(\overline{g})\right)/\mathrm{C}_{\overline{G}}(\overline{g}).$$

By the definition of $k_{\mathcal{P}}(\overline{g})$ this proves

$$\lambda_\chi(\mathfrak{Cl}_G(g)^+) = k_{\mathcal{P}}(\overline{g})^*\nu(z)^*\lambda_{\mathcal{D}}(\mathrm{rep}(\overline{g})). \qquad \square$$

The following small technical remark will be useful.

Lemma 3.4. *Let Q be a p-subgroup of a finite group G, $b \in \mathrm{Bl}(\mathrm{N}_G(Q))$, and $B' \in \mathrm{Bl}(G)$ such that some defect group of B' contains Q. Then the following statements are equivalent:*

(i) $b^G = B'$,

(ii) $\lambda_{b^G}(\mathfrak{Cl}_G(g)^+) = \lambda_{B'}(\mathfrak{Cl}_G(g)^+)$ *for every* $g \in \mathrm{C}_G(Q)$.

Proof. By [25, Theorem (4.14)] the block b^G is defined. It is clear that the equation in (ii) holds whenever $b^G = B'$.

The blocks b^G and B' have defect groups containing Q. By assumption we have
$$\lambda_{b^G}(\mathfrak{Cl}_G(g)^+) = \lambda_{B'}(\mathfrak{Cl}_G(g)^+) \quad \text{for every } g \in \mathrm{C}_G(Q).$$
There exists some $g \in \mathrm{C}_G(Q)$ with $\lambda_{b^G}(\mathfrak{Cl}_G(g)^+) \neq 0$ and $D \in \mathrm{Syl}_p(\mathrm{C}_G(g))$, where D is a defect group of B' with $Q \leq D$. By [25, Theorem (4.4)] this implies that D contains a defect group of b^G. Analogous considerations imply that D is a defect group of B. Hence by [25, Exercise (4.5)] we have $b^G = B'$. □

The following lemma is an analogue of [24, Lemma 5.3.1] for Brauer characters.

Lemma 3.5. *Let $H \leq G$ and $\phi \in \mathrm{IBr}(H)$ such that $\phi^G \in \mathrm{IBr}(G)$. Then $\mathrm{bl}(\phi)^G$ is defined and $\mathrm{bl}(\phi^G) = \mathrm{bl}(\phi)^G$.*

Proof. Let $\mathcal{D}$ be an $\mathbb{F}$-representation affording ϕ. We construct a representation $\widetilde{\mathcal{D}}$ of G affording ϕ^G as in [7, (10.1)]. Let $1, g_2, \dots, g_s \in G$ be representatives of the H-cosets in G. Let
$$\widehat{\mathcal{D}}_{i,j}(g) := \begin{cases} \mathcal{D}(g_i^{-1} g g_j) & \text{if } (g_i^{-1} g g_j) \in H, \\ 0 & \text{if } (g_i^{-1} g g_j) \notin H. \end{cases}$$
Then $\widetilde{\mathcal{D}} : G \to \mathrm{GL}_{\phi(1)}(\mathbb{F})$ is given by
$$\widetilde{\mathcal{D}}(g) = \begin{pmatrix} \widehat{\mathcal{D}}_{1,1}(g) & \dots & \widehat{\mathcal{D}}_{1,s}(g) \\ \vdots & & \vdots \\ \widehat{\mathcal{D}}_{s,1}(g) & \dots & \widehat{\mathcal{D}}_{s,s}(g) \end{pmatrix}.$$
Accordingly the value of the scalar associated with $\widetilde{\mathcal{D}}(\mathfrak{Cl}_G(g)^+)$ coincides with the scalar associated with $\widehat{\mathcal{D}}_{1,1}(\mathfrak{Cl}_G(x)^+) = \mathcal{D}((\mathfrak{Cl}_G(x) \cap H)^+)$. This implies
$$\lambda_{\phi^G}(\mathfrak{Cl}_G(g)^+) = \lambda_\phi((\mathfrak{Cl}_G(g) \cap H)^+) \text{ for every } g \in G. \qquad \square$$

The next statement is a key result in the reduction.

Proposition 3.6. *Let $N \lhd G$ and $H \leq G$ such that $NH = G$. Let $\widetilde{\theta} \in \mathrm{IBr}(G)$ with $\theta := \widetilde{\theta}_N \in \mathrm{IBr}(N)$, and $\widetilde{\theta}' \in \mathrm{IBr}(H)$ with $\theta' := \widetilde{\theta}'_M \in \mathrm{IBr}(M)$ for $M := N \cap H$. Assume that the characters satisfy*
$$\mathrm{bl}(\widetilde{\theta}_J) = \mathrm{bl}(\widetilde{\theta}'_{J\cap H})^J \quad \textit{for every } N \leq J \leq G.$$
Then, for every $\overline{\eta} \in \mathrm{IBr}(G/N)$ and its lift $\eta \in \mathrm{IBr}(G)$ the block $\mathrm{bl}(\widetilde{\theta}'\eta_H)^G$ is defined and
$$\mathrm{bl}(\widetilde{\theta}\eta) = \mathrm{bl}(\widetilde{\theta}'\eta_H)^G.$$

Proof. Let $\mathcal{Q}$, $\mathcal{D}$ and $\mathcal{D}'$ be $\mathbb{F}$-representations affording η, $\widetilde{\theta}$ and $\widetilde{\theta}'$, respectively. Let $\overline{G} := G/N$ and $\epsilon : G \to \overline{G}$ be the canonical epimorphism.

For the proof it is sufficient to show

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = \lambda_{\widetilde{\theta}'\eta_H}((\mathfrak{Cl}_G(g) \cap H)^+) \quad \text{for every } g \in G, \tag{3.1}$$

as this proves that the block $\mathrm{bl}(\widetilde{\theta}'\eta_H)^G$ is defined and $\mathrm{bl}(\widetilde{\theta}\eta) = \mathrm{bl}(\widetilde{\theta}'\eta_H)^G$.

By Lemma 2.5 we have

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = \lambda_{\widetilde{\theta}_{L_{\epsilon(g)}}}(\mathfrak{Cl}_{L_{\epsilon(g)}}(g)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\epsilon(g))) \quad \text{for every } g \in G,$$

where $L_y := \epsilon^{-1}(\mathrm{C}_{G/N}(y))$ for every $y \in \overline{G}$.

First we verify equation (3.1) for elements $g \in G$ with $\mathfrak{Cl}_G(g) \cap H = \emptyset$. For such an element g we have $\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = 0$, as $\mathrm{bl}(\widetilde{\theta}_{L_{\epsilon(g)}}) = \mathrm{bl}(\widetilde{\theta}'_{L_{\epsilon(g)} \cap H})^{L_{\epsilon(g)}}$ implies

$$\lambda_{\widetilde{\theta}_{L_{\epsilon(g)}}}(\mathfrak{Cl}_{L_{\epsilon(g)}}(g)^+) = \lambda^G_{\widetilde{\theta}'_{L_{\epsilon(g)}}}(\mathfrak{Cl}_{L_{\epsilon(g)}}(g)^+) = \lambda_{\widetilde{\theta}'_{L_{\epsilon(g)} \cap H}}(0) = 0.$$

For the remaining case let $g \in H$ and $\overline{g} = \epsilon(g)$. Note that we can choose an H-transversal $\mathbb{T}$ in $\mathfrak{Cl}_G(g) \cap H$ with

$$\epsilon(t) = \epsilon(g) \quad \text{for every } t \in \mathbb{T}.$$

Let $\epsilon' : H \to H/M$ and $L'_y := \epsilon'^{-1}(\mathrm{C}_{H/M}(y))$ for every $y \in H/M$. Then Lemma 2.5 (a) implies

$$\lambda_{\widetilde{\theta}'\eta_H}(\mathfrak{Cl}_H(t)^+) = \lambda_{\widetilde{\theta}'_{L'_{\epsilon'(t)}}}(\mathfrak{Cl}_{L'_{\epsilon'(t)}}(t)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\epsilon'(t))^+) \quad \text{for every } t \in \mathbb{T}$$

and because $\epsilon'(\mathbb{T}) = \epsilon'(g)$ this shows

$$\lambda_{\widetilde{\theta}'\eta_H}(\mathfrak{Cl}_H(t)^+) = \lambda_{\widetilde{\theta}'_{L'_{\epsilon'(g)}}}(\mathfrak{Cl}_{L'_{\epsilon'(g)}}(t)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\epsilon(g))^+).$$

We can compute $\lambda_{\widetilde{\theta}'\eta_H}((\mathfrak{Cl}_G(g) \cap H)^+)$:

$$\begin{aligned}
\lambda_{\widetilde{\theta}'\eta_H}((\mathfrak{Cl}_G(g) \cap H)^+) &= \sum_{t \in \mathbb{T}} \lambda_{\widetilde{\theta}'_{L'_{\epsilon'(t)}}}(\mathfrak{Cl}_{L'_{\epsilon'(t)}}(t)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\epsilon(t))^+) \\
&= \sum_{t \in \mathbb{T}} \lambda_{\widetilde{\theta}'_{L'_{\overline{g}}}}(\mathfrak{Cl}_{L'_{\overline{g}}}(t)^+)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+) \\
&= \left(\sum_{t \in \mathbb{T}} \lambda_{\widetilde{\theta}'_{L'_{\overline{g}}}}(\mathfrak{Cl}_{L'_{\overline{g}}}(t)^+)\right)\lambda_{\overline{\eta}}(\mathfrak{Cl}_{\overline{G}}(\overline{g})^+).
\end{aligned}$$

Because $L_{\overline{g}} \cap H = L'_{\overline{g}}$, the assumption $\mathrm{bl}(\widetilde{\theta}'_{L'_{\overline{g}}})^{L_{\overline{g}}} = \mathrm{bl}(\widetilde{\theta}_{L_{\overline{g}}})$ implies

$$\lambda_{\widetilde{\theta}_{L_{\overline{g}}}}(\mathfrak{Cl}_{L_{\overline{g}}}(g)^+) = \lambda_{\widetilde{\theta}'_{L'_{\overline{g}}}}((\mathfrak{Cl}_{L_{\overline{g}}}(g) \cap L'_{\overline{g}})^+).$$

Observe that $\mathbb{T}$ is also an $L'_{\overline{g}}$-transversal in $\mathfrak{Cl}_{L_{\overline{g}}}(g) \cap L'_{\overline{g}}$, and hence

$$\lambda_{\widetilde{\theta}'_{L'_{\overline{g}}}}((\mathfrak{Cl}_{L_{\overline{g}}}(g) \cap L'_{\overline{g}})^+) = \sum_{t \in \mathbb{T}} \lambda_{\widetilde{\theta}'_{L'_{\overline{g}}}}(\mathfrak{Cl}_{L'_{\overline{g}}}(t)^+).$$

This shows

$$\lambda_{\widetilde{\theta}\eta}(\mathfrak{Cl}_G(g)^+) = \lambda_{\widetilde{\theta}'\eta_H}((\mathfrak{Cl}_G(g) \cap H)^+) \quad \text{for every } g \in H,$$

and $\mathrm{bl}(\widetilde{\theta}'\eta_H)^G = \mathrm{bl}(\widetilde{\theta}\eta)$. □

The following statement helps determine the blocks and induced blocks of the direct product of two groups.

Proposition 3.7. *For $i = 1, 2$ let N_i be a finite group and $Q_i \in \mathrm{Rad}(N_i)$. Suppose $N_i \lhd G_i$ with $G_i = N_i \mathrm{N}_{N_i}(Q_i)$. Let $\widetilde{\theta}_i \in \mathrm{IBr}(G_i)$ with $\theta_i := (\widetilde{\theta}_i)_{N_i} \in \mathrm{IBr}(N_i)$ and $\widetilde{\theta}'_i \in \mathrm{IBr}(\mathrm{N}_{G_i}(Q_i))$ with $\theta'_i := (\widetilde{\theta}'_i)_{\mathrm{N}_{N_i}(Q_i)} \in \mathrm{IBr}(\mathrm{N}_{N_i}(Q_i)$. Assume*

$$\mathrm{bl}(\widetilde{\theta}_{J_i}) = \mathrm{bl}(\widetilde{\theta}'_{\mathrm{N}_{J_i}(Q_i)})^{J_i} \quad \textit{for every } J_i \textit{ with } N_i \le J_i \le G_i.$$

Let $N := N_1 \times N_2$ and $G := G_1 \times G_2$. Then

$$\mathrm{bl}((\widetilde{\theta}_1 \times \widetilde{\theta}_2)_J) = \mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}'_2)_{\mathrm{N}_J(Q)})^J \quad \textit{for every } J \textit{ with } N \le J \le G.$$

Proof. In order to prove the statement for J with $N \le J \le G$ we show that

$$\phi := (\widetilde{\theta}'_1 \times \widetilde{\theta}_2)_{\mathrm{N}_J(Q_1)} \in \mathrm{IBr}(\mathrm{N}_J(Q_1))$$

satisfies

$$\mathrm{bl}((\widetilde{\theta}_1 \times \widetilde{\theta}_2)_J) = \mathrm{bl}(\phi)^J \quad \text{and} \quad \mathrm{bl}(\phi) = \mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}'_2)_{\mathrm{N}_J(Q)})^{\mathrm{N}_J(Q_1)}.$$

Let $\widehat{\theta}_1 := \widetilde{\theta}_1 \times 1_{G_2} \in \mathrm{IBr}(G)$. This character is an extension of $\theta_1 \in \mathrm{IBr}(N_1)$, and analogously the character $\widehat{\theta}'_1 := \widetilde{\theta}'_1 \times 1_{G_2} \in \mathrm{IBr}(\mathrm{N}_G(Q_1))$ is an extension of $\theta'_1 \in \mathrm{IBr}(\mathrm{N}_{N_1}(Q_1))$, since $\mathrm{N}_G(Q_1) = \mathrm{N}_{G_1}(Q_1) \times G_2$. Let $\mathrm{pr}_1 : G \to G_1$ be the canonical epimorphism. By assumption

$$\mathrm{bl}((\widetilde{\theta}_1)_{J_1}) = \mathrm{bl}((\widetilde{\theta}'_1)_{\mathrm{N}_{J_1}(Q)})^{J_1} \quad \text{for every } J_1 \text{ with } N_1 \le J_1 \le G_1.$$

Since $\widehat{\theta}_1 = \widetilde{\theta}_1 \circ \mathrm{pr}_1$ and $\widehat{\theta}'_1 = \widetilde{\theta}'_1 \circ (\mathrm{pr}_1)_{\mathrm{N}_G(Q_1)}$, this implies, by Corollary 2.11, the equality

$$\mathrm{bl}((\widehat{\theta}_1)_{\widehat{J}_1}) = \mathrm{bl}((\widehat{\theta}'_1)_{\mathrm{N}_{\widehat{J}_1}(Q_1)})^{\widehat{J}_1} \quad \text{for every } \widehat{J}_1 \text{ with } N_1 \le \widehat{J}_1 \le J.$$

Let $\widehat{\theta}_2 := 1_{G_1} \times \widetilde{\theta}_2 \in \mathrm{IBr}(G_1 \times G_2)$. Then $N_1 \leq \ker((\widehat{\theta}_2)_J)$. Hence we can apply Proposition 3.6 and

$$\mathrm{bl}((\widehat{\theta}_1)_J (\widehat{\theta}_2)_J) = \mathrm{bl}((\widehat{\theta}'_1)_{\mathrm{N}_J(Q_1)} (\widehat{\theta}_2)_{\mathrm{N}_J(Q_1)})^J.$$

This is equivalent to

$$\mathrm{bl}((\widetilde{\theta}_1 \times \widetilde{\theta}_2)_J) = \mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}_2)_{\mathrm{N}_J(Q_1)})^J. \tag{3.2}$$

Let $\widehat{\theta}'_2 := 1_{G_1} \times \widetilde{\theta}'_2 \in \mathrm{IBr}(G_1 \times \mathrm{N}_{G_2}(Q_2))$. As above, by Corollary 2.11, the assumption on the characters $\widetilde{\theta}_2$ and $\widetilde{\theta}'_2$ implies

$$\mathrm{bl}((\widehat{\theta}_2)_{\widehat{J}_2}) = \mathrm{bl}((\widehat{\theta}'_2)_{\mathrm{N}_{\widehat{J}_2}(Q_2)})^{\widehat{J}_2} \quad \text{for every } \widehat{J}_2 \text{ with } N_2 \leq \widehat{J}_2 \leq \mathrm{N}_J(Q_1).$$

We can apply Proposition 3.6 and obtain

$$\mathrm{bl}((\widehat{\theta}'_1)_{\mathrm{N}_J(Q_1)} (\widehat{\theta}_2)_{\mathrm{N}_J(Q_1)}) = \mathrm{bl}((\widehat{\theta}'_1)_{\mathrm{N}_J(Q)} (\widehat{\theta}'_2)_{\mathrm{N}_J(Q)})^{\mathrm{N}_J(Q_1)},$$

and equivalently

$$\mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}_2)_{\mathrm{N}_J(Q_1)}) = \mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}'_2)_{\mathrm{N}_J(Q)})^{\mathrm{N}_J(Q_1)}.$$

Together with (3.2) this shows

$$\begin{aligned} \mathrm{bl}((\widetilde{\theta}_1 \times \widetilde{\theta}_2)_J) &= \mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}_2)_{\mathrm{N}_J(Q_1)})^J \\ &= (\mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}'_2)_{\mathrm{N}_J(Q)})^{\mathrm{N}_J(Q_1)})^J \\ &= \mathrm{bl}((\widetilde{\theta}'_1 \times \widetilde{\theta}'_2)_{\mathrm{N}_J(Q)})^J. \end{aligned}$$

□

Another ingredient is a character correspondence generalizing the Glauberman correspondence, which is sometimes called DGN-correspondence. The construction of this bijection can be found in [24, Theorem 5.12.1]. According to [27, Theorem 4.2], based on [28] there exists a bijection between certain characters lying over DGN-correspondents.

Suppose that $N \lhd G$, and that Q/N is a normal p-subgroup of G/N. Let $\theta \in \mathrm{dz}_G(N)$. Let D be a defect group of the unique block B of Q covering $\mathrm{bl}(\theta)$. Then $Q = ND$, $G = N\mathrm{N}_G(D)$ and $D \cap N = 1$. Then the Brauer first main correspondent $b \in \mathrm{Bl}(\mathrm{N}_G(D))$ of B covers a unique block c of $\mathrm{C}_N(D)$. This block c has defect zero and its ordinary character is called the *Dade–Glauberman–Nagao correspondent* of θ with respect to the group D. We denote this character by $\mathrm{DGN}_D(\theta) \in \mathrm{dz}(\mathrm{C}_N(D))$.

The following statement is a "blockwise" version of [27, Theorem 4.2]. This is obtained using a Morita equivalence from [20]. The result can also be verified by using the work [19] about blocks above nilpotent blocks.

Theorem 3.8. *Suppose that $N \lhd G$ and, in addition, that Q/N is a normal p-subgroup of G/N. Let $\theta \in \mathrm{dz}_G(N)$. Let D be a defect group of the unique block of Q covering* $\mathrm{bl}(\theta)$. *Let $\overline{\mathrm{DGN}}_D(\theta) \in \mathrm{Irr}(\mathrm{N}_Q(D)/D)$ be the character associated with $\mathrm{DGN}_D(\theta)$. Then there exists a G-invariant extension $\widetilde{\theta} \in \mathrm{Irr}(Q)$ of θ and a bijection*

$$\Delta_\theta : \mathrm{rdz}(G \mid \widetilde{\theta}) \longrightarrow \mathrm{dz}(\mathrm{N}_G(D)/D \mid \overline{\mathrm{DGN}}_D(\theta)),$$

such that for all $\psi \in \mathrm{rdz}(G \mid \widetilde{\theta})$ the blocks satisfy

$$\mathrm{bl}(\psi) = \mathrm{bl}(\psi'),$$

where $\psi' \in \mathrm{Irr}(\mathrm{N}_G(D))$ is the lift of $\Delta_\theta(\psi)$.

Proof. Our situation satisfies the assumptions made in [20, Section 11]. By [20, Corollary 11.3] there exists a bijection

$$\Delta_0 : \mathrm{Irr}(G \mid \theta) \longrightarrow \mathrm{Irr}(\mathrm{N}_G(D) \mid \mathrm{DGN}_D(\theta)),$$

such that $\mathrm{bl}(\psi) = \mathrm{bl}(\Delta_0(\psi))^G$. The character $\mathrm{DGN}_D(\theta)$ extends to the character $\mathrm{DGN}_D(\theta) \times 1_D \in \mathrm{Irr}(\mathrm{C}_N(D) \times D)$. The characters of $\mathrm{Irr}(\mathrm{N}_G(D) \mid \mathrm{DGN}_D(\theta) \times 1_D)$ can be identified with characters in $\mathrm{Irr}(\mathrm{N}_G(D)/D \mid \overline{\mathrm{DGN}}_D(\theta))$. Furthermore, by [20, Theorem 4.3] there exists some $\widetilde{\theta} \in \mathrm{Irr}(ND \mid \theta)$ such that

$$\Delta_0^{-1}(\mathrm{Irr}(\mathrm{N}_G(D) \mid \mathrm{DGN}_D(\theta) \times 1_D)) = \mathrm{Irr}(G \mid \widetilde{\theta}).$$

By comparing the degrees we see that $\widetilde{\theta}$ is an extension of θ to ND. By further properties of character correspondences induced by magic presentations, this character is G-invariant. We obtain the required bijection by restricting the bijection Δ_0 to $\mathrm{rdz}(G \mid \widetilde{\theta})$. □

4 The inductive BAW condition

In this section we state the inductive blockwise Alperin weight condition, which we abbreviate to ‘inductive BAW condition’, and some of its consequences. This condition is assumed in Theorem A and can be obtained from the condition on AWC-good groups presented in [27, Section 3] by adding two compatibility requirements. These requirements give rise to information about blocks.

Note that the results of this sections have “relative analogues”, adapted to considering only blocks with specific defect groups. This can be found in the second half of Section 5.

A group X is called the *universal p'-covering group* of a perfect group S if it is the maximal perfect central extension of S by an abelian p'-group.

Definition 4.1 (Inductive BAW condition for S and p). Let p be a prime, S be a non-abelian simple group and X be the universal p'-covering group of S. We say that the *inductive BAW condition holds for S and p* if the following statements are satisfied:

(i) There exist subsets $\mathrm{IBr}(X \mid Q) \subseteq \mathrm{IBr}(X)$ for $Q \in \mathrm{Rad}(X)$ with the following properties:
   (1) $\mathrm{IBr}(X \mid Q)^a = \mathrm{IBr}(X \mid Q^a)$ for every $Q \in \mathrm{Rad}(X)$ and $a \in \mathrm{Aut}(X)$, and
   (2) $\mathrm{IBr}(X) = \dot{\bigcup}_{Q \in \mathrm{Rad}(X)/\sim_X} \mathrm{IBr}(X \mid Q)$.

(ii) For every $Q \in \mathrm{Rad}(X)$ there exists a bijection

$$\Omega_Q^X : \mathrm{IBr}(X \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_X(Q)/Q),$$

with the following properties for every $\nu \in \mathrm{Irr}(\mathrm{Z}(X))$, $\chi \in \mathrm{IBr}(X \mid Q)$ and $a \in \mathrm{Aut}(X)$:
   (1) all characters of $\Omega_Q^X(\mathrm{IBr}(X \mid Q) \cap \mathrm{IBr}(X \mid \nu^0))$ lift to characters in $\mathrm{Irr}(\mathrm{N}_X(Q) \mid \nu)$,
   (2) $\Omega_Q^X(\chi)^a = \Omega_{Q^a}^X(\chi^a)$,
   (3) $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$, where $\chi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ is the lift of $\Omega_Q^X(\chi)$.

(iii) For every $\chi \in \mathrm{IBr}(X \mid Q)$, there exists a group $A(\chi)$ and characters $\Upsilon(\chi) \in \mathrm{IBr}(A(\chi))$ and $\Upsilon(\Omega_Q^X(\chi))$ with the following properties:
   (1) for $Z := \ker(\chi_{\mathrm{Z}(X)})$ and $\overline{X} := X/Z$ the group $A := A(\chi)$ satisfies $\overline{X} \lhd A$, $A/\mathrm{C}_A(\overline{X}) \cong \mathrm{Aut}(X)_\chi$, $\mathrm{C}_A(\overline{X}) = \mathrm{Z}(A)$ and $p \nmid |\mathrm{Z}(A)|$,
   (2) $\Upsilon(\chi) \in \mathrm{IBr}(A)$ is an extension of the character of $\overline{X}$ associated with χ,
   (3) $\Upsilon(\Omega_Q^X(\chi)) \in \mathrm{IBr}(\mathrm{N}_A(\overline{Q}))$ is an extension of $\overline{\chi}'$, where $\overline{Q} := QZ/Z$, $\overline{\chi}' \in \mathrm{IBr}(\mathrm{N}_{\overline{X}}(\overline{Q}))$ is associated with $\Omega_Q^X(\chi)^0 \in \mathrm{IBr}(\mathrm{N}_X(Q)/Q)$,
   (4) $\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$ for every subgroup H satisfying $\mathrm{N}_{\overline{X}}(Q) \leq H \leq \mathrm{N}_A(Q)$.

(iv) The map Ω_1^X satisfies

$$\Omega_1^X(\psi^0) = \psi \quad \text{for every } \psi \in \mathrm{dz}(X)$$

and

$$\Upsilon(\chi) = \Upsilon(\Omega_1^X(\chi)) \quad \text{for every } \chi \in \mathrm{IBr}(X \mid 1).$$

This condition on simple groups has similarities to other conjectured properties of simple groups occurring in [15] and [27].

Remark 4.2. Let S be a non-abelian simple group. If S is AWC-good in the sense of [27, Section 3], then the inductive BAW condition is satisfied for S and p if in addition the bijections $\{\Omega_Q^X\}_{Q\in \mathrm{Rad}(X)}$ satisfy Definition 4.1 (ii) (3) and the property described in Definition 4.1 (iii) (4) holds for every $\chi \in \mathrm{IBr}(X)$. Note that in the situation of Definition 4.1 (iii) the characters $\Upsilon(\chi)$ and $\Upsilon(\Omega_Q^X(\chi))$ satisfy

$$\mathrm{IBr}\big(\Upsilon(\chi)_{\mathrm{Z}(A)}\big) = \mathrm{IBr}\big(\Upsilon(\Omega_Q^X(\chi))_{\mathrm{Z}(A)}\big) \tag{4.1}$$

because of Definition 4.1 (iii) (4). Hence considerations as in [33, Section 2] show that the condition in Definition 4.1 (iii) is a refinement of the conditions (3.a)–(3.d) from [27].

Obviously, the inductive BAW condition for S and p implies that S is AWC-good for p in the sense of [27].

The condition in Definition 4.1 (iii) can be reformulated in various ways. It can be compared to the (cohom)-condition in the inductive McKay conditions of [15, Section 10 (5)–(8)]. Its reformulations have been studied in [33, Section 2]. Furthermore, [34, condition 7.2 (iii)] is a similar condition on irreducible ordinary characters and was used in the context of the Alperin–McKay conjecture.

The added condition in Definition 4.1 (ii) (3) has already been suggested in [27, Remark 3.1] and also been used in [32]. The condition in Definition 4.1 (iii) (4) seems to be new in this context but is essential in the proofs.

In a first step we state the implications of the inductive BAW condition for the universal p'-covering group of a direct product $S^r = S\times\cdots\times S$.

Proposition 4.3 (Consequences for S^r). *Let S be a non-abelian simple group, for which the inductive BAW condition holds. Let r be a positive integer. Then X^r is the universal p'-covering group of $S^r = S\times\cdots\times S$ and the following holds:*

(a) *There exist subsets* $\mathrm{IBr}(X^r \mid Q) \subseteq \mathrm{IBr}(X^r)$ *for* $Q \in \mathrm{Rad}(X^r)$ *with the following properties:*
   (i) $\mathrm{IBr}(X^r \mid Q)^a = \mathrm{IBr}(X^r \mid Q^a)$ *for every* $Q \in \mathrm{Rad}(X^r)$, $a \in \mathrm{Aut}(X^r)$,
   (ii) $\mathrm{IBr}(X^r) = \dot{\bigcup}_{Q\in \mathrm{Rad}(X^r)/\sim_{X^r}} \mathrm{IBr}(X^r \mid Q)$.

(b) *For every* $Q \in \mathrm{Rad}(X^r)$ *there exists a bijection*

$$\Omega_Q^{X^r} : \mathrm{IBr}(X^r \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_{X^r}(Q)/Q),$$

with the following properties for every $\nu \in \mathrm{Irr}(\mathrm{Z}(X^r))$, $\chi \in \mathrm{IBr}(X^r \mid Q)$ *and* $a \in \mathrm{Aut}(X^r)$*:*
   (i) *all characters of* $\Omega_Q^{X^r}(\mathrm{IBr}(X^r \mid Q)\cap \mathrm{IBr}(X^r \mid \nu^0))$ *lift to characters in* $\mathrm{Irr}(\mathrm{N}_{X^r}(Q) \mid \nu)$,
   (ii) $\Omega_{Q^a}^{X^r}(\chi^a) = \Omega_Q^{X^r}(\chi)^a$,
   (iii) $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^{X^r}$, *where* $\chi' \in \mathrm{Irr}(\mathrm{N}_{X^r}(Q))$ *is the lift of* $\Omega_Q^{X^r}(\chi)$.

(c) *For every* $\chi \in \mathrm{IBr}(X^r \mid Q)$ *there exists a group* $A(\chi)$ *and characters* $\Upsilon(\chi) \in \mathrm{IBr}(A(\chi))$ *and* $\Upsilon(\Omega_Q^{X^r}(\chi))$ *such that:*

  (i) *For* $Z := \ker(\chi_{\mathrm{Z}(X^r)})$ *and* $\overline{X^r} := X^r/Z$ *the group* $A := A(\chi)$ *satisfies* $\overline{X^r} \lhd A$, $A/\mathrm{C}_A(\overline{X^r}) \cong \mathrm{Aut}(X^r)_\chi$, $\mathrm{C}_A(\overline{X^r}) = \mathrm{Z}(A)$ *and* $p \nmid |\mathrm{Z}(A)|$.

  (ii) $\Upsilon(\chi) \in \mathrm{IBr}(A)$ *is an extension of the character of* $\overline{X^r}$ *associated with* χ.

  (iii) $\Upsilon(\Omega_Q^{X^r}(\chi)) \in \mathrm{IBr}(\mathrm{N}_A(\overline{Q}))$ *is an extension of* $\overline{\chi}'$, *where* $\overline{Q} := QZ/Z$, $\overline{\chi}' \in \mathrm{IBr}(\mathrm{N}_{\overline{X^r}}(\overline{Q}))$ *is associated with* $\Omega_Q^{X^r}(\chi)^0 \in \mathrm{IBr}(\mathrm{N}_{X^r}(Q)/Q)$.

  (iv) $\mathrm{bl}(\Upsilon(\chi)_{\overline{X^r}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^{X^r}(\chi))_H)^{\overline{X^r}H}$ *for every subgroup* H *satisfying* $\mathrm{N}_{\overline{X^r}}(\overline{Q}) \leq H \leq \mathrm{N}_A(\overline{Q})$.

(d) *The map* $\Omega_1^{X^r}$ *satisfies*

$$\Omega_1^{X^r}(\psi^0) = \psi \quad \text{for every } \psi \in \mathrm{dz}(X^r)$$

and

$$\Upsilon(\chi) = \Upsilon(\Omega_1^{X^r}(\chi)) \quad \text{for every } \chi \in \mathrm{IBr}(X^r \mid 1).$$

We start by verifying the first two parts of the proposition.

Proof of Proposition 4.3 (a)–(b) (ii). Every character $\chi \in \mathrm{IBr}(X^r)$ can be written as $\chi_1 \times \cdots \times \chi_r$ with $\chi_i \in \mathrm{IBr}(X)$.

Let $Q \in \mathrm{Rad}(X^r)$. According to [27, Lemma 2.3 (b)] there exist radical subgroups $Q_i \in \mathrm{Rad}(X)$ such that $Q = Q_1 \times \cdots \times Q_r$. We set

$$\mathrm{IBr}(X^r \mid Q) := \mathrm{IBr}(X \mid Q_1) \times \cdots \times \mathrm{IBr}(X \mid Q_r).$$

It is clear that the sets $\mathrm{IBr}(X^r \mid Q)$ satisfy the requirements from (a) as the analogous properties hold for the sets $\mathrm{IBr}(X \mid Q_i)$ according to Definition 4.1 (i).

We observe that $\mathrm{N}_{X^r}(Q) = \mathrm{N}_X(Q_1) \times \cdots \times \mathrm{N}_X(Q_r)$ and define the bijection $\Omega_Q^{X^r} : \mathrm{IBr}(X^r \mid Q) \to \mathrm{dz}(\mathrm{N}_{X^r}(Q)/Q)$ by

$$\Omega_Q^{X^r}(\chi) := \Omega_{Q_1}^X(\chi_1) \times \cdots \times \Omega_{Q_r}^X(\chi_r),$$

for every $\chi = \chi_1 \times \cdots \times \chi_r \in \mathrm{IBr}(X^r \mid Q)$ with $\chi_i \in \mathrm{IBr}(X \mid Q_i)$ for $1 \leq i \leq r$. The properties of $\Omega_{Q_i}^X$ from Definition 4.1 (ii) imply the statements in (b) (i) and (b) (ii) about $\Omega_Q^{X^r}$. □

Before proving the remaining parts of Proposition 4.3, we give a reformulation of Proposition 4.3 (c) in terms of projective representations.

Note that $\mathrm{Aut}(X) = \mathrm{Aut}(S)$ and hence, by abuse of notation, we shall write $\mathrm{N}_{\mathrm{Aut}(S)}(Q)$ for the stabilizer of $Q \leq X$ in $\mathrm{Aut}(S)$. For a finite group G and elements $x, y \in G$ we denote by $\mathrm{T}_G(x, y)$ the set $\{g \in G \mid x^g = y\}$.

Proposition 4.4. *Assume the situation of Proposition* 4.3 *and assume that the sets* $\mathrm{IBr}(X^r \mid Q)$ *(*$Q \in \mathrm{Rad}(X^r)$*) and bijections* $\Omega_Q^{X^r}$ *as in Proposition* 4.3 (a)–(b) (ii) *exist. Let* $\chi \in \mathrm{IBr}(X^r \mid Q)$ *and* $\overline{A} := \mathrm{Aut}(S^r)_\chi$*. Then the following statements are equivalent:*

(i) *The statement in Proposition* 4.3 (c) *holds for* χ.

(ii) *There exists a* $\mathrm{Z}(X^r)$*-section* $\mathrm{rep} : S^r \to X^r$*, a projective* $\mathbb{F}$*-representation* $\mathcal{P}$ *of* $\overline{A}$ *and a projective* $\mathbb{F}$*-representation* $\mathcal{P}'$ *of* $\mathrm{N}_{\overline{A}}(Q)$ *with the following properties:*

(1) $\mathcal{P}_{S^r}$ *is obtained from* χ *using* rep.

(2) ${\mathcal{P}'}_{\mathrm{N}_{S^r}(Q)}$ *is obtained from the lift* $\chi' \in \mathrm{IBr}(\mathrm{N}_{X^r}(Q))$ *of* $\Omega_Q^{X^r}(\chi)^0$ *using* $\mathrm{rep}_{\mathrm{N}_{S^r}(Q)}$.

(3) *The factor sets* α *and* α' *of* $\mathcal{P}$ *and* $\mathcal{P}'$ *satisfy*

$$\alpha' = \alpha_{\mathrm{N}_{\overline{A}}(Q)\times \mathrm{N}_{\overline{A}}(Q)}.$$

(4) *Let* $Z := \langle \alpha(a, a') \mid a, a' \in \mathrm{Aut}(S^r)_\chi\rangle \leq \mathbb{F}^*$*, and let* $\overline{J}$ *be a group with* $S^r \leq \overline{J} \leq A$*, and* $\overline{H} := \mathrm{N}_{\overline{J}}(Q)$*. For every* $\overline{a} \in \overline{J}$ *and* $\overline{h}, \overline{h}' \in \overline{H}$ *let*

$$\mathrm{Z}_{\overline{a}}^{\overline{J}}(\overline{h}) := \left\{\zeta \in Z \;\middle|\; \zeta \mathcal{P}(\overline{h}) = \mathcal{P}(\overline{a})^{\mathcal{P}(y)} \text{ for some } y \in \mathrm{T}_{\overline{J}}(\overline{a}, \overline{h})\right\},$$

$$\mathrm{Z}_{\overline{h}'}^{\overline{H}}(\overline{h}) := \left\{\zeta \in Z \;\middle|\; \zeta \mathcal{P}'(\overline{h}) = \mathcal{P}'(\overline{h}')^{\mathcal{P}'(y)} \text{ for some } y \in \mathrm{T}_{\overline{H}}(\overline{h}', \overline{h})\right\},$$

and $\mathbb{T}(\overline{a})$ *be a representative set of* $\overline{H}$*-orbits in* $\mathfrak{Cl}_{\overline{J}}(\overline{a}) \cap \overline{H}$*. Then*

$$k_{\mathcal{P}_{\overline{J}}}(\overline{a})^* \lambda_{\mathcal{P}_{\overline{J}}}(\overline{a}) = \sum_{\substack{t\in\mathbb{T}(\overline{a})\\ \zeta\in \mathrm{Z}_{\overline{a}}^{\overline{J}}(t)/\mathrm{Z}_{\overline{h}}^{\overline{H}}(\overline{h})}} \zeta k_{\mathcal{P}'_{\overline{H}}}(t)^* \lambda_{{\mathcal{P}'}_{\overline{H}}}(t) \quad \text{for every } \overline{a} \in \overline{J}$$

with the notation of Proposition 3.3.

Proof. For (i) ⇒ (ii) we may assume that for χ there exists a group $A := A(\chi)$ and characters $\widetilde{\chi} := \Upsilon(\chi)$ and $\widetilde{\chi}' := \Upsilon(\Omega_Q^{X^r}(\chi))$ with properties described in Proposition 4.3 (c). Without loss of generality we can assume that $\widetilde{\chi}_{\mathrm{Z}(A)}$ and $\widetilde{\chi}'_{\mathrm{Z}(A)}$ are faithful.

Recall $\overline{A} = A/\mathrm{Z}(A)$. Hence there exists some $\mathrm{Z}(A)$-section $\mathrm{rep} : \overline{A} \to A$ with $\mathrm{rep}(S^r) \subseteq \overline{X^r}$, where $Z := \mathrm{Z}(\ker(\chi_{\mathrm{Z}(X^r)}))$ and $\overline{X^r} := X^r/Z$. Abusing notation, we denote by $\mathrm{N}_{\mathrm{Aut}(S^r)_\chi}(Q)$ the stabilizer of Q in $\mathrm{Aut}(S^r)_\chi = \mathrm{Aut}(X^r)_\chi$. Let $\mathcal{P}$ be the projective representation of $\overline{A} = \mathrm{Aut}(S)_\chi$ obtained from $\Upsilon(\chi)$ using rep, and $\mathcal{P}'$ be the projective representation of $\mathrm{N}_{\mathrm{Aut}(S^r)_\chi}(Q) = \mathrm{N}_{\overline{A}}(Q)$ obtained from $\widetilde{\chi}' = \Upsilon(\Omega_Q^X(\chi))$ using $\mathrm{rep}_{\mathrm{N}_{\overline{A}}(Q)}$. By this construction it is clear that the projective representations $\mathcal{P}$ and $\mathcal{P}'$ together with the $Z(X^r)$-section rep_{S^r} satisfy the first two requirements.

The fact $\mathrm{bl}(\widetilde{\chi}) = \mathrm{bl}(\widetilde{\chi}')^A$, together with $p \nmid \mathrm{Z}(A)$, implies

$$\mathrm{IBr}(\widetilde{\chi}_{\mathrm{Z}(A)}) = \mathrm{IBr}(\widetilde{\chi}'_{\mathrm{Z}(A)}).$$

Hence by Remark 3.2 the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$\alpha' = \alpha_{\mathrm{N}_{\overline{A}}(Q)\times \mathrm{N}_{\overline{A}}(Q)}.$$

Now it remains to verify the statement in (iv) for any $\overline{J}$ with $S^r \leq \overline{J} \leq \overline{A}$. Let $\epsilon : A \to \overline{A}$ be the canonical epimorphism. Furthermore, consider $J := \epsilon^{-1}(\overline{J})$, $\overline{H} := \mathrm{N}_{\overline{J}}(Q)$, and $H := J \cap \mathrm{N}_A(Q)$. The property $\mathrm{bl}(\widetilde{\chi}_J) = \mathrm{bl}(\widetilde{\chi}'_H)^J$ implies

$$\lambda_{\widetilde{\chi}_J}(\mathfrak{Cl}_J(a)^+) = \lambda_{\widetilde{\chi}'_H}((\mathfrak{Cl}_J(a) \cap H)^+) \quad \text{for every } a \in J.$$

Let $\overline{a} \in \overline{J}$ and $a := \mathrm{rep}(\overline{a})$. Proposition 3.3 implies

$$\lambda_{\widetilde{\chi}_J}(\mathfrak{Cl}_J(a)^+) = k_{\mathcal{P}_{\overline{J}}}(\overline{a})^* \lambda_{\mathcal{P}_{\overline{J}}}(\overline{a}),$$

where $k_{\mathcal{P}_{\overline{J}}}(\overline{a})$ and $\lambda_{\mathcal{P}_{\overline{J}}}(\overline{a})$ are defined as in that proposition.

In our next step we compute $\lambda_{\widetilde{\chi}'_H}((\mathfrak{Cl}_J(a) \cap H)^+)$ using $\mathcal{P}'$: By the definition of $\mathrm{Z}^{\overline{J}}_{\overline{a}}(\overline{g})$ we have

$$\mathcal{D}(\mathfrak{Cl}_J(a)) = \left\{\zeta \mathcal{P}(\overline{g}) \;\middle|\; \overline{g} \in \mathfrak{Cl}_{\overline{J}}(\overline{a}),\, \zeta \in \mathrm{Z}^{\overline{J}}_{\overline{a}}(\overline{g})\right\},$$

even when seen as a multiset. This implies, on the other hand,

$$\mathfrak{Cl}_J(a) = \left\{(\nu^*)^{-1}(\zeta)\,\mathrm{rep}(\overline{g}) \;\middle|\; \overline{g} \in \mathfrak{Cl}_{\overline{J}}(\overline{a}),\, \zeta \in \mathrm{Z}^{\overline{J}}_{\overline{a}}(\overline{g})\right\}.$$

An analogous statement holds for the H-conjugacy classes, for every $\overline{h} \in \overline{H}$, $z \in Z$ and $h := \mathrm{rep}(\overline{h})$ we have

$$\mathfrak{Cl}_H(zh) = \left\{z\,(\nu^*)^{-1}(\zeta)\ \mathrm{rep}(\overline{g}) \;\middle|\; \zeta \in \mathrm{Z}^{\overline{H}}_{\overline{h}}(\overline{g}),\, \overline{g} \in \mathfrak{Cl}_{\overline{H}}(\overline{h})\right\}.$$

Together with the above formulas we obtain

$$\mathfrak{Cl}_J(a) \cap H = \dot{\bigcup_{\substack{t \in \mathbb{T}(\overline{a}) \\ \zeta \in \mathrm{Z}^{\overline{J}}_{\overline{a}}(t)/\,\mathrm{Z}^{\overline{H}}_t(t)}}} \mathfrak{Cl}_{\overline{H}}((\nu^*)^{-1}(\zeta)\,\mathrm{rep}(t)).$$

According to Proposition 3.3 this implies

$$\lambda_{\widetilde{\chi}'_H}((\mathfrak{Cl}_J(a) \cap H)^+) = \sum_{\substack{t \in \mathbb{T}(\overline{a}) \\ \zeta \in \mathrm{Z}^{\overline{J}}_{\overline{a}}(t)/\,\mathrm{Z}^{\overline{H}}_t(t)}} \zeta k_{\mathcal{P}'_{\overline{H}}}(t)^* \lambda_{\mathcal{P}'_{\overline{H}}}(t).$$

Hence $\lambda_{\widetilde{\chi}_J}(\mathfrak{Cl}_J(a)^+) = \lambda_{\widetilde{\chi}'_H}((\mathfrak{Cl}_J(a) \cap H)^+)$ is equivalent to

$$k_{\mathcal{P}_{\overline{J}}}(\overline{a})^* \lambda_{\mathcal{P}_{\overline{J}}}(\overline{a}) = \sum_{\substack{t \in \mathbb{T}(\overline{a}) \\ \zeta \in Z_{\overline{a}}^{\overline{J}}(t)/Z_t^{\overline{H}}(t)}} \zeta k_{\mathcal{P}'_{\overline{H}}}(t)^* \lambda_{\mathcal{P}'_{\overline{H}}}(t).$$

Accordingly the section rep and the projective representations $\mathcal{P}$ and $\mathcal{P}'$ have the properties described in (ii).

For (ii) $\Rightarrow$ (i) we choose A to be the central extension of $\overline{A} = \mathrm{Aut}(S^r)_\chi$ by Z defined by the factor set α, more precisely let $A = \overline{A} \times Z$ as set with

$$(\overline{a}, \zeta)(\overline{a}', \zeta') = (\overline{a}\overline{a}', \zeta\zeta'\alpha(\overline{a}, \overline{a}')) \quad \text{for every } (\overline{a}, \zeta), (\overline{a}', \zeta') \in A.$$

By this construction $\mathcal{P}$ lifts to a representation $\mathcal{D}$ of the group $A(\chi)$, see [24, Lemma 3.5.19 (iii)] or the proof of [25, Theorem (8.28)]. Let $Z := \ker(\chi_{Z(X^r)})$. Since $\mathcal{P}_{S^r}$ is obtained from χ using rep, there is an isomorphism between $\overline{X^r} := X^r/Z$ and $\widetilde{X}_0 := \langle (\overline{a}, 1) \mid \overline{a} \in S^r \rangle \lhd A$ via $\mathrm{rep}(\overline{a}) \mapsto (\overline{a}, 1)$. In the following we identify $\overline{X^r}$ with $\widetilde{X}_0$. The representation $\mathcal{D}$ affords a character $\widetilde{\chi} \in \mathrm{IBr}(A)$ that is an extension of the character of $\overline{X^r}$ associated with χ.

Because $\alpha' = \alpha_{\mathrm{N}_{\overline{A}}(Q) \times \mathrm{N}_{\overline{A}}(Q)}$, the projective representation $\mathcal{P}'$ can be lifted to a representation $\mathcal{D}'$ of $\mathrm{N}_A(Q)$. Let $\chi' \in \mathrm{IBr}(\mathrm{N}_{\overline{X^r}}(Q))$ be the lift of $\Omega_Q^{X^r}(\chi)^0$. Since $\mathcal{P}'_{\mathrm{N}_{S^r}(Q)}$ is obtained from χ' using $\mathrm{rep}_{\mathrm{N}_{S^r}(Q)}$, the character $\widetilde{\chi}'$ afforded by $\mathcal{D}'$ is an extension of χ'.

Using Proposition 3.3 similar considerations as above can be made and show that the characters $\widetilde{\chi}$ and $\widetilde{\chi}'$ satisfy

$$\mathrm{bl}(\widetilde{\chi}_J) = \mathrm{bl}(\widetilde{\chi}'_{\mathrm{N}_J(Q)})^J \quad \text{for every } \overline{X^r} \le J \le A.$$

Hence Proposition 4.3 (c) holds for χ with

$$A(\chi) := A, \quad \Upsilon(\chi) := \widetilde{\chi} \quad \text{and} \quad \Upsilon(\Omega_Q^{X^r}(\chi)) := \widetilde{\chi}'. \qquad \square$$

The above is key to proving the following statement.

Corollary 4.5. *In the situation of Proposition* 4.3 *assume that there exist the sets* $\mathrm{IBr}(X^r \mid Q)$ *(*$Q \in \mathrm{Rad}(X^r)$*) and bijections* $\Omega_Q^{X^r}$ *as in Proposition* 4.3 (a)–(b) (ii). *Let* $\chi \in \mathrm{IBr}(X^r)$ *and* $a \in \mathrm{Aut}(X^r)$. *Then the condition in Proposition* 4.3 (c) *holds for* χ *if and only if it holds for* χ^a.

Proof. There exist a $Z(X^r)$-section $\mathrm{rep} : S^r \to X^r$ and projective representations $\mathcal{P}$ of $\overline{A} := \mathrm{Aut}(S^r)_\chi$ and $\mathcal{P}'$ of $\mathrm{N}_{\overline{A}}(Q) = \mathrm{Aut}(S^r)_{Q,\chi}$ as described in Proposition 4.4 (ii). Let $\chi' \in \mathrm{IBr}(\mathrm{N}_{X^r}(Q))$ be the lift of $\Omega_Q^{X^r}(\chi)^0$.

We can conjugate $\mathrm{rep} : S^r \to X^r$, $\mathcal{P}$ and $\mathcal{P}'$ by a. Let $\mathrm{rep}_1 : S^r \to X^r$ be defined by

$$\mathrm{rep}_1(s) = \mathrm{rep}(s^{a^{-1}})^a \quad \text{for every } s \in S^r$$

and $\mathcal{P}_1$ be the projective representation of $\overline{A}^a$ given by

$$\mathcal{P}_1(s) = \mathcal{P}(s^{a^{-1}}) \quad \text{for every } s \in \overline{A}^a.$$

Then $(\mathcal{P}_1)_{S^r}$ is obtained from χ^a using rep_1, since

$$\mathrm{tr}(\mathcal{P}_1(s)) = \chi(\mathrm{rep}(s^{a^{-1}})) = \chi^a(\mathrm{rep}_1(s)) \quad \text{for every } s \in S^r.$$

Analogously let $\mathcal{P}_1'$ be the projective representation of $\mathrm{N}_{\overline{A}^a}(Q^a)$ given by

$$\mathcal{P}_1'(s) = \mathcal{P}'(s^{a^{-1}}) \quad \text{for every } s \in \mathrm{N}_{\overline{A}^a}(Q^a).$$

Hence $(\mathcal{P}_1')_{\mathrm{N}_{S^r}(Q^a)}$ is obtained from χ'^a using $(\mathrm{rep}_1)_{\mathrm{N}_{S^r}(Q^a)}$. By these definitions it is clear that the factor sets of $\mathcal{P}_1$ and $\mathcal{P}_1'$ satisfy the equation from Proposition 4.4 (ii) (2).

Let $S^r \leq \overline{J} \leq \overline{A}$. Furthermore, by definition $\mathcal{P}_1$ and $\mathcal{P}_1'$ satisfy

$$\lambda_{(\mathcal{P}_1)_{\overline{J}^a}}(s) = \lambda_{\mathcal{P}_{\overline{J}}}(s^{a^{-1}}) \quad \text{for every } s \in \overline{J}^a$$

and

$$\lambda_{(\mathcal{P}_1')_{\mathrm{N}_{\overline{J}^a}(Q^a)}}(s) = \lambda_{(\mathcal{P}')_{\mathrm{N}_{\overline{J}}(Q)}}(s^{a^{-1}}) \quad \text{for every } s \in \mathrm{N}_{\overline{J}^a}(Q^a).$$

Since $k_{\mathcal{P}_1}(s) = k_{\mathcal{P}}(s^{a^{-1}})$ and $k_{\mathcal{P}_1'}(s) = k_{\mathcal{P}'}(s^{a^{-1}})$, we see that the equation from Proposition 4.4 (ii) (4) also holds for $\mathcal{P}_1$ and $\mathcal{P}_1'$. Hence $\mathcal{P}_1$ and $\mathcal{P}_1'$ have the properties described in Proposition 4.4 (ii) for χ^a, and hence χ^a satisfies Proposition 4.3 (c) by Proposition 4.4. □

Now we are able to prove the remaining part of Proposition 4.3.

Proof of Proposition 4.3. Observe that by Proposition 3.7 the constructed bijections $\{\Omega_Q^{X^r}\}$ satisfy Proposition 4.3 (b) (iii). Hence it suffices to verify that Proposition 4.3 (c) holds.

Let $\chi \in \mathrm{IBr}(X^r \mid Q)$. According to Corollary 4.5 we can assume that χ satisfies $\mathrm{Aut}(X^r)_\chi = (\mathrm{Aut}(X)_{\chi_1} \times \cdots \times \mathrm{Aut}(X)_{\chi_r}) \rtimes (\mathfrak{S}_r)_\chi$, as every character is conjugate to one with this property. Analogously we can assume that Q coincides with $Q_1 \times \cdots \times Q_r$ where all groups Q_i and Q_j are either equal or not conjugate in $\mathrm{Aut}(X)$.

For χ we set

$$A := (A(\chi_1) \times \cdots \times A(\chi_r)) \rtimes (\mathfrak{S}_r)_\chi.$$

The character

$$\Upsilon_0(\chi) := \Upsilon(\chi_1) \times \cdots \times \Upsilon(\chi_r) \in \mathrm{IBr}(A(\chi_1) \times \cdots \times A(\chi_r))$$

is an extension of χ. Let V_i be the $A(\chi_i)$-modules affording $\Upsilon(\chi_i)$. The character $\Upsilon_0(\chi)$ can be extended to A, since the natural permutation action of $(\mathfrak{S}_r)_\chi$ on the modules V_i allows us to regard $V_1 \otimes \cdots \otimes V_r$ as A-module. Let us denote the afforded character by $\Upsilon(\chi) \in \mathrm{IBr}(A \mid \chi)$.

Our assumption on Q implies

$$\mathrm{N}_A(Q) = \big(\mathrm{N}_{A(\chi_1)}(Q_1) \times \cdots \times \mathrm{N}_{A(\chi_r)}(Q_r)\big) \rtimes (\mathfrak{S}_r)_\chi.$$

The lift of $\Omega_Q^{X^r}(\chi)^0 = \Omega_{Q_1}^X(\chi_1)^0 \times \cdots \times \Omega_{Q_r}^X(\chi_r)^0$ to $\mathrm{N}_{X^r}(Q)$ has an extension to

$$\mathrm{N}_{A(\chi_1)}(Q_1) \times \cdots \times \mathrm{N}_{A(\chi_r)}(Q_r),$$

namely the character $\Upsilon(\Omega_{Q_1}^X(\chi_1)) \times \cdots \times \Upsilon(\Omega_{Q_r}^X(\chi_r))$. Again, using the natural permutation action of $(\mathfrak{S}_r)_\chi$ on the modules this character extends to $\mathrm{N}_A(Q)$. We denote this character by $\Upsilon(\Omega_Q^{X^r}(\chi)) \in \mathrm{IBr}(\mathrm{N}_A(Q))$.

Let $\overline{X}_i := X/\ker(\chi_{i\,\mathrm{Z}(X)})$ and $\widehat{X} := \overline{X}_1 \times \cdots \times \overline{X}_r$. In the next step we check that for every H with $\mathrm{N}_{\widehat{X}}(Q) \leq H \leq \mathrm{N}_A(Q)$ we have

$$\mathrm{bl}(\Upsilon(\chi)_{\widehat{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^{X^r}(\chi))_H)^{\widehat{X}H}.$$

According to Lemma 3.4 and [25, Theorem (9.5)] it is sufficient to verify this equality in the case where $H \leq \mathrm{C}_A(Q)\mathrm{N}_{\overline{X^r}}(Q)$. We first consider the case where $Q_i \neq 1$ for all $1 \leq i \leq r$. Then

$$H \leq \mathrm{C}_A(Q)\mathrm{N}_{\widehat{X}}(Q) \leq \mathrm{N}_{A(\chi_1)}(Q_1) \times \cdots \times \mathrm{N}_{A(\chi_r)}(Q_r).$$

Successively applying Proposition 3.7 proves

$$\mathrm{bl}(\widetilde{\Upsilon}(\chi)_{\widehat{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^{X^r}(\chi))_H)^{\widehat{X}H}.$$

For $Q = 1$ we have $\Upsilon(\chi_i) = \Upsilon(\Omega_1^X(\chi_i))$ by Definition 4.1 (iv) and may choose $\Upsilon(\chi) = \Upsilon(\Omega_1^{X^r}(\chi))$ for every $\chi \in \mathrm{IBr}(X^r \mid 1)$. For all the remaining characters a combination of the results in the two extreme cases implies

$$\mathrm{bl}(\Upsilon(\chi)_{\widehat{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^{X^r}(\chi))_H)^{\widehat{X}H}$$

by Proposition 3.7. Hence the group $A(\chi) := A/\ker(\chi_{\mathrm{Z}(\widehat{X})})$ together with the characters $\Upsilon(\chi)$ and $\Upsilon(\Omega_Q^{X^r}(\chi))$ has the required properties, by Corollary 2.11 together with [24, Theorem 5.8.8]. □

The following statement helps to see how the inductive BAW condition predicts the representation theory of an arbitrary finite group G with a normal subgroup that is a central extension of S^r.

Proposition 4.6. *Let $K \lhd G$ such that $[K,K] = K$, $K \cap \mathrm{Z}(G) = \mathrm{Z}(K)$, $p \nmid |\mathrm{Z}(K)|$ and $K/\mathrm{Z}(K) \cong S^r$ for some non-abelian simple group S satisfying the inductive BAW condition from Definition* 4.1. *Then:*

(a) *There exist subsets* $\mathrm{IBr}(K \mid Q) \subseteq \mathrm{IBr}(K)$ *for* $Q \in \mathrm{Rad}(K)$ *with the following properties:*

  (i) $\mathrm{IBr}(K \mid Q)^a = \mathrm{IBr}(K \mid Q^a)$ *for every* $Q \in \mathrm{Rad}(K)$ *and* $a \in G$,

  (ii) $\mathrm{IBr}(K) = \dot{\bigcup}_{Q \in \mathrm{Rad}(K)/\sim_K} \mathrm{IBr}(K \mid Q)$.

(b) *For every* $Q \in \mathrm{Rad}(K)$ *there exists a bijection*

$$\Omega_Q^K : \mathrm{IBr}(K \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_K(Q)/Q)$$

with the following properties for every $\nu \in \mathrm{Irr}(\mathrm{Z}(K))$, $\chi \in \mathrm{IBr}(K \mid Q)$ *and* $a \in G$:

- *all characters of* $\Omega_Q^K(\mathrm{IBr}(K \mid Q) \cap \mathrm{IBr}(K \mid \nu^0))$ *lift to characters in* $\mathrm{Irr}(\mathrm{N}_K(Q) \mid \nu)$,
- $\Omega_{Q^a}^K(\chi^a) = \Omega_Q^K(\chi)^a$,
- $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^K$, *where* $\chi' \in \mathrm{Irr}(\mathrm{N}_K(Q))$ *is the lift of* $\Omega_Q^K(\chi)$.

(c) *Let* $\chi \in \mathrm{IBr}(K \mid Q)$ *with trivial* $\ker(\chi_{\mathrm{Z}(K)})$ *and* $J := G_\chi$. *Then there exists a finite cyclic* p'*-group* C *and a cocycle* $\beta : J \times J \to C$ *such that the group* $\widehat{J}$ *associated with* β *and* C, *i.e.* $\widehat{J} = J \times C$ *as a set and*

$$(j,\zeta)(j',\zeta') = (jj', \zeta\zeta'\beta(j,j')),$$

has the following properties:

  (i) $\iota : K \to K_0 := \{(k,1) \mid k \in K\}$ *is an isomorphism,*

  (ii) $\chi_0 := \chi \circ \iota^{-1} \in \mathrm{IBr}(K_0)$ *extends to a character* $\widetilde{\chi}_0 \in \mathrm{IBr}(\widehat{J})$,

  (iii) *setting* $Q_0 := \iota(Q)$ *the character* $\chi_0' \in \mathrm{IBr}(\mathrm{N}_{K_0}(Q_0))$ *associated with* $\Omega_Q^K(\chi)^0$ *extends to a character* $\widetilde{\chi}_0' \in \mathrm{IBr}(\mathrm{N}_{\widehat{J}}(Q_0))$,

  (iv) *we have* $\mathrm{bl}((\widetilde{\chi}_0)_{K_0 H}) = \mathrm{bl}((\widetilde{\chi}_0')_H)^{K_0 H}$ *for every subgroup* H *satisfying* $\mathrm{N}_{K_0}(Q) \leq H \leq \mathrm{N}_{\widehat{J}}(Q)$.

Proof. As K is a perfect group and a central extension of S^r by a p'-group, there exists some epimorphism

$$\epsilon : X^r \longrightarrow K,$$

where X^r is the universal p'-covering group of S^r. By properties of universal covering groups every automorphism of K is induced by a unique automorphism of X^r. Hence the sets $\mathrm{IBr}(K \mid Q)$ and the bijections Ω_Q^K can easily be obtained from the ones given in Proposition 4.3.

Hence it remains to prove part (c). Let $\chi \in \mathrm{IBr}(K \mid Q)$ with trivial $\ker(\chi_{\mathrm{Z}(K)})$. In the following we assume without loss of generality $G_\chi = G$. As a first step we construct a projective $\mathbb{F}$-representation $\mathcal{P}$ of G associated with χ, and a projective $\mathbb{F}$-representation $\mathcal{P}'$ of $\mathrm{N}_G(Q)$ associated with χ', the lift of $\Omega_Q^K(\chi)^0$.

Since $\chi_{\mathrm{Z}(K)}$ is faithful, it follows that χ corresponds to a character $\chi_0 \in \mathrm{IBr}(X^r)$ with $\chi_0 = \chi \circ \epsilon$, and

$$K \cong \overline{X^r} := X^r / \ker((\chi_0)_{\mathrm{Z}(X^r)}).$$

Hence there exists a natural embedding $\iota : K \to A$ where $A := A(\chi_0)$ is the group from Proposition 4.3 (c).

We fix an $\mathrm{Inn}(S^r)$-transversal $\mathbb{T}$ in $\mathrm{Aut}(S^r)_{\chi_0}$ with $\mathbb{T} \subseteq \mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0}}(\widehat{Q})$, where $\widehat{Q}$ is the Sylow p-group in $\epsilon^{-1}(Q)$. This is possible since the properties of the subsets $\mathrm{IBr}(K \mid Q_1)$ and the bijections $\Omega_{Q_1}^{X^r}$ ($Q_1 \in \mathrm{Rad}(K)$) from Proposition 4.3 imply

$$\mathrm{Inn}(S^r)\mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0}}(\widehat{Q}) = \mathrm{Aut}(S^r)_{\chi_0}.$$

Let $\mathrm{Aut}(X^r)_{\chi_0,G}$ be the group of automorphisms of $S^r = K/Z$ induced by G. Let

$$\mathrm{rep}_A : \mathrm{Aut}(X^r)_{\chi_0} \to A$$

be a $\mathrm{C}_A(\overline{X^r})$-section, and

$$\mathrm{rep}_G : \mathrm{Aut}(X^r)_{\chi_0,G} \to G$$

be a $\mathrm{C}_G(K)$-section such that

- $\mathrm{rep}_G(\mathrm{Inn}(X^r)) \subseteq K$,
- $\iota \circ \mathrm{rep}_G(s) = \mathrm{rep}_A(s)$ for every $s \in \mathrm{Inn}(X^r)$,
- $\mathrm{rep}_A(st) = \mathrm{rep}_A(s)\,\mathrm{rep}_A(t)$ for every $s \in \mathrm{Inn}(X^r)$ and $t \in \mathbb{T}$,
- $\mathrm{rep}_G(st) = \mathrm{rep}_G(s)\,\mathrm{rep}_G(t)$ for every $s \in \mathrm{Inn}(X^r)$ and $t \in \mathbb{T} \cap \mathrm{Aut}(S^r)_{\chi_0,G}$.

One can easily construct such sections by successively fixing the values of the sections on $\mathrm{Inn}(X^r)$ and $\mathbb{T}$, see also the proof of [34, Lemma 3.6]. Let $\nu : \mathrm{Z}(K) \to \mathbb{F}$ be the morphism associated with $\chi_{\mathrm{Z}(K)}$. Fix further a set $\mathbb{M} \subseteq \mathrm{C}_G(K)$, such its elements represent the $\mathrm{Z}(K)$-cosets in $\mathrm{C}_G(K)$, and a map $\mu : \mathrm{C}_G(K) \to \mathbb{F}$ with

$$\mu(zt) = \nu(z) \quad \text{for every } z \in \mathrm{Z}(K) \text{ and } t \in \mathbb{M}.$$

Let $\mathcal{D}$ be an $\mathbb{F}$-representation of $A = A(\chi_0)$ affording $\widetilde{\chi} = \Upsilon(\chi_0)$ and denote by $\mathcal{P} := \mathcal{D} \circ \mathrm{rep}_A$ a projective $\mathbb{F}$-representation of $\mathrm{Aut}(S^r)_{\chi_0}$ obtained from $\widetilde{\chi}$ using rep_A. We define $\mathcal{Q} : G \to \mathrm{GL}_{\chi(1)}(\mathbb{F})$ via the following formula:

$$\mathcal{Q}(c\,\mathrm{rep}_G(a)) = \mu(c)\mathcal{P}(a) \quad \text{for every } c \in \mathrm{C}_G(K) \text{ and } a \in \mathrm{Aut}(S^r)_{\chi_0,G}.$$

Straight-forward calculations show that $\mathcal{Q}$ is a projective representation because of the construction, see also [34, 3.9].

By construction $\mathcal{Q}_K$ is a representation affording χ. Furthermore, the projective representation $\mathcal{Q}$ satisfies

$$\mathcal{Q}(kg) = \mathcal{Q}(k)\mathcal{Q}(g) \quad \text{and} \quad \mathcal{Q}(gk) = \mathcal{Q}(g)\mathcal{Q}(k) \quad \text{for every } g \in G \text{ and } k \in K.$$

The first equation follows from the properties of rep_G and rep_A. By the construction it is clear that for every $g \in G$ the map

$$\zeta_g : K \longrightarrow \mathbb{F} \quad \text{defined by } \mathcal{Q}(k)^{\mathcal{Q}(g)} = \zeta_g(k)\mathcal{Q}(k^g)$$

is a linear character. Because $K = [K, K]$, the character ζ_g is trivial. This implies the second equation, see the proof of [34, Lemma 3.10] for a similar construction. Hence $\mathcal{Q}$ is a projective representation of G associated with χ.

Let $\mathcal{D}'$ be an $\mathbb{F}$-representation of $\mathrm{N}_{A(\chi_0)}(\widehat{Q})$ affording $\Upsilon(\Omega^{X^r}_{\widehat{Q}}(\chi_0))$ and let

$$\mathcal{P}' := \mathcal{D}' \circ (\mathrm{rep}_A)_{\mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0}}(\widehat{Q})}$$

be a projective $\mathbb{F}$-representation of $\mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0}}(\widehat{Q})$ obtained from $\Upsilon(\Omega^{X^r}_{\widehat{Q}}(\chi_0))$ using $(\mathrm{rep}_A)_{\mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0}}(\widehat{Q})}$. Then

$$\mathcal{Q}' : \mathrm{N}_G(Q) \to \mathrm{GL}_{\Omega^K_Q(\chi)(1)}(\mathbb{F})$$

defined by

$$\mathcal{Q}'(c\,\mathrm{rep}_G(a)) = \mu(c)\mathcal{P}'(a) \quad \text{for every } c \in \mathrm{C}_G(K) \text{ and } a \in \mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0,G}}(\widehat{Q})$$

is a projective representation. A similar construction has been given in [34, 3.11].

Because $p \nmid |\mathrm{C}_{A(\chi_0)}(\overline{X^r})|$, the fact $\mathrm{bl}(\Upsilon(\chi_0)) = \mathrm{bl}(\Upsilon(\Omega^{X^r}_{\widehat{Q}}(\chi_0)))^{A(\chi_0)}$ implies

$$\mathrm{IBr}(\Upsilon(\chi_0)_{\mathrm{C}_{A(\chi_0)}(\overline{X^r})}) = \mathrm{IBr}(\Upsilon(\Omega^{X^r}_{\widehat{Q}}(\chi_0))_{\mathrm{C}_{A(\chi_0)}(\overline{X^r})}).$$

Hence the factor sets α and α' of $\mathcal{P}$ and $\mathcal{P}'$ satisfy

$$\alpha' = \alpha_{\mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0}}(\widehat{Q}) \times \mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0}}(\widehat{Q})}$$

according to Remark 3.2. By definition the factor set β of $\mathcal{Q}$ satisfies

$$\beta(c\,\mathrm{rep}_G(a), c'\,\mathrm{rep}_G(a')) = \alpha(a, a')\frac{\mu(c)\mu(c')}{\mu(c'')},$$

where $a, a' \in \mathrm{Aut}(S^r)_{\chi_0,G}$ and $c, c', c'' \in \mathrm{C}_G(K)$ with

$$(c\,\mathrm{rep}_G(a))(c'\,\mathrm{rep}_G(a')) = c''\,\mathrm{rep}_G(aa').$$

Analogously the factor set β' of $\mathcal{Q}'$ satisfies

$$\beta'(c\,\mathrm{rep}_G(a), c'\,\mathrm{rep}_G(a')) = \alpha'(a, a')\frac{\mu(c)\mu(c')}{\mu(c'')},$$

where $a, a' \in \mathrm{N}_{\mathrm{Aut}(S^r)_{\chi_0,G}}(\widehat{Q})$ and $c, c', c'' \in \mathrm{C}_G(K)$ with

$$(c\,\mathrm{rep}_G(a))(c'\,\mathrm{rep}_G(a')) = c''\,\mathrm{rep}_G(aa').$$

Hence the factor sets of $\mathcal{Q}$ and $\mathcal{Q}'$ satisfy

$$\beta' = \beta_{\mathrm{N}_G(Q)\times\mathrm{N}_G(Q)},$$

see also [34, (3.30)]. This implies that ${\mathcal{Q}'}_{\mathrm{N}_K(Q)}$ is a representation of $\mathrm{N}_K(Q)$ and affords $\Omega_Q^K(\chi)$ by construction. Furthermore, $\mathcal{Q}'$ satisfies

$$\mathcal{Q}'(kg) = \mathcal{Q}'(k)\mathcal{Q}'(g) \quad \text{and} \quad \mathcal{Q}'(gk) = \mathcal{Q}'(g)\mathcal{Q}'(k)$$

for every $g \in \mathrm{N}_G(Q)$ and $k \in \mathrm{N}_K(Q)$, as $\mathcal{Q}$ has a similar property.

In our final step we prove that some subgroup C with $C \geq \langle \beta(g, g') \mid g, g' \in G\rangle$ and the factor set β have the properties described in (c). The factor set β defines the group $\widehat{G}$, with $\widehat{G} = G \times C$ as set and

$$(g, \zeta)(g', \zeta') = (gg', \zeta\zeta'\beta(g, g')) \quad \text{for every } g, g' \in G \text{ and } \zeta, \zeta' \in C.$$

By this definition $\mathcal{Q}$ lifts to a representation $\mathcal{R}$ of $\widehat{G}$, which is given by

$$\mathcal{R}(g, \zeta) = \zeta\mathcal{Q}(g) \quad \text{for every } (g, \zeta) \in \widehat{G}.$$

Analogously because of the equation $\beta' = \beta_{\mathrm{N}_G(Q)\times\mathrm{N}_G(Q)}$ the projective representation $\mathcal{Q}'$ lifts to a representation $\mathcal{R}'$ of $\mathrm{N}_{\widehat{G}}(Q_0)$ with $Q_0 := \langle (q, 1) \mid q \in Q\rangle$, where $\mathcal{R}'$ is defined by

$$\mathcal{R}'(g, \zeta) = \zeta\mathcal{Q}'(g) \quad \text{for every } (g, \zeta) \in \mathrm{N}_{\widehat{G}}(Q_0).$$

Let C_A be a subgroup containing the scalars associated with $\mathcal{D}(\mathrm{Z}(A(\chi_0)))$. Note that $A(\chi_0)$ is the central extension of $\mathrm{Aut}(S^r)_{\chi_0}$ by C_A given by α and we write for $z\,\mathrm{rep}_A(a)$ ($a \in \mathrm{Aut}(S^r)_{\chi_0}$, $z \in \mathrm{Z}(A(\chi_0))$) the pair (a, ζ), where ζ is the scalar associated with $\mathcal{D}(z)$. We may choose C to be C_A.

Note that the map $\kappa : \widehat{G} \to A(\chi_0)$ given by $(c\,\mathrm{rep}_G(a), \zeta) \mapsto (a, \mu(c)\zeta)$ for $a \in \mathrm{Aut}(S^r)_{\chi_0,G}$, $c \in \mathrm{C}_G(K)$ and $\zeta \in C$ is a morphism since

$$(c\,\mathrm{rep}_G(a), \zeta)(c'\,\mathrm{rep}_G(a'), \zeta') = (c''\,\mathrm{rep}_G(aa'), \alpha(a, a')\frac{\mu(c)\mu(c')}{\mu(c'')}\zeta\zeta') \quad (4.2)$$

for $a, a' \in \mathrm{Aut}(S^r)_{\chi_0,G}$ and $c, c', c'' \in \mathrm{C}_G(K)$ such that

$$(c\,\mathrm{rep}_G(a))(c'\,\mathrm{rep}_G(a')) = c''\,\mathrm{rep}_G(aa').$$

Accordingly $\widehat{G}/\ker(\kappa)$ and a subgroup of $A(\chi_0)$ are isomorphic. Computations show that $\mathcal{D}_{\kappa(\widehat{G})} \circ \kappa$ and $\mathcal{R}$ coincide, analogously

$$\mathcal{R}' = \mathcal{D}'_{\kappa(\mathrm{N}_{\widehat{G}}(Q_0))} \circ \kappa_{\mathrm{N}_{\widehat{G}}(Q_0)}.$$

As the characters $\Upsilon(\chi_0)$ and $\Upsilon(\Omega_Q^X(\chi_0))$ afforded by $\mathcal{D}$ and $\mathcal{D}'$ satisfy

$$\mathrm{bl}(\Upsilon(\Omega_{\widehat{Q}}^{X^r}(\chi_0))_H)^{\overline{X^r}H} = \mathrm{bl}(\Upsilon(\chi_0)_{\overline{X^r}H})$$

for every H with $N_{\overline{X^r}}(\kappa(Q_0)) \leq H \leq N_{A(\chi_0)}(\kappa(Q_0))$, the characters $\widetilde{\chi}_0 \in \mathrm{IBr}(\widehat{G})$ afforded by $\mathcal{R}$ and $\widetilde{\chi}'_0 \in \mathrm{IBr}(N_{\widehat{G}}(Q_0))$ afforded by $\mathcal{R}'$ satisfy

$$\mathrm{bl}((\widetilde{\chi}'_0)_H)^{K_0 H} = \mathrm{bl}((\widetilde{\chi}_0)_{K_0 H}) \quad \text{for every } H \text{ with } N_{K_0}(Q_0) \leq H \leq N_{\widehat{G}}(Q_0),$$

where $K_0 = \langle (k, 1) \mid k \in K \rangle$. This finishes the proof of (c). □

Proposition 4.6, especially its part (c) is important for proving the following bijection which plays a key role in the proof of Theorem A.

Corollary 4.7. *Like in Proposition* 4.6*, let $K \lhd G$ such that $[K, K] = K$, $Z(K) = K \cap Z(G)$, $p \nmid |Z(K)|$ and $K/Z(K) \cong S^r$ for some non-abelian simple group S satisfying the inductive BAW condition. Let $Q \in \mathrm{Rad}(K)$, $\mathrm{IBr}(K \mid Q)$ be the set from Proposition* 4.6 (a)*, and let Ω_Q^K be the bijection from Proposition* 4.6 (b)*. Let $\chi \in \mathrm{IBr}(K)$ with trivial $\ker(\chi_{Z(K)})$ and such that $\chi \in \mathrm{IBr}(K \mid Q)$. Furthermore, let $\chi' \in \mathrm{Irr}(N_K(Q))$ be the lift of $\Omega_Q^K(\chi)$. Then there exists a bijection*

$$\Pi_\chi : \mathrm{IBr}(G_\chi \mid \chi) \longrightarrow \mathrm{IBr}(N_{G_\chi}(Q) \mid (\chi')^0)$$

such that $\mathrm{bl}(\theta) = \mathrm{bl}(\Pi_\chi(\theta))^{G_\chi}$ for every $\theta \in \mathrm{IBr}(G_\chi \mid \chi)$.

Proof. We use Proposition 4.6. Let $\widehat{J}$, K_0, $\widetilde{\chi}_0 \in \mathrm{IBr}(\widehat{J})$ and $\widetilde{\chi}'_0 \in \mathrm{IBr}(N_{\widehat{J}}(Q_0))$ be the groups and characters associated with χ and Q as in Proposition 4.6 (c). The characters in $\mathrm{IBr}(G_\chi \mid \chi)$ correspond naturally to the ones in $\mathrm{IBr}(\widehat{J} \mid \chi_0 \times 1_C)$. According to [25, Corollary (8.20)] the set $\mathrm{IBr}(\widehat{J} \mid \chi_0 \times 1_C)$ coincides with

$$\left\{\widetilde{\chi}_0 \eta \;\middle|\; \eta \in \mathrm{IBr}(\widehat{J} \mid \nu^0) \text{ with } K_0 \leq \ker(\eta)\right\},$$

where $\nu^{-1} \in \mathrm{IBr}((\widetilde{\chi}_0)_C)$.

Let $\chi'_0 \in \mathrm{IBr}(N_{K_0}(Q_0))$ be associated with χ'. Instead of $\mathrm{IBr}(N_J(Q) \mid \chi')$ we can consider $\mathrm{IBr}(N_{\widehat{J}}(Q_0) \mid \chi'_0 \times 1_C)$ and like before $\mathrm{IBr}(N_{\widehat{J}}(Q_0) \mid \chi'_0 \times 1_C)$ coincides with

$$\left\{\widetilde{\chi}'_0 \eta \;\middle|\; \eta \in \mathrm{IBr}(N_{\widehat{J}}(Q_0) \mid \nu') \text{ with } N_{K_0}(Q_0) \leq \ker(\eta)\right\},$$

where $\nu'^{-1} \in \mathrm{IBr}((\widetilde{\chi}'_0)_C)$. Observe that $\nu = \nu'$ because of Proposition 4.6 (c) (iv). Furthermore, the properties of the set $\mathrm{IBr}(K \mid Q)$ and the bijection Ω_Q^K imply

$$J = K N_J(Q).$$

Hence the map $\widehat{\Pi}_{\chi_0} : \mathrm{IBr}(\widehat{J} \mid \chi_0 \times 1_C) \longrightarrow \mathrm{IBr}(N_{\widehat{J}}(Q_0) \mid (\chi'_0)^0 \times 1_C)$ given by

$$\widetilde{\chi}_0 \eta \longmapsto \widetilde{\chi}'_0 \eta_{N_{\widehat{J}}(Q_0)} \quad \text{for every } \eta \in \mathrm{IBr}(\widehat{J} \mid \nu^0) \text{ with } K_0 \leq \ker(\eta)$$

is a well-defined bijection. Because of Proposition 4.6 (c) (iv) and Proposition 3.6 the map $\widehat{\Pi}_{\chi_0}$ satisfies

$$\mathrm{bl}(\theta) = \mathrm{bl}(\widehat{\Pi}_{\chi_0}(\theta))^{\widehat{J}}$$

for every $\theta \in \mathrm{IBr}(\widehat{J} \mid \chi_0 \times 1_C)$. For the associated characters $\overline{\theta} \in \mathrm{IBr}(G_\chi)$ and

$\overline{\theta}' \in \mathrm{IBr}(\mathrm{N}_{G_\chi}(Q))$, Corollary 2.11 together with [24, Theorem 5.8.8] implies

$$\mathrm{bl}(\overline{\theta}) = \mathrm{bl}(\overline{\theta}')^{G_\chi},$$

since C is a p'-group. This gives a bijection

$$\Pi_\chi : \mathrm{IBr}(J \mid \chi) \longrightarrow \mathrm{IBr}(\mathrm{N}_J(Q) \mid (\chi')^0)$$

with the claimed properties. □

5 The reduction

In this section we prove first Theorem A and then sketch the adaptations of the approach that imply Theorem B.

For the inductive proof of Theorem A we use Conjecture 5.1, a relative version of Conjecture 1.1 using the sets introduced in Notation 2.14. After determining properties of a minimal situation to consider, we apply the bijections from Theorem 3.8 and Corollary 4.7. The proof of [27, Theorem 5.1] serves as guideline for our approach. Here in addition we also care about blocks and have to keep track of additional information.

Conjecture 5.1. Let $Z \lhd G$ and let $B \in \mathrm{Bl}(G)$ such that there exists a character $\nu \in \mathrm{dz}_G(Z)$ with $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G \mid \nu)$. Then

$$|\mathrm{IBr}(B)| = \sum_{Q/Z \in \mathrm{Rad}(G/Z)/\sim_G} |\mathrm{rdz}(\mathrm{N}_G(Q), B \mid \Theta_Q(\nu))|,$$

where $\Theta_Q(\nu)$ is an $\mathrm{N}_G(Q)$-invariant extension of ν to Q.

Note that in the situation of Conjecture 5.1, (G, Z, ν, B) forms a block quadruple. The set $\mathrm{rdz}(\mathrm{N}_G(Q), B \mid \Theta_Q(\nu))$ has been introduced in Notation 2.14. By Theorem 2.16 a character $\Theta_Q(\nu)$ exists for every Q with $Q/Z \in \mathrm{Rad}(G/Z)$ and $|\mathrm{rdz}(\mathrm{N}_G(Q), B \mid \Theta_Q(\nu))|$ does not depend on the choice of $\Theta_Q(\nu)$. Hence the statement of Conjecture 5.1 makes sense.

Obviously this conjecture implies Conjecture 1.1 in the case where $Z = 1$. But in fact both conjectures are equivalent.

Lemma 5.2. *Let* (G, Z, ν, B) *and* (G_1, Z_1, ν_1, B_1) *be isomorphic block quadruples, such that* (G_1, Z_1, ν_1, B_1) *is central and faithful. Then Conjecture* 5.1 *holds for* (G, Z, ν, B) *if and only if Conjecture* 1.1 *holds for* B_1.

Proof. According to Corollary 2.10 and Corollary 2.13, Conjecture 5.1 holds for (G, Z, ν, B) if and only if it is true for (G_1, Z_1, ν_1, B_1). Hence it is sufficient to verify that Conjecture 5.1 is true for (G_1, Z_1, ν_1, B_1) if and only if Conjecture 1.1 holds for B_1.

For every radical p-subgroup $Q/Z_1 \in \mathrm{Rad}(G_1/Z_1)$ the group Q satisfies $Q = Z_1 \times Q_p$ where $Q_p \in \mathrm{Syl}_p(Q)$. Let $\Theta_Q(\nu_1) := \nu_1 \times 1_{Q_p} \in \mathrm{Irr}(Q)$. The group Q_p is then a radical p-subgroup of G_1, the characters in $\mathrm{rdz}(\mathrm{N}_{G_1}(Q), B_1 \mid \Theta_Q(\nu_1))$ are the lifts of the characters in $\mathrm{dz}(\mathrm{N}_{G_1}(Q_p)/Q_p, B_1)$. This implies

$$\begin{aligned} |\mathrm{IBr}(B_1)| &= \sum_{Q/Z_1 \in \mathrm{Rad}(G_1/Z_1)/\sim_{G_1}} |\mathrm{rdz}(\mathrm{N}_{G_1}(Q), B_1 \mid \Theta_Q(\nu_1))| \\ &= \sum_{R \in \mathrm{Rad}(G_1)/\sim_{G_1}} |\mathrm{dz}(\mathrm{N}_{G_1}(R)/R, B_1)|. \end{aligned}$$

Hence Conjecture 5.1 for the block quadruple (G_1, Z_1, ν_1, B_1) is equivalent to Conjecture 1.1 for B_1. □

Theorem A follows from the following statement. Recall that for $K \lhd G$ and $b \in \mathrm{Bl}(K)$ we denote by $\mathrm{Bl}(G \mid b)$ the set of blocks of G covering b.

Theorem 5.3. *Let $\mathcal{S}$ be the set of all non-abelian simple groups S that satisfy the inductive BAW condition from Definition* 4.1 *for p. Let G be a finite group and $B \in \mathrm{Bl}(G)$. Assume that every non-abelian simple group involved in G is contained in $\mathcal{S}$. Then*

$$|\mathrm{IBr}(B)| = \sum_{R \in \mathrm{Rad}(G)/\sim_G} |\mathrm{dz}(\mathrm{N}_G(R)/R, B)|.$$

Theorem 5.3 is proven by induction on $|G : \mathrm{Z}(G)|$ in a series of results. Note that we keep some notation during the intermediate steps and recall them only when necessary.

The inductive hypothesis used in the procedure is the following.

Hypothesis 5.4. Conjecture 1.1 holds for every group G_1 and block $B_1 \in \mathrm{Bl}(G_1)$ whenever $|G_1 : \mathrm{Z}(G_1)| < |G : \mathrm{Z}(G)|$ and all non-abelian simple groups involved in $G_1/\mathrm{Z}(G_1)$ are contained in $\mathcal{S}$.

Proposition 5.5. *Without loss of generality we can assume that G is non-abelian, $\mathrm{O}_p(G) = 1$ and $\nu \in \mathrm{Irr}(\mathrm{Z}(G))$ with $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G \mid \nu)$ is faithful.*

Proof. The equation from Theorem 5.3 is clear for abelian groups. Hence we may assume that G is non-abelian.

Let $K := \mathrm{O}_p(G)$, the maximal normal p-subgroup of G, and $\overline{G} := G/K$. According to Lemma 2.4 (a) we have

$$|\mathrm{IBr}(B)| = \sum_i |\mathrm{IBr}(\overline{B}_i)|,$$

where $\overline{B}_1, \dots, \overline{B}_s \in \mathrm{Bl}(\overline{G})$ are the blocks dominated by B. If K is non-central, Hypothesis 5.4 implies that the blocks $\overline{B}_i$ satisfy

$$|\mathrm{IBr}(\overline{B}_i)| = \sum_{\overline{R} \in \mathrm{Rad}(\overline{G})/\sim_{\overline{G}}} |\mathrm{dz}(\mathrm{N}_{\overline{G}}(\overline{R})/\overline{R}, \overline{B}_i)| \quad \text{for every } 1 \leq i \leq s.$$

Let $R \in \mathrm{Rad}(G)$ and $\phi \in \mathrm{dz}(\mathrm{N}_G(R)/R, B)$. Then $K \lhd R$, and ϕ defines some $\overline{\phi} \in \mathrm{dz}(\mathrm{N}_{\overline{G}}(\overline{R})/\overline{R})$ with $\overline{R} := R/K \in \mathrm{Rad}(G/K)$. By the definition of the blocks $\overline{B}_i$ there exists an integer $1 \leq i \leq s$ such that $\overline{\phi} \in \mathrm{dz}(\mathrm{N}_{\overline{G}}(\overline{R})/\overline{R}, \overline{B}_i)$. On the other hand every character in $\mathrm{dz}(\mathrm{N}_{\overline{G}}(\overline{R})/\overline{R}, \overline{B}_i)$ lifts to a character in $\mathrm{dz}(\mathrm{N}_G(R)/R, B)$, see Lemma 2.4 (b). This implies

$$\begin{aligned} |\mathrm{IBr}(B)| &= \sum_{i=1}^{s} |\mathrm{IBr}(\overline{B}_i)| = \sum_{i=1}^{s} \sum_{\overline{R} \in \mathrm{Rad}(\overline{G})/\sim_{\overline{G}}} |\mathrm{dz}(\mathrm{N}_{\overline{G}}(\overline{R})/\overline{R}, \overline{B}_i)| \\ &= \sum_{R \in \mathrm{Rad}(G)/\sim_G} |\mathrm{dz}(\mathrm{N}_G(R)/R, B)|. \end{aligned}$$

Let K be the Sylow p-subgroup of $\mathrm{Z}(G)$. Then similar considerations as above show that it suffices to prove Conjecture 1.1 in the case where $\mathrm{O}_p(G) = 1$.

Then $\mathrm{Z}(G)$ is a p'-group. If $\nu \in \mathrm{Irr}(\mathrm{Z}(G))$ is not faithful, the group $K := \ker(\nu)$ is non-trivial and a p'-group. Let $\overline{B}$ be the block of $G/\ker(\nu)$ dominated by B and $\overline{\nu} \in \mathrm{Irr}(\mathrm{Z}(G)/K)$ associated with ν. Considerations as above show that Conjecture 1.1 for $\overline{B}$ holds if and only if it holds for B. □

Lemma 5.6. *Without loss of generality we can assume that there exists a non-central normal subgroup $K \lhd G$ such that*

- *K is a p'-group with $\mathrm{Z}(G) \leq K$, or*
- *the quotient $K/\mathrm{Z}(K)$ is isomorphic to S^r for some non-abelian simple group $S \in \mathcal{S}$, $p \nmid |\mathrm{Z}(K)|$, $\mathrm{Z}(K) = \mathrm{Z}(G) \cap K$ and $K = [K, K]$.*

Proof. Recall that $\mathrm{O}_p(G) = 1$ by Proposition 5.5. As the generalized Fitting subgroup of the non-abelian group G is non-central, $\mathrm{O}_{p'}(G)$ or $\mathrm{E}(G)$ is non-central, where $\mathrm{O}_{p'}(G)$ is the maximal normal p'-subgroup and $\mathrm{E}(G)$ the layer of G. (See [14, p. 274] for a definition.)

If $\mathrm{O}_{p'}(G) \not\leq \mathrm{Z}(G)$, we set $K = O_{p'}(G)$. Then K is a p'-group with $\mathrm{Z}(G) \leq K$.

Otherwise we have $\mathrm{O}_{p'}(G) = \mathrm{Z}(G)$ and $\mathrm{E}(G) \not\leq \mathrm{Z}(G)$. Recall that the quotient $E(G)/\mathrm{Z}(E(G))$ is the direct product of non-abelian simple groups. Let S be one of the occurring simple groups and r its multiplicity. Then there exists a subgroup $K \leq \mathrm{E}(G)$ with $K \lhd G$, and $K/Z(K) \cong S^r$ for some integer r. Then we have $Z(K) \leq \mathrm{O}_p(G)\,\mathrm{O}_{p'}(G) = \mathrm{O}_{p'}(G) = \mathrm{Z}(G)$. If K is not perfect, we replace K by $[K, K]$. Hence we have a group with the required properties. □

In the following let $K \lhd G$ be a group as above. Recall that for $\eta \in \mathrm{IBr}(K)$ and $B \in \mathrm{Bl}(G)$ we denote by $\mathrm{IBr}(B \mid \eta)$ the set $\mathrm{IBr}(G \mid \eta) \cap \mathrm{IBr}(B)$. Let $b \in \mathrm{Bl}(K)$ be covered by B, $\mathbb{A}$ be a G_b-transversal of $\mathrm{IBr}(b)$ and

$$\Gamma_\eta := \left\{c \in \mathrm{Bl}(G_\eta \mid b) \;\middle|\; c^G = B\right\} \quad \text{for } \eta \in \mathbb{A}.$$

Observe that according to Lemma 3.5, for every block $c \in \mathrm{Bl}(G_\eta \mid b)$ the block c^G is defined, as there exists some $\phi \in \mathrm{IBr}(c) \cap \mathrm{IBr}(G_\eta \mid \eta)$ with $\phi^G \in \mathrm{IBr}(G)$. Hence the set Γ_η is well-defined.

Lemma 5.7. *With the above,*

$$|\mathrm{IBr}(B)| = \sum_{\substack{\eta \in \mathbb{A} \\ c \in \Gamma_\eta}} |\mathrm{IBr}(c \mid \eta)|.$$

Proof. The group G_b acts on $\mathrm{IBr}(b)$ by conjugation and $\mathbb{A}$ is chosen with respect to this action. We have

$$|\mathrm{IBr}(B)| = \sum_{\eta \in \mathbb{A}} |\mathrm{IBr}(B \mid \eta)|$$

by Clifford theory.

In the following we describe the characters in $\mathrm{IBr}(B \mid \eta)$ for a fixed $\eta \in \mathbb{A}$. According to the Clifford correspondence and Lemma 3.5, the set Γ_η satisfies

$$\mathrm{IBr}(B \mid \eta) = \left\{\phi^G \;\middle|\; \phi \in \mathrm{IBr}(c \mid \eta) \text{ for some } c \in \Gamma_\eta\right\}$$

and hence

$$|\mathrm{IBr}(B)| = \sum_{\substack{\eta \in \mathbb{A} \\ c \in \Gamma_\eta}} |\mathrm{IBr}(c \mid \eta)|. \qquad \square$$

Lemma 5.8. *There exist sets* $\mathrm{Irr}(b \mid E) \subset \mathrm{IBr}(b)$ *for* $E \in \mathrm{Rad}(K)$ *such that*

- $\mathrm{IBr}(b \mid E)^a = \mathrm{IBr}(b \mid E^a)$ *for every* $E \in \mathrm{Rad}(K)$ *and* $a \in G_b$,
- $\mathrm{IBr}(b) = \dot\bigcup_{E \in \mathrm{Rad}(K)/\sim_K} \mathrm{IBr}(b \mid E)$.

Proof. If K is a p'-group or the defect group of b is trivial, we define $\mathrm{IBr}(b \mid E)$ for $E \in \mathrm{Rad}(K)$ by

$$\mathrm{IBr}(b \mid E) = \begin{cases} \mathrm{IBr}(b) & \text{if } E = 1, \\ \emptyset & \text{otherwise.} \end{cases}$$

Otherwise we have $K/\mathrm{Z}(K) \cong S^r$ for some integer r and some non-abelian simple group $S \in \mathcal{S}$. By assumption the inductive BAW condition from Definition 4.1 holds for S and p. The assumptions on K made in Proposition 4.6 are satisfied and there exists a partitioning of $\mathrm{IBr}(K)$ into sets $\mathrm{IBr}(K \mid E)$, where $E \in \mathrm{Rad}(K)$. The subsets $\mathrm{IBr}(b \mid E)$ defined by

$$\mathrm{IBr}(b \mid E) := \mathrm{IBr}(K \mid E) \cap \mathrm{IBr}(b) \quad \text{for every } E \in \mathrm{Rad}(K)$$

have the required properties by Proposition 4.6 (a).

If $K/Z(K) \cong S^r$ and b has trivial defect group, the two definitions of the subsets $\mathrm{IBr}(b \mid E) \subseteq \mathrm{IBr}(b)$ coincide. □

From now on we use the subsets $\mathrm{IBr}(b \mid E)$ for $E \in \mathrm{Rad}(K)$ from Lemma 5.8.

Lemma 5.9. *Then there exists an $\mathrm{N}_{G_b}(E)$-equivariant bijection*

$$\Omega_E^K : \mathrm{IBr}(b \mid E) \longrightarrow \mathrm{dz}(\mathrm{N}_K(E)/E, b)$$

such that

$$\sum_{c \in \Gamma_\eta} |\mathrm{IBr}(c \mid \eta)| = \sum_{c' \in \Gamma'_\eta} |\mathrm{IBr}(c' \mid \eta'^0)|, \tag{5.1}$$

where $\eta \in \mathrm{IBr}(b \mid E)$, $\eta' \in \mathrm{Irr}(\mathrm{N}_K(E))$ is the lift of $\Omega_E^K(\eta)$, and

$$\Gamma'_\eta := \left\{c' \in \mathrm{Bl}(\mathrm{N}_{G_\eta}(E) \mid \mathrm{bl}(\eta')) \;\middle|\; c'^{G_\eta} \in \Gamma_\eta\right\}.$$

Note that by [25, Theorem (4.14)] *the block c'^{G_η} is defined for every*

$$c' \in \mathrm{Bl}(\mathrm{N}_{G_\eta}(E) \mid \mathrm{bl}(\eta')).$$

Proof. Assume first that $K/Z \not\cong S^r$ or b has trivial defect group. Then we have $\mathrm{IBr}(b) = \{\eta\}$ and $E = 1$. Then the canonical map

$$\Omega_E^K : \{\eta\} \longrightarrow \mathrm{dz}(\mathrm{N}_K(E)/E, b)$$

is an $\mathrm{N}_{G_b}(E)$-equivariant bijection. Since $\Gamma'_\eta = \Gamma_\eta$ and $\eta = \eta'^0$, equation (5.1) holds.

Otherwise $K/Z \cong S^r$. By the definition of the set $\mathrm{IBr}(b \mid E)$ the bijection from Proposition 4.6 (b) induces a bijection

$$\Omega_E^K : \mathrm{IBr}(b \mid E) \longrightarrow \mathrm{dz}(\mathrm{N}_K(E)/E, b).$$

Since the bijection from Proposition 4.6 (b) is $\mathrm{N}_G(E)$-equivariant, the defined bijection Ω_E^K is $\mathrm{N}_{G_b}(E)$-equivariant.

In the next step we verify (5.1). Let $\eta \in \mathrm{IBr}(b \mid E)$. Then $\eta' \in \mathrm{Irr}(\mathrm{N}_K(E))$, the lift of $\Omega_E^K(\eta)$ is $\mathrm{N}_{G_\eta}(E)$-invariant and the characters η and η' satisfy

$$\mathrm{bl}(\eta) = \mathrm{bl}(\eta')^K$$

by Proposition 4.6 (b).

By Proposition 5.5 the character $\nu \in \mathrm{Irr}(\mathrm{Z}(G))$ satisfying $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G \mid \nu)$ is faithful. Observe that $\eta \in \mathrm{Irr}(K \mid \nu_{\mathrm{Z}(K)})$ and hence $\ker(\eta_{\mathrm{Z}(K)})$ is trivial. By Corollary 4.7 there exists a bijection

$$\Pi_\eta : \mathrm{IBr}(G_\eta \mid \eta) \longrightarrow \mathrm{IBr}(\mathrm{N}_{G_\eta}(E) \mid \eta'^0)$$

with $\mathrm{bl}(\theta) = \mathrm{bl}(\Pi_\eta(\theta))^{G_\eta}$ for every $\theta \in \mathrm{IBr}(G_\eta \mid \eta)$.

Let $\Gamma'_\eta := \{c \in \mathrm{Bl}(\mathrm{N}_{G_\eta}(E) \mid \mathrm{bl}(\eta')) \mid c^{G_\eta} \in \Gamma_\eta\}$. The properties of Π_η imply

$$\dot{\bigcup_{c \in \Gamma_\eta}} \Pi_\eta(\mathrm{IBr}(c \mid \eta)) = \dot{\bigcup_{c' \in \Gamma'_\eta}} \mathrm{IBr}(c' \mid \eta'^0)$$

and hence

$$\sum_{c \in \Gamma_\eta} |\mathrm{IBr}(c \mid \eta)| = \sum_{c' \in \Gamma'_\eta} |\mathrm{IBr}(c' \mid \eta'^0)|. \qquad \square$$

In what follows we use the bijections Ω_E^K from Lemma 5.9 and a K-transversal $\mathbb{B}$ of $\mathrm{Rad}(K)$. In order to use Hypothesis 5.4, we have to introduce further notation. Let $E(\eta) \in \mathbb{B}$ be the group with $\eta \in \mathrm{IBr}(b \mid E(\eta))$ and let Γ'_η be the set from Lemma 5.9 associated with η and $E(\eta)$.

Thereby we use the following notation: For $\eta \in \mathbb{A}$ let $\mathbb{T}(\eta)$ be an $\mathrm{N}_{G_\eta}(E)$-transversal of the following set of groups:

$$\{Q \mid Q/\mathrm{N}_K(E) \in \mathrm{Rad}(\mathrm{N}_{G_\eta}(E)/\mathrm{N}_K(E))\},$$

where $E \in \mathbb{B}$ with $\eta \in \mathrm{IBr}(b \mid E)$.

For $\eta \in \mathbb{A}$ and $Q \in \mathbb{T}(\eta)$ let $\Psi_Q(\eta') \in \mathrm{Irr}(Q)$ be an $\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q)$-invariant extension of $\eta' \in \mathrm{Irr}(\mathrm{N}_K(E))$, which is the lift of $\Omega_E^K(\eta)$. Recall that

$$\Omega_E^K(\eta) \in \mathrm{dz}(\mathrm{N}_K(E)/E).$$

Since Ω_E^K is $\mathrm{N}_{G_b}(E)$-equivariant, $\Omega_E^K(\eta)$ is $\mathrm{N}_{G_\eta}(E)$-invariant. By Theorem 2.16 there exists an $\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q)$-invariant extension of $\Omega_E^K(\eta)$ to Q/E and this lifts to some $\Psi_Q(\eta') \in \mathrm{Irr}(Q)$, an extension of η'.

For $\eta \in \mathbb{A}$, $E \in \mathbb{B}$ with $\eta \in \mathrm{IBr}(b \mid E)$ and $Q \in \mathbb{T}(\eta)$ let

$$\Gamma''_{\eta,Q} := \{c'' \in \mathrm{Bl}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q) \mid \mathrm{bl}(\Psi_Q(\eta'))) \mid (c'')^{\mathrm{N}_{G_\eta}(E)} \in \Gamma'_\eta\}.$$

Recall that for $X \lhd Y$, $\theta \in \mathrm{Irr}(X)$ and $c \in \mathrm{Bl}(Y)$ we denote by $\mathrm{rdz}(c \mid \theta)$ the set $\mathrm{rdz}(X \mid \theta) \cap \mathrm{Irr}(c)$.

Lemma 5.10. *Let* $\eta \in \mathrm{IBr}(b)$ *and let* $E \in \mathbb{B}$ *be such that* $\eta \in \mathrm{IBr}(b \mid E)$. *Further, let* $\eta' \in \mathrm{Irr}(\mathrm{N}_K(E))$ *be the lift of* $\Omega_E^K(\eta)$. *For any* $c \in \Gamma'_\eta$ *we have*

$$|\mathrm{IBr}(c \mid \eta'^0)| = \sum_{Q \in \mathbb{T}(\eta)} |\mathrm{rdz}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q), c \mid \Psi_Q(\eta'))|. \tag{5.2}$$

Furthermore,

$$\sum_{c \in \Gamma'_\eta} |\mathrm{IBr}(c \mid \eta'^0)| = \sum_{\substack{Q \in \mathbb{T}(\eta) \\ c'' \in \Gamma''_{\eta,Q}}} |\mathrm{rdz}(c'' \mid \Psi_Q(\eta'))|. \tag{5.3}$$

Proof. We prove the claimed formula using the inductive hypothesis.

By the properties of the sets from Lemma 5.8 the group G_η satisfies

$$G_\eta = K\mathrm{N}_{G_\eta}(E).$$

Let $c \in \Gamma'_\eta$ and $\overline{c} \in \mathrm{Bl}(\mathrm{N}_{G_\eta}(E)/E)$ such that $\overline{c}$ is contained in c. For every $Q \in \mathbb{T}(\eta)$ we denote by

$$\overline{\Psi_Q(\eta')} \in \mathrm{Irr}(Q/E)$$

the character given by $\Psi_Q(\eta')$.

We want to apply Hypothesis 5.4. To this end, let $\nu \in \mathrm{Irr}(\mathrm{Z}(G))$ be the character with $\mathrm{Irr}(B) \subseteq \mathrm{Irr}(G \mid \nu)$, $\overline{Z} := \mathrm{Z}(G)E/E$ and $\overline{\nu} \in \mathrm{Irr}(\overline{Z})$ the character associated with ν. Denote by $\Omega_E^K(\eta) \cdot \overline{\nu}$ the unique extension of

$$\Omega_E^K(\eta) \in \mathrm{Irr}(\overline{\mathrm{N}_K(E)})$$

contained in $\mathrm{Irr}(\overline{\mathrm{N}_K(E)}\,\overline{Z} \mid \overline{\nu})$, where $\overline{\mathrm{N}_K(E)} := \mathrm{N}_K(E)/E$. Let

$$\overline{\mathrm{N}_{G_\eta}(E)} = \mathrm{N}_{G_\eta}(E)/E.$$

Then

$$(\overline{\mathrm{N}_{G_\eta}(E)}, \overline{\mathrm{N}_K(E)}\,\overline{Z}, \Omega_E^K(\eta) \cdot \overline{\nu}, \overline{c})$$

is a block quadruple. Every non-abelian simple group involved in the quotient $\overline{\mathrm{N}_{G_\eta}(E)}/(\overline{\mathrm{N}_K(E)}\,\overline{Z})$ lies in $\mathcal{S}$ and

$$|\overline{\mathrm{N}_{G_\eta}(E)} : (\overline{\mathrm{N}_K(E)}\,\overline{Z})| = |G_\eta : (K\,\mathrm{Z}(G))| < |G : \mathrm{Z}(G)|.$$

According to Lemma 5.2 and Corollary 2.13, Hypothesis 5.4 implies that Conjecture 5.1 holds for

$$(\overline{\mathrm{N}_{G_\eta}(E)}, \overline{\mathrm{N}_K(E)}\,\overline{Z}, \Omega_E^K(\eta) \cdot \overline{\nu}, \overline{c}).$$

For $Q \in \mathbb{T}(\eta)$ let $\overline{Q} := Q\,\mathrm{Z}(G)/E$. Then the group $\overline{Q}/(\overline{\mathrm{N}_K(E)}\,\overline{Z})$ is a radical subgroup of $\overline{\mathrm{N}_{G_\eta}(E)}/(\overline{\mathrm{N}_K(E)}\,\overline{Z})$. Furthermore

$$\{\overline{Q}/(\overline{\mathrm{N}_K(E)}\,\overline{Z}) \mid Q \in \mathbb{T}(\eta)\}$$

is a $\overline{\mathrm{N}_{G_\eta}(E)}$-transversal of radical subgroups of $\overline{\mathrm{N}_{G_\eta}(E)}/(\overline{\mathrm{N}_K(E)}\,\overline{Z})$. We denote by $\overline{\Psi_Q(\eta')} \cdot \overline{\nu}$ the extension of $\overline{\Psi_Q(\eta')}$ contained in $\mathrm{Irr}(\overline{Q} \mid \overline{\nu})$. This is an $\mathrm{N}_{\overline{\mathrm{N}_{G_\eta}(E)}}(\overline{Q})$-invariant extension of $\Omega_E^K(\eta) \cdot \overline{\nu}$. Hence Conjecture 5.1 states

$$|\mathrm{IBr}(\overline{c} \mid \Omega_E^K(\eta)^0 \cdot \overline{\nu}^0)| = \sum_{Q \in \mathbb{T}(\eta)} |\mathrm{rdz}(\mathrm{N}_{\overline{\mathrm{N}_{G_\eta}(E)}}(\overline{Q}), \overline{c} \mid \overline{\Psi_Q(\eta')} \cdot \overline{\nu})|.$$

Since any character $\phi \in \mathrm{Irr}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q))$ with

$$\mathrm{bl}(\phi)^{\mathrm{N}_{G_\eta}(E)} = c$$

satisfies $\nu \in \mathrm{Irr}(\phi_{Z(G)})$, we can take the sum over all blocks $\overline{c} \in \mathrm{Bl}(\overline{\mathrm{N}_{G_\eta}(E)})$ that are contained in c and obtain

$$|\mathrm{IBr}(c \mid \eta'^0)| = \sum_{Q \in \mathbb{T}(\eta)} |\mathrm{rdz}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q), c \mid \Psi_Q(\eta'))|.$$

The characters counted on the right hand side are contained in the blocks of

$$\Gamma''_{c,Q} := \left\{c'' \in \mathrm{Bl}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q) \mid \mathrm{bl}(\Psi_Q(\eta'))) \;\middle|\; (c'')^{\mathrm{N}_{G_\eta}(E)} = c\right\}.$$

This proves

$$|\mathrm{rdz}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q), c \mid \Psi_Q(\eta'))| = \sum_{c'' \in \Gamma''_{c,Q}} |\mathrm{rdz}(c'' \mid \Psi_Q(\eta'))|.$$

Hence when considering all blocks of Γ'_η, we obtain

$$\sum_{c \in \Gamma'_\eta} |\mathrm{rdz}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q), c \mid \Psi_Q(\eta'))| = \sum_{c'' \in \Gamma''_{\eta,Q}} |\mathrm{rdz}(c'' \mid \Psi_Q(\eta'))|,$$

where

$$\Gamma''_{\eta,Q} := \left\{c'' \in \mathrm{Bl}(\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q) \mid \mathrm{bl}(\Psi_Q(\eta'))) \;\middle|\; (c'')^{\mathrm{N}_{G_\eta}(E)} \in \Gamma'_\eta\right\}.$$

Together with the first part of the proof we obtain

$$\sum_{c \in \Gamma'_\eta} |\mathrm{IBr}(c \mid \eta'^0)| = \sum_{\substack{Q \in \mathbb{T}(\eta) \\ c'' \in \Gamma''_{\eta,Q}}} |\mathrm{rdz}(c'' \mid \Psi_Q(\eta'))|. \qquad \square$$

For $\eta \in \mathbb{A}$ and $E \in \mathbb{B}$ with $\eta \in \mathrm{IBr}(b \mid E)$ and for $Q \in \mathbb{T}(\eta)$ we can associate to $\Psi_Q(\eta')$ a character $\overline{\Psi_Q(\eta')} \in \mathrm{Irr}(Q/E)$, where $\eta' \in \mathrm{Irr}(\mathrm{N}_K(E))$ is the lift of $\Omega_E^K(\eta)$.

For $\eta \in \mathbb{A}$ and $Q \in \mathbb{T}(\eta)$ let $D(\eta, Q) \leq Q$ be such that

$$\overline{D(\eta, Q)} := D(\eta, Q)/E$$

is a defect group of $\mathrm{bl}(\overline{\Psi_Q(\eta')})$. Since $\Omega_E^K(\eta) \in \mathrm{dz}(\mathrm{N}_K(E)/E)$, we can apply the Dade–Glauberman–Nagao correspondence described before Theorem 3.8 with respect to $\overline{D(\eta, Q)}$ and obtain

$$\mathrm{DGN}_{\overline{D(\eta,Q)}}(\Omega_E^K(\eta)) \in \mathrm{dz}(\mathrm{N}_K(D(\eta, Q))/E),$$

the $\overline{D(\eta, Q)}$-correspondent of $\Omega_E^K(\eta)$. We denote by $\overline{\mathrm{DGN}}_{\overline{D(\eta,Q)}}(\Omega_E^K(\eta))$ the associated character of $\mathrm{N}_K(D(\eta, Q))/D(\eta, Q)$.

The following statement relates some characters of $c \in \Gamma''_{\eta,Q}$ to characters belonging to the blocks in $\Gamma'''_{\eta,D(\eta,Q)}$, where

$$\Gamma'''_{\eta,D(\eta,Q)} \subseteq \mathrm{Bl}(\mathrm{N}_{G_\eta}(D(\eta, Q))/D(\eta, Q) \mid \mathrm{bl}(\overline{\mathrm{DGN}}_{\overline{D(\eta,Q)}}(\Omega_E^K(\eta))))$$

is defined as the set of blocks dominated by one in

$$\left\{c \in \mathrm{Bl}(\mathrm{N}_{G_\eta}(D(\eta, Q))) \;\middle|\; c^{\mathrm{N}_K(E)\mathrm{N}_{G_\eta}(D(\eta,Q))} \in \Gamma''_{\eta,Q}\right\}.$$

Lemma 5.11. *Let $\eta \in \mathbb{A}$ and $Q \in \mathbb{T}(\eta)$. Then*

$$\sum_{c \in \Gamma''_{\eta,Q}} |\mathrm{rdz}(c \mid \Psi_Q(\eta'))| = \sum_{c' \in \Gamma'''_{\eta,D(\eta,Q)}} |\mathrm{dz}(c' \mid \overline{\mathrm{DGN}}_{\overline{D(\eta,Q)}}(\Omega_E^K(\eta)))|. \tag{5.4}$$

Proof. Let $D := D(\eta, Q)$, $\overline{D} := \overline{D(\eta, Q)}$, $\overline{\eta}' := \Omega_E^K(\eta)$ and $\eta' \in \mathrm{Irr}(\mathrm{N}_K(E))$ be the lift of $\overline{\eta}'$. Then $\Psi_Q(\eta') \in \mathrm{Irr}(Q)$ defines a character $\overline{\Psi_Q(\eta')} \in \mathrm{Irr}(Q/E)$. Since $\Omega_E^K(\eta)$ is a defect zero character, it follows that D satisfies $\mathrm{N}_K(E)D = Q$ and $D \cap K = E$, see [25, Theorem (9.13)] and [24, Lemma 5.12.2]. A Frattini argument shows

$$\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q) = \mathrm{N}_K(E)\mathrm{N}_{G_\eta}(D).$$

According to Theorem 3.8 we can choose an $\mathrm{N}_{\mathrm{N}_{G_\eta}(E)}(Q)$-invariant extension $\Psi'_Q(\eta') \in \mathrm{Irr}(Q)$ of η' such that the following bijection exists:

$$\Delta_{\eta'} : \mathrm{rdz}\big(\mathrm{N}_K(E)\mathrm{N}_{G_\eta}(D) \mid \Psi'_Q(\eta')\big) \longrightarrow \mathrm{dz}\big(\mathrm{N}_{G_\eta}(D)/D \,|\, \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}')\big)$$

and

$$\mathrm{bl}(\psi) = \mathrm{bl}(\psi')^{\mathrm{N}_K(E)\mathrm{N}_{G_\eta}(D)}$$

for every character $\psi \in \mathrm{rdz}(\mathrm{N}_K(E)\mathrm{N}_{G_\eta}(D) \mid \Psi'_Q(\eta'))$, where $\psi' \in \mathrm{Irr}(\mathrm{N}_{G_\eta}(D))$ is the lift of $\Delta_{\eta'}(\psi)$. Hence every character of

$$\Delta_{\eta'}\left(\mathrm{rdz}(\mathrm{N}_K(E)\mathrm{N}_{G_\eta}(D) \mid \Psi'_Q(\eta')) \cap \bigcup_{c \in \Gamma''_{\eta,Q}} \mathrm{IBr}(c)\right)$$

lies in a block $\overline{c} \in \mathrm{Bl}(\mathrm{N}_{G_\eta}(D)/D)$ such that

- $\overline{c}$ covers $\mathrm{bl}(\overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}'))$,
- $(c)^{\mathrm{N}_K(E)\mathrm{N}_{G_\eta}(D)} \in \Gamma''_{\eta,Q}$, where $c \in \mathrm{Bl}(\mathrm{N}_{G_\eta}(D))$ dominates $\overline{c}$.

These are exactly the blocks of $\Gamma'''_{\eta,D}$. As $\Delta_{\eta'}$ is a bijection, it follows that every character of $\mathrm{dz}(c' \mid \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}'))$ is contained in a block of the mentioned set. This proves

$$\sum_{c\in\Gamma''_{\eta,Q}} |\mathrm{rdz}(c \mid \Psi'_Q(\eta'))| = \sum_{c'\in\Gamma'''_{\eta,D}} |\mathrm{dz}(c' \mid \overline{\mathrm{DGN}}_D(\overline{\eta}'))|.$$

According to Theorem 2.16 the sets $\mathrm{rdz}(c \mid \Psi'_Q(\eta'))$ and $\mathrm{rdz}(c \mid \Psi_Q(\eta'))$ have the same cardinality for every $c \in \Gamma''_{\eta,Q}$. □

Lemma 5.12. *Let $\eta \in \mathbb{A}$, $Q \in \mathbb{T}(\eta)$, $D := D(\eta, Q)$ and $\overline{D} := \overline{D(\eta, Q)}$. Then*

$$\sum_{c\in\Gamma'''_{\eta,D}} \left|\mathrm{dz}\big(c \mid \overline{\mathrm{DGN}}_{\overline{D}}(\Omega^K_E(\eta))\big)\right| = \left|\mathrm{dz}\big(\mathrm{N}_G(D)/D, B \mid \overline{\mathrm{DGN}}_{\overline{D}}(\Omega^K_E(\eta))\big)\right|.$$

Proof. Let $\overline{\eta}' := \Omega^K_E(\eta) \in \mathrm{dz}(\mathrm{N}_K(E)/E)$. The equivariance of the DGN-correspondence implies

$$(\mathrm{N}_G(D))_{\overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}')} = \mathrm{N}_{G_\eta}(D).$$

Let $c \in \Gamma'''_{\eta,D}$. As before, we denote by $b'' \in \mathrm{Bl}(\mathrm{N}_K(D))$ the block that dominates $\mathrm{bl}(\mathrm{DGN}_{\overline{D}}(\overline{\eta}'))$. As c covers b'', the block $c^{\mathrm{N}_G(D)}$ is defined by Lemma 5.3.1 (ii) of [24]. Furthermore induction defines a bijection between

$$\mathrm{dz}\big(\mathrm{N}_{G_\eta}(D)/D \mid \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}')\big) \cap \Bigg(\bigcup_{c\in\Gamma'''_{\eta,D}} \mathrm{IBr}(c)\Bigg)$$

and

$$\mathrm{dz}\big(\mathrm{N}_G(D)/D \mid \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}')\big) \cap \Bigg(\bigcup_{c\in\Gamma'''_{\eta,D}} \mathrm{IBr}(c^{\mathrm{N}_G(D)/D})\Bigg).$$

We observe that the blocks in $\{c^{\mathrm{N}_G(D)/D} \mid c \in \Gamma'''_{\eta,D}\}$ are the blocks covered by a block in $\{c \in \mathrm{Bl}(\mathrm{N}_G(D)|b'') \mid c^G = B\}$, where the block $b'' \in \mathrm{Bl}(\mathrm{N}_K(D))$ dominates $\mathrm{bl}(\mathrm{DGN}_{\overline{D}}(\overline{\eta}'))$. Then $\mathrm{dz}(\mathrm{N}_G(D)/D, B \mid \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}'))$ coincides with

$$\mathrm{dz}\big(\mathrm{N}_G(D)/D \mid \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}')\big) \cap \Bigg(\bigcup_{c\in\Gamma'''_{\eta,D}} \mathrm{IBr}(c^{\mathrm{N}_G(D)/D})\Bigg).$$ □

The next statement collects the above results and expresses $|\mathrm{IBr}(B)|$ in terms of local data.

Proposition 5.13. *With the notation from above*

$$|\mathrm{IBr}(B)| = \sum_{\substack{\eta\in\mathbb{A}\\ Q\in\mathbb{T}(\eta)}} \Big|\mathrm{dz}\big(\mathrm{N}_G(D(\eta,Q))/D(\eta,Q), B \mid \overline{\mathrm{DGN}}_{\overline{D(\eta,Q)}}(\Omega_E^K(\eta))\big)\Big|.$$

Proof. Lemma 5.7 implies

$$|\mathrm{IBr}(B)| = \sum_{\substack{\eta\in\mathbb{A}\\ c\in\Gamma_\eta}} |\mathrm{IBr}(c \mid \eta)|.$$

Then Lemma 5.9 shows that for fixed $\eta \in \mathbb{A}$ we have

$$\sum_{c\in\Gamma_\eta} |\mathrm{IBr}(c \mid \eta)| = \sum_{c'\in\Gamma'_\eta} |\mathrm{IBr}(c' \mid \eta'^0)|,$$

where $\eta' \in \mathrm{Irr}(\mathrm{N}_K(E))$ is the lift of $\overline{\eta}' := \Omega_E^K(\eta)$. Applying the inductive hypothesis lead in Lemma 5.10 to the equation

$$\sum_{c\in\Gamma'_\eta} |\mathrm{IBr}(c \mid \eta'^0)| = \sum_{\substack{Q\in\mathbb{T}(\eta)\\ c''\in\Gamma''_{\eta,Q}}} |\mathrm{rdz}(c'' \mid \Psi_Q(\eta'))|,$$

where $\Psi_Q(\eta')$ denotes certain extensions of η'.

For $Q \in \mathbb{T}(\eta)$ let $D := D(\eta, Q)$ and $\overline{D}$ be defined as above. Then Lemma 5.11 implies

$$\sum_{c\in\Gamma''_{\eta,Q}} |\mathrm{rdz}(c \mid \Psi_Q(\eta'))| = \sum_{c'\in\Gamma'''_{\eta,D}} |\mathrm{dz}(c' \mid \overline{\mathrm{DGN}}_{\overline{D}}(\Omega_E^K(\eta)))|.$$

According to Lemma 5.12 we have

$$\sum_{c'\in\Gamma'''_{\eta,D}} |\mathrm{dz}(c' \mid \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}'))| = |\mathrm{dz}(\mathrm{N}_G(D)/D, B \mid \overline{\mathrm{DGN}}_{\overline{D}}(\overline{\eta}'))|.$$

Altogether this proves the claimed equation. □

In the next step we show that the characters counted in Proposition 5.13 lie in disjoint G-conjugacy classes.

Lemma 5.14. *Let ϕ_1, ϕ_2 be characters such that, with the notation from above, there exist $\eta_i \in \mathbb{A}$ and $Q_i \in \mathbb{T}(\eta_i)$ with*

$$\phi_i \in \mathrm{dz}(\mathrm{N}_G(D_i)/D_i, B \mid \overline{\mathrm{DGN}}_{D_i/E_i}(\Omega^K_{E_i}(\eta_i))),$$

where $E_i \in \mathbb{B}$ with $\eta_i \in \mathrm{IBr}(b \mid E_i)$ and $D_i := D(\eta_i, Q_i)$. If ϕ_1, ϕ_2 are G-conjugate, then $\phi_1 = \phi_2$.

Proof. By the construction $(\phi_i)_{\mathrm{N}_K(D_i)D_i/D_i}$ contains $\overline{\mathrm{DGN}}_{D_i/E_i}(\Omega^K_{E_i}(\eta_i))$ as constituent.

Assume that ϕ_1 and ϕ_2 are G-conjugate. Because

$$\mathrm{N}_K(D_i) \lhd \mathrm{N}_G(D_i),$$

this implies that $\overline{\mathrm{DGN}}_{D_1/E_1}(\Omega^K_{E_1}(\eta_1))$ and $\overline{\mathrm{DGN}}_{D_2/E_2}(\Omega^K_{E_2}(\eta_2))$ are G-conjugate, as well. By the equivariance properties of the DGN-correspondence and the G_b-equivariance of the bijections $\{\Omega^K_E\}_{E \in \mathrm{Rad}(K)}$ the characters η_1, η_2 have to be G-conjugate. Since $\eta_1, \eta_2 \in \mathrm{Irr}(b)$, $\eta_1, \eta_2 \in \mathbb{A}$ and $\mathbb{A}$ is a G_b-transversal in $\mathrm{IBr}(b)$, this shows $\eta_1 = \eta_2$. Then the associated groups E_1 and E_2 coincide.

Let $n \in G$ such that $\phi_1^n = \phi_2$. Then $D_1^n = D_2$. Because $D_1 \cap K = E_1$ and $D_2 \cap K = E_2$, this element n can be chosen to be contained in $\mathrm{N}_{G_{\eta_1}}(E_1)$. The groups Q_1 and Q_2 satisfy

$$Q_1^n = (\mathrm{N}_K(E_1)D_1)^n = \mathrm{N}_K(E_1)D_2 = Q_2.$$

Together with $Q_1, Q_2 \in \mathbb{T}(\eta_1) = \mathbb{T}(\eta_2)$ this implies $Q_1 = Q_2$ by the definition of $\mathbb{T}(\eta_1)$. Then we have $n \in \mathrm{N}_{G_\eta}(Q_1)$. As the groups D_i are uniquely determined by $Q_1 = Q_2$ and $\eta_1 = \eta_2$, this proves $D_1 = D_2$, and $n \in \mathrm{N}_G(D_1)$. As ϕ_1 and ϕ_2 are characters of $\mathrm{N}_G(D_1) = \mathrm{N}_G(D_2)$, this implies $\phi_1^n = \phi_1 = \phi_2$. □

It remains to prove that Proposition 5.13 implies Theorem 5.3.

Proposition 5.15. *Then* $|\mathrm{IBr}(B)| = \sum_{R \in \mathrm{Rad}(G)/\sim_G} |\mathrm{dz}(\mathrm{N}_G(R)/R, B)|$.

Proof. We have to prove that

$$\sum_{\substack{\eta \in \mathbb{A} \\ Q \in \mathbb{T}(\eta)}} \left|\mathrm{dz}\big(\mathrm{N}_G(D(\eta, Q))/D(\eta, Q), B \mid \overline{\mathrm{DGN}}_{\overline{D(\eta,Q)}}(\eta')\big)\right|$$

from Proposition 5.13 coincides with

$$\sum_{R \in \mathrm{Rad}(G)/\sim_G} |\mathrm{dz}(\mathrm{N}_G(R)/R, B)|.$$

The later can be seen as the number of G-orbits of pairs (D, ϕ) with $D \in \mathrm{Rad}(G)$ and $\phi \in \mathrm{dz}(\mathrm{N}_G(D)/D, B)$. According to Lemma 5.14 no two characters counted in the first sum are G-conjugate.

Hence for the remaining it is sufficient to show that for every $D \in \mathrm{Rad}(G)$ every character $\phi \in \mathrm{dz}(\mathrm{N}_G(D)/D, B)$ is conjugate to a character counted in the equation of Proposition 5.13, in short we have counted all G-orbits of (D, ϕ) associated with B in the equation from Proposition 5.13.

Let $D \in \mathrm{Rad}(G)$ and $\phi \in \mathrm{dz}(\mathrm{N}_G(D)/D, B)$. We can choose a defect zero character in $\mathrm{Irr}(\phi_{\mathrm{N}_K(D)D/D})$ and it defines a defect zero character $\overline{\chi}$ of $\mathrm{N}_K(D)/E$ for $E := K \cap D$. This character $\overline{\chi}$ is a Dade–Glauberman–Nagao correspondent with respect to $\overline{D} := D/E$, i.e.

$$\mathrm{DGN}_{\overline{D}}^{-1}(\overline{\chi}) \in \mathrm{dz}(\mathrm{N}_K(E)/E)$$

is well-defined and $\mathrm{DGN}_{\overline{D}}^{-1}(\overline{\chi})$ is contained in $\mathrm{dz}(\mathrm{N}_K(E)/E)$. Let $\chi \in \mathrm{Irr}(\mathrm{N}_K(E))$ be the lift of $\mathrm{DGN}_{\overline{D}}^{-1}(\overline{\chi})$. By [25, Theorem (4.14)] the block $\mathrm{bl}(\chi)^K$ is defined. Straight-forward calculations with central functions show that $\mathrm{bl}(\chi)^K$ is covered by B, since $\phi \in \mathrm{dz}(\mathrm{N}_G(D)/D, B)$.

By [25, Corollary (9.3)] there exists some $g \in G$ such that $\mathrm{bl}(\chi^g)^K = b$. Then

$$\mathrm{DGN}_{\overline{D}}^{-1}(\overline{\chi}^g) \in \mathrm{dz}(\mathrm{N}_K(E)/E, b).$$

The bijection $\Omega_{E^g}^K$ can be applied to $\mathrm{DGN}_{\overline{D}^g}^{-1}(\overline{\chi}^g)$ and we obtain a character

$$\eta_1 := (\Omega_{E^g}^K)^{-1}(\mathrm{DGN}_{\overline{D}^g}^{-1}(\overline{\chi}^g)) \in \mathrm{IBr}(b \mid E^g).$$

For some $g_1 \in G_b$ we have $\eta_0 := \eta_1^{g_1} \in \mathbb{A}$. For $E_2 := (K \cap D)^{gg_1}$ the character $\eta_2 := (\Omega_{E_2}^K)^{-1}(\mathrm{DGN}_{\overline{D}^{gg_1}}^{-1}(\overline{\chi}^{gg_1}))$ is defined.

We choose $k \in K$ such that $E_0 := E_2^k \in \mathbb{B}$. Then we observe for $D_2 := D^{gg_1k}$ that $(\mathrm{N}_K(E_0)D_2)/\mathrm{N}_K(E_0)$ is a radical subgroup of $\mathrm{N}_{G_{\eta_0}}(E_0)/\mathrm{N}_K(E_0)$ by Theorems 2.16 and 3.8. Hence we can choose an element $n \in \mathrm{N}_{G_{\eta_0}}(E_0)$ such that

$$Q_0 := (\mathrm{N}_K(E_0)D^{gg_1k})^n \in \mathbb{T}(\eta_0).$$

The block of Q_0/E_0 covering $\mathrm{bl}(\Omega_{E_0}^K(\eta_0))$ has defect group D^{gg_1kn}/E_0. Hence there exists some $q \in Q_0$ such that $D_0 := D^{gg_1knq}$ coincides with $D(\eta_0, Q_0)$. This process yields

$$\phi^{gg_1knq} \in \mathrm{dz}\big(\mathrm{N}_G(D_0)/D_0, B \mid \overline{\mathrm{DGN}}_{\overline{D}_0}(\Omega_{E_0}^K(\eta_0))\big),$$

where $\eta_0 \in \mathbb{A}$, $Q_0 \in \mathbb{T}(\eta_0)$, $D_0 := D(\eta_0, Q_0)$ and $\overline{D}_0 := D_0/E_0$. All in all this proves that a G-conjugate of ϕ has been counted in Proposition 5.13. □

The above result implies Theorem A.

As already mentioned in the introduction this method is also useful when considering blocks with specific defect groups. In this situation only a weaker version of the inductive BAW condition has to be assumed.

In order to distinguish the characters, weights and defect zero characters that are relevant to the consideration, we use a set $\mathcal{R}$ of p-groups and associate to it specific blocks, radical subgroups and Brauer characters. We say that a p-group is contained in $\mathcal{R}$ if it is isomorphic to a group from $\mathcal{R}$.

Notation 5.16. Let $\mathcal{R}$ be a set of p-groups. For a finite group G we denote by $\mathrm{Rad}_{\mathcal{R}}(G)$ the set of radical subgroups of G that are involved in one of $\mathcal{R}$. We write $\mathrm{Bl}_{\mathcal{R}}(G)$ for the set of blocks of G, having a defect group in $\mathcal{R}$, and

$$\mathrm{IBr}_{\mathcal{R}}(G) = \bigcup_{B \in \mathrm{Bl}_{\mathcal{R}}(G)} \mathrm{IBr}(B).$$

For a finite group G and a p-subgroup Q we fix

$$\mathrm{dz}(\mathrm{N}_G(Q)/Q, \mathrm{Bl}_{\mathcal{R}}(G)) := \bigcup_{B \in \mathrm{Bl}_{\mathcal{R}}(G)} \mathrm{dz}(\mathrm{N}_G(Q)/Q, B).$$

(Clearly such a set is empty when $Q \notin \mathrm{Rad}_{\mathcal{R}}(G)$.)

For the set $\mathcal{R}_p$ of all p-groups the above sets coincide with the original ones, i.e.

$$\mathrm{Rad}_{\mathcal{R}_p}(G) = \mathrm{Rad}(G), \quad \mathrm{Bl}_{\mathcal{R}_p}(G) = \mathrm{Bl}(G) \quad \text{and} \quad \mathrm{IBr}_{\mathcal{R}_p}(G) = \mathrm{IBr}(G).$$

Definition 5.17 (Inductive BAW condition for S with respect to $\mathcal{R}$). Let p be a prime and $\mathcal{R}$ be a set of p-groups. Let S be a non-abelian simple group and X be the universal p'-covering group of S. We say that the *inductive BAW condition holds for S with respect to $\mathcal{R}$* if the following statements are satisfied:

(i) There exist subsets $\mathrm{IBr}_{\mathcal{R}}(X \mid Q) \subseteq \mathrm{IBr}_{\mathcal{R}}(X)$ for $Q \in \mathrm{Rad}(X)$ with the following properties:

  (1) $\mathrm{IBr}_{\mathcal{R}}(X \mid Q)^a = \mathrm{IBr}_{\mathcal{R}}(X \mid Q^a)$ for every $Q \in \mathrm{Rad}(X)$, $a \in \mathrm{Aut}(X)$,

  (2) $\mathrm{IBr}_{\mathcal{R}}(X) = \dot{\bigcup}_{Q \in \mathrm{Rad}_{\mathcal{R}}(X)/\sim_X} \mathrm{IBr}_{\mathcal{R}}(X \mid Q)$.

(ii) For every $Q \in \mathrm{Rad}(X)$ there exists a bijection

$$\Omega_Q^X : \mathrm{IBr}_{\mathcal{R}}(X \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_X(Q)/Q, \mathrm{Bl}_{\mathcal{R}}(X)),$$

with the following properties for every $\nu \in \mathrm{Irr}(\mathrm{Z}(X))$, $\chi \in \mathrm{IBr}_{\mathcal{R}}(X \mid Q)$ and $a \in \mathrm{Aut}(X)$:

  (1) all characters of $\Omega_Q^X(\mathrm{IBr}_{\mathcal{R}}(X \mid Q) \cap \mathrm{IBr}(X \mid \nu^0))$ lift to characters in $\mathrm{Irr}(\mathrm{N}_X(Q) \mid \nu)$,

  (2) $\Omega_Q^X(\chi)^a = \Omega_{Q^a}^X(\chi^a)$,

  (3) $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$, where $\chi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ is the lift of $\Omega_Q^X(\chi)$.

(iii) For every character $\chi \in \mathrm{IBr}_{\mathcal{R}}(X \mid Q)$, there exists a group $A(\chi)$ and characters $\Upsilon(\chi) \in \mathrm{IBr}(A(\chi))$ and $\Upsilon(\Omega_Q^X(\chi))$ with the following properties:

(1) for $Z := \ker(\chi_{\mathrm{Z}(X)})$ and $\overline{X} := X/Z$ the group $A := A(\chi)$ satisfies $\overline{X} \lhd A$, $A/\mathrm{C}_A(\overline{X}) \cong \mathrm{Aut}(X)_\chi$, $\mathrm{C}_A(\overline{X}) = \mathrm{Z}(A)$ and $p \nmid |\mathrm{Z}(A)|$,

(2) $\Upsilon(\chi) \in \mathrm{IBr}(A)$ is an extension of the character of $\overline{X}$ associated with χ,

(3) $\Upsilon(\Omega_Q^X(\chi)) \in \mathrm{IBr}(\mathrm{N}_A(\overline{Q}))$ is an extension of $\overline{\chi}'^0$, where $\overline{Q} := QZ/Z$, $\overline{\chi}' \in \mathrm{Irr}(\mathrm{N}_{\overline{X}}(\overline{Q}))$ is associated with $\Omega_Q^X(\chi) \in \mathrm{Irr}(\mathrm{N}_X(Q)/Q)$,

(4) $\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$ for every subgroup H satisfying $\mathrm{N}_{\overline{X}}(\overline{Q}) \leq H \leq \mathrm{N}_A(\overline{Q})$.

(iv) If $1 \in \mathcal{R}$, the map Ω_1^X satisfies

$$\Omega_1^X(\psi^0) = \psi \quad \text{for every } \psi \in \mathrm{dz}(X)$$

and

$$\Upsilon(\chi) = \Upsilon(\Omega_1^X(\chi)) \quad \text{for every } \chi \in \mathrm{IBr}_{\mathcal{R}}(X \mid 1).$$

We should observe the following implications.

Remark 5.18. The inductive BAW condition for p from Definition 4.1 is equivalent to the inductive BAW condition with respect to the set of all p-groups. Furthermore, if D is not isomorphic to a defect group of any block of X, then the inductive BAW condition holds for S with respect to $\{D\}$. If the inductive BAW condition holds for S with respect to $\mathcal{R}_1$ and $\mathcal{R}_2$, then it holds with respect to $\mathcal{R}_1 \cup \mathcal{R}_2$. Hence proving that a group satisfies the inductive BAW condition can be done by proving that the group satisfies the inductive BAW condition with respect to smaller sets of p-groups. The sets $\mathrm{Irr}_{\mathcal{R}}(X \mid Q)$ and $\mathrm{dz}(\mathrm{N}_X(Q)/Q, \mathrm{Bl}_{\mathcal{R}}(X))$ are empty whenever $Q \notin \mathrm{Rad}_{\mathcal{R}}(X)$. Accordingly it is sufficient to define the set $\mathrm{IBr}(X \mid Q)$ and the bijection Ω_Q^X with the required properties for $Q \in \mathrm{Rad}_{\mathcal{R}}(X)$.

Now the results of Section 4 can be transferred. We obtain the following analogue of Proposition 4.6 and Corollary 4.7.

Proposition 5.19. *Let $\mathcal{R}$ be a set of p-groups that is closed under taking subgroups. Let $K \lhd G$ such that $[K, K] = K$, $K \cap \mathrm{Z}(G) = \mathrm{Z}(K)$, $p \nmid |\mathrm{Z}(K)|$ and $K/\mathrm{Z}(K) \cong S^r$ for some non-abelian simple group S satisfying the inductive BAW condition with respect to $\mathcal{R}$. Then the following conditions hold:*

(a) *There exist subsets* $\mathrm{IBr}_{\mathcal{R}}(K \mid Q) \subset \mathrm{IBr}_{\mathcal{R}}(K)$ *for* $Q \in \mathrm{Rad}(K)$ *with*

(i) $\mathrm{IBr}_{\mathcal{R}}(K \mid Q)^a = \mathrm{IBr}_{\mathcal{R}}(K \mid Q^a)$ *for every* $Q \in \mathrm{Rad}(K)$ *and* $a \in G$,

(ii) $\mathrm{IBr}_{\mathcal{R}}(K) = \dot{\bigcup}_{Q \in \mathrm{Rad}(K)/\sim_K} \mathrm{IBr}_{\mathcal{R}}(K \mid Q)$.

(b) *For every* $Q \in \mathrm{Rad}(K)$ *there exists a bijection*

$$\Omega_Q^K : \mathrm{IBr}_{\mathcal{R}}(K \mid Q) \longrightarrow \mathrm{dz}(\mathrm{N}_K(Q)/Q, \mathrm{Bl}_{\mathcal{R}}(K))$$

with the following properties for every $\nu \in \mathrm{Irr}(\mathrm{Z}(K))$, $\chi \in \mathrm{IBr}_{\mathcal{R}}(K \mid Q)$ *and* $a \in G$*:*

- *all characters of* $\Omega_Q^K(\mathrm{IBr}_{\mathcal{R}}(K \mid Q) \cap \mathrm{IBr}(K \mid \nu^0))$ *lift to characters in* $\mathrm{Irr}(\mathrm{N}_K(Q) \mid \nu)$,
- $\Omega_{Q^a}^K(\chi^a) = \Omega_Q^K(\chi)^a$,
- $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^K$, *where* $\chi' \in \mathrm{Irr}(\mathrm{N}_K(Q))$ *is the lift of* $\Omega_Q^K(\chi)$.

(c) *Let* $\chi \in \mathrm{IBr}_{\mathcal{R}}(K)$ *with trivial* $\ker(\chi_{\mathrm{Z}(K)})$, $Q \in \mathrm{Rad}(K)$ *with* $\chi \in \mathrm{IBr}(K \mid Q)$ *and* $J := G_\chi$. *Let* $\chi' \in \mathrm{Irr}(\mathrm{N}_K(Q))$ *be the lift of* $\Omega_Q^K(\chi)$. *Then there exists a bijection*

$$\Pi_\chi : \mathrm{IBr}(J \mid \chi) \longrightarrow \mathrm{IBr}(\mathrm{N}_J(Q) \mid \chi'^0),$$

where $\mathrm{bl}(\theta) = \mathrm{bl}(\Pi_\chi(\theta))^J$ *for every* $\theta \in \mathrm{IBr}(J \mid \chi)$.

Proof. The proofs of Proposition 4.4 and Proposition 4.6 apply. A careful reading of the arguments shows that all occurring characters lie in blocks whose defect groups are in $\mathcal{R}$. Part (c) follows from the proof of Corollary 4.7. □

Using the above we prove the relative analogue of Theorem 5.3. This implies Theorem B.

Theorem 5.20. *Let* $\mathcal{R}$ *be a set of* p*-groups that is closed under taking subgroups and quotients and let* $\mathcal{S}_{\mathcal{R}}$ *be the set of all non-abelian simple groups that satisfy the inductive BAW condition with respect to* $\mathcal{R}$. *Let* G *be a finite group and* $B \in \mathrm{Bl}(G)$. *Assume that a defect group of* B *is contained in* $\mathcal{R}$ *and that every non-abelian simple group involved in* G *is contained in* $\mathcal{S}_{\mathcal{R}}$. *Then*

$$|\mathrm{IBr}(B)| = \sum_{R \in \mathrm{Rad}(G)/\sim_G} |\mathrm{dz}(\mathrm{N}_G(R)/R, B)|.$$

Proof. This follows from a transfer of the proof of Theorem 5.3. We only sketch some of the main necessary adaptations.

Note that blocks belonging to isomorphic block quadruples have isomorphic defect groups by Corollary 2.10. Note that the blocks $\overline{B}_i$ occurring in the proof of Proposition 5.5 have defect groups contained in the one of B. Hence they are contained in $\mathcal{R}$.

Another key ingredient is that the blocks in Γ_η and Γ'_η have defect groups that are subgroups of the one of B, and are again contained in $\mathcal{R}$.

When applying the inductive hypothesis in Lemma 5.10, the block $\overline{c}$ from the proof has a defect group contained in $\mathcal{R}$ as it is covered by one in Γ'_η. The remaining considerations can be made without any modifications. □

6 The inductive BAW condition for some simple groups

In this section we verify the inductive BAW condition from Definition 4.1 for various series of simple groups. Before considering specific groups, we show some helpful statements, which simplify the verification for non-abelian simple groups having additional properties. In [27, Sections 6–8] it was proven that several simple groups are AWC-good, especially all non-abelian finite simple groups having an abelian Sylow 2-subgroup and all simple groups of Lie type whenever the prime p is the characteristic of the underlying field. According to Remark 4.2 the inductive BAW condition is a refinement of AWC-goodness. We revisit the proofs of [27] and observe that the constructed bijections often have the compatibility property required in 4.1 (ii) (3). On the other hand in all situations the structure of the group of automorphisms implies that the extensions of the characters from Definition 4.1 (iii) (1)–(3) also satisfy Definition 4.1 (iii) (4).

Lemma 6.1. *Let p be a prime, S be a non-abelian simple group and X be its universal p'-covering group. Assume that S is AWC-good for p in the sense of* [27, Section 3] *and the used bijections also satisfy Definition* 4.1 (ii) (3). *Then the condition from Definition* 4.1 (iii) *holds for every character $\chi \in \mathrm{IBr}(X)$ with cyclic $\mathrm{Aut}(S)_\chi/S$. If moreover $\mathrm{Aut}(S)/S$ is cyclic, the inductive BAW condition holds for S and p.*

Proof. By assumption there exist subsets $\mathrm{IBr}(X \mid Q) \subseteq \mathrm{IBr}(X)$ for $Q \in \mathrm{Rad}(X)$ with the conditions from Definition 4.1 (i). Furthermore, there exist bijections $\Omega_Q^X : \mathrm{IBr}(X \mid Q) \to \mathrm{dz}(\mathrm{N}_X(Q)/Q)$ for $Q \in \mathrm{Rad}(X)$ with the required equivariance properties and such that $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$ for every $\chi \in \mathrm{IBr}(X \mid Q)$, where $\chi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ is the lift of $\Omega_Q^X(\chi)$.

Let $Q \in \mathrm{Rad}(X)$ and $\chi \in \mathrm{IBr}(X \mid Q)$. We first construct the group $A(\chi)$ for χ. As, by assumption, the group $\mathrm{Aut}(S)_\chi/S$ is cyclic there exists $\phi \in \mathrm{Aut}(X)_\chi$ with $\langle S, \phi\rangle = \mathrm{Aut}(S)_\chi$.

Let $Z := \ker(\chi_{\mathrm{Z}(X)})$, $\overline{X} := X/Z$ and $A_0 = \overline{X} \rtimes \langle\phi\rangle$, where ϕ is seen as automorphism of $\overline{X}$. As $\mathrm{Z}(\overline{X})$ is a p'-group, the group $A(\chi) := A_0/\mathrm{Z}(A_0)_p$ has the group $\overline{X}$ as normal subgroup, where $\mathrm{Z}(A_0)_p$ is the Sylow p-subgroup of $\mathrm{Z}(A_0)$. By straight-forward computations the group $A(\chi)$ has the group-theoretic properties from Definition 4.1 (iii) (1).

Let $A_{p'}$ be the group satisfying $\overline{X} \leq A_{p'} \leq A(\chi)$ such that $A_{p'}/\overline{X}$ is the Hall p'-group of $A(\chi)/\overline{X}$.

As $A(\chi)/\overline{X}$ is cyclic, it follows from [25, Theorem (8.12)] that the character χ extends to $A(\chi)$. Let $\chi' \in \operatorname{Irr}(\mathrm{N}_{\overline{X}}(Q))$ be associated to $\Omega_Q^{\overline{X}}(\chi)$. Because of the equivariance of $\Omega_Q^{\overline{X}}$ the character χ' is $\mathrm{N}_{A(\chi)}(Q)$-invariant. Hence there exists an extension $\widetilde{\chi}' \in \operatorname{IBr}(\mathrm{N}_{A_{p'}}(Q))$ of χ'^0.

Let $B = \mathrm{bl}(\chi)$ and $\widetilde{b} = \mathrm{bl}(\widetilde{\chi}')$. The block $\widetilde{b}^{A_{p'}}$ is defined according to [25, Theorem (4.14)]. Computing the values of the central function $\lambda_{\widetilde{b}^{A_{p'}}}$ on $A_{p'}$-conjugacy classes in $\overline{X}$ shows that $\widetilde{b}^{A_{p'}}$ covers B. Hence we can choose an extension $\widetilde{\chi} \in \operatorname{IBr}(\widetilde{b}^{A_{p'}})$ of χ such that $\mathrm{bl}(\widetilde{\chi}) = \mathrm{bl}(\widetilde{\chi}')^{A_{p'}}$.

Now let us consider the restrictions $\widetilde{\chi}_{\overline{X}H}$ and $\widetilde{\chi}'_H$ for any subgroup H with $\mathrm{N}_{\overline{X}}(Q) \leq H \leq \mathrm{N}_{A_{p'}}(Q)$. If $h \in \overline{X}H$, then $\mathfrak{Cl}_{A_{p'}}(h)$ is the disjoint union of the $\overline{X}H$-conjugacy classes $\mathfrak{Cl}_{\overline{X}H}(h_1), \dots, \mathfrak{Cl}_{\overline{X}H}(h_j)$ for some $h_1, \dots, h_j \in \overline{X}H$, where j is a p'-number because $p \nmid |A_{p'}/(\overline{X}H)|$. By definition of the central function we have

$$\lambda_{\widetilde{\chi}}(\mathfrak{Cl}_{A_{p'}}(h)^+) = j^* \lambda_{\widetilde{\chi}_{\overline{X}H}}(\mathfrak{Cl}_{\overline{X}H}(h_i)^+) \quad \text{for every } 1 \leq i \leq j.$$

We can determine $\lambda_{\widetilde{\chi}'_H}((\mathfrak{Cl}_{\overline{X}H}(h) \cap H)^+)$ using similar considerations:

$$\lambda_{\widetilde{\chi}'}((\mathfrak{Cl}_{A_{p'}}(h) \cap H)^+) = j^* \lambda_{\widetilde{\chi}'_H}((\mathfrak{Cl}_{\overline{X}H}(h_i) \cap H)^+) \quad \text{for every } 1 \leq i \leq j.$$

This implies that $\mathrm{bl}(\widetilde{\chi}_{\overline{X}H}) = \mathrm{bl}(\widetilde{\chi}'_H)^{\overline{X}H}$.

Let $\Upsilon(\chi)$ be an extension of $\widetilde{\chi}$ to $A(\chi)$ and $\Upsilon(\Omega_Q^X(\chi))$ an extension of $\widetilde{\chi}'$ to $\mathrm{N}_{A(\chi)}(Q)$. As $\mathrm{C}_{A(\chi)}(\overline{X})$ is a p'-group, the above implies

$$\operatorname{IBr}(\Upsilon(\chi)_{\mathrm{C}_{A(\chi)}(\overline{X})}) = \operatorname{IBr}(\Upsilon(\Omega_Q^X(\chi))_{\mathrm{C}_{A(\chi)}(\overline{X})}).$$

In order to verify that the characters $\Upsilon(\chi)$ and $\Upsilon(\Omega_Q^X(\chi))$ satisfy the equation in Definition 4.1 (iii) (4), let H be a group with $\mathrm{N}_{\overline{X}}(Q) \leq H \leq \mathrm{N}_{A(\chi)}(Q)$. Then $\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H})$ is the unique block of H covering

$$\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H \cap A_{p'}}),$$

and analogously $\mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)$ is the unique block covering

$$\mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_{H \cap A_{p'}}).$$

By [25, Theorem (4.14)] the block $\mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$ is defined, and considering the associated central functions implies that the block $\mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$ covers $\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H \cap A_{p'}})$. Hence

$$\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$$

and Definition 4.1 (iii) (4) holds for χ.

This shows that for χ, Definition 4.1 (iii) holds. In the case where $\operatorname{Aut}(S)/S$ is cyclic this proves that the inductive BAW condition holds for S and p. □

We next consider the inductive BAW condition with respect to $\mathcal{R}_c$, the set of all cyclic p-groups. We verify the inductive BAW condition with respect to $\mathcal{R}_c$ for S whenever S is a non-abelian simple group with cyclic $\mathrm{Aut}(S)/S$. For other non-abelian simple groups the first two parts of the BAW condition can be verified.

Proposition 6.2. *Let S be a finite non-abelian simple group with cyclic* $\mathrm{Aut}(S)/S$ *and $\mathcal{R}_c$ be the set of cyclic p-groups. Then S satisfies the inductive BAW condition with respect to $\mathcal{R}_c$.*

Proof. Let X be the universal p'-covering group of the given group S. For every cyclic $Q \in \mathrm{Rad}_{\mathcal{R}_c}(X)$ we define $\mathrm{IBr}_{\mathcal{R}_c}(X \mid Q)$ to be the set of all irreducible Brauer characters lying in a block with defect group Q.

Let $D \in \mathrm{Rad}_{\mathcal{R}_c}(X)$ and $B \in \mathrm{Bl}(X)$ be a block with defect group D. It is clear that the set $\mathrm{dz}(\mathrm{N}_X(Q)/Q, B)$ is empty for every group $Q \leq X$ that is not conjugate to some subgroup of D, see [25, Corollary (9.25)]. Let $\overline{\phi} \in \mathrm{dz}(\mathrm{N}_X(Q)/Q, B)$ and let $\phi \in \mathrm{Irr}(\mathrm{N}_X(Q))$ be its lift. Then $\mathrm{bl}(\phi)$ has D as defect group according to [25, Corollary (9.25)]. A block with cyclic defect group $\mathrm{bl}(\phi)$ contains only characters with height zero, see [10, Chapter VII, Theorem 2.16]. Accordingly the set $\mathrm{dz}(\mathrm{N}_X(Q)/Q, \mathrm{Bl}_{\{D\}}(X))$ is empty unless D and Q are isomorphic.

The proofs of [27, Theorem 7.1 and Corollary 7.2] show that there exists an $\mathrm{Aut}(X)_B$-equivariant bijection

$$\Omega_B : \mathrm{IBr}(B) \longrightarrow \mathrm{dz}(\mathrm{N}_X(D)/D, B).$$

Let $Q \in \mathrm{Rad}_{\mathcal{R}_c}(X)/{\sim_{\mathrm{Aut}(X)}}$ and let $\mathbb{T}$ be an $\mathrm{Aut}(X)_Q$-transversal of the set of blocks of X with defect group Q. Then we define

$$\Omega^X_{Q^a} : \mathrm{IBr}_{\mathcal{R}_c}(X \mid Q^a) \longrightarrow \mathrm{dz}(\mathrm{N}_X(Q^a)/Q^a, \mathrm{Bl}_{\mathcal{R}_c}(X))$$

by $\Omega^X_{Q^a}(\chi^a) = \Omega_B(\chi)^a$ for every $\chi \in \mathrm{Irr}(B)$, $a \in \mathrm{Aut}(X)$ and $B \in \mathbb{T}$. As Ω_B is $\mathrm{Aut}(X)_B$-equivariant, the maps satisfy

$$(\Omega^X_Q(\chi))^a = \Omega^X_{Q^a}(\chi^a)$$

for every $\chi \in \mathrm{IBr}_{\mathcal{R}_c}(X \mid Q)$ and $a \in \mathrm{Aut}(X)$. Furthermore, by the considerations from above the maps are bijections. Then the proof of Lemma 6.1 shows that S satisfies the inductive BAW condition with respect to $\mathcal{R}_c$. □

This statement can be applied to the following simple groups of Lie type of small rank and over a field of characteristic 2.

Corollary 6.3. *The inductive BAW condition holds for* $S \in \{{}^2\mathrm{B}_2(2^{2a+1}), \mathrm{SL}_2(2^n)\}$ *($a \geq 1$, $n \geq 2$) and primes $p > 2$.*

Proof. The group $\mathrm{SL}_2(4)$ is isomorphic to the alternating group $\mathfrak{A}_5$. Hence the statement for $\mathrm{SL}_2(4)$ follows from [21]. For the remaining simple groups $\mathrm{Out}(S)$ is cyclic. For $p > 2$ the statement follows from Proposition 6.2, as every Sylow p-subgroup $P \in \mathrm{Syl}_p(S)$ is cyclic. □

Also for the small Ree groups the inductive BAW condition can be verified.

Proposition 6.4. *The Ree simple groups* $S = {}^2\mathsf{G}_2(3^{2n+1})$ *(*$n \geq 1$*) satisfy the inductive BAW condition for* $p \neq 3$.

Proof. By [27, Proposition 8.4], the group S is AWC-good for every prime p. The Schur multiplier of S is trivial and accordingly the universal p'-covering group coincides with S. Note that $\mathrm{Aut}(S)/S$ is cyclic. According to Proposition 6.2 the inductive BAW condition holds with respect to $\mathcal{R}_c$. Since for $p > 3$ every Sylow p-subgroup of S is cyclic, this implies the inductive BAW condition for S and $p > 3$.

For $p = 2$ by Lemma 6.1 it is sufficient to verify that the sets $\mathrm{IBr}(X \mid Q)$ and the bijections Ω_Q^K from the proof of [27, Proposition 8.4] satisfy $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$ for every $Q \in \mathrm{Rad}(X)$ and $\chi \in \mathrm{IBr}(X \mid Q)$, where $\chi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ is the lift of $\Omega_Q^X(\chi)$. Accordingly $\mathrm{IBr}(X \mid Q)$ is the set of all Brauer characters of X lying in a block with defect group Q.

If $Q = 1$, the bijection Ω_Q^X is defined by $\Omega_Q^X(\psi^0) = \psi$ for every $\psi \in \mathrm{dz}(X)$. Obviously

$$\mathrm{bl}(\chi) = \mathrm{bl}(\Omega_Q^X(\chi))^X.$$

For $Q = \mathrm{C}_2$ and $Q = \mathrm{C}_2 \times \mathrm{C}_2$ the bijection Ω_Q^X given in the proof of [27, Proposition 8.4] satisfies by definition

$$\mathrm{bl}(\chi')^X = \mathrm{bl}(\chi) \quad \text{for every } \chi \in \mathrm{IBr}(X \mid Q),$$

where χ' denotes the lift of $\Omega_Q^X(\chi)$. For $Q \in \mathrm{Syl}_2(X)$ the set $\mathrm{IBr}(X \mid Q)$ is defined to be the Brauer characters contained in the principal block of X, as this is the only block with this defect group. Via Brauer correspondence we know that there is exactly one 2-block of $\mathrm{N}_X(Q)$ with defect group Q. Since

$$\Omega_Q^X(\mathrm{IBr}(X \mid Q)) = \mathrm{dz}(\mathrm{N}_X(Q)/Q),$$

every character of $\mathrm{dz}(\mathrm{N}_X(Q)/Q)$ lifts to one in the principal block of $\mathrm{N}_X(Q)$. As the principal block of $\mathrm{N}_X(Q)$ is Brauer's first main correspondent of the principal block of X, the required property of Ω_Q^X follows. □

In the next statement we adapt [27, Proposition 8.5].

Proposition 6.5. *The simple groups* $S = \mathrm{PSL}_2(q)$ *with* $q \equiv \pm 3 \mod 8$ *satisfy the inductive BAW condition for* $p \nmid q$.

Proof. In the case where $S = \mathrm{PSL}_2(5)$ the statement is known from Corollary 6.3, since $\mathrm{PSL}_2(5)$ and $\mathrm{SL}_2(4)$ are isomorphic.

By [27, Proposition 8.5], S is AWC-good for p. The group $X = \mathrm{SL}_2(q)$ is the universal p'-covering group of S whenever p is odd. The group $\mathrm{Out}(S)$ is cyclic. For $p > 2$ the statement holds by Proposition 6.2, as every Sylow p-subgroup of S is cyclic.

Let $p = 2$ and $q \geq 11$. The universal p'-covering group X of S coincides with S. For $Q \in \mathrm{Rad}(X)$ let $\mathrm{IBr}(X \mid Q)$ and Ω_Q^X be defined as in the proof of [27, Proposition 8.5]. According to Lemma 6.1, for the proof it is sufficient to verify

$$\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$$

for every $Q \in \mathrm{Rad}(X)$ and $\chi \in \mathrm{IBr}(X \mid Q)$, where $\chi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ is the lift of $\Omega_Q^X(\chi)$.

For $Q = 1$ this follows from $\Omega_Q^X(\psi^0) = \psi$ for every $\psi \in \mathrm{dz}(X)$.

For $Q = \mathrm{C}_2$ the set $\mathrm{IBr}(S \mid Q)$ is defined to be the Brauer characters lying in blocks whose defect group is isomorphic to C_2. We define $\psi_s \in \mathrm{IBr}(X \mid Q)$, $\delta_s \in \mathrm{dz}(\mathrm{N}_X(Q)/Q)$ and $g \in X$ as in part (iv) of the proof of [27, Proposition 8.5]. Straight-forward computations of the central function λ_{δ_s} and λ_{ψ_s} on the conjugacy class containing g^{2k} show that ψ_s and δ_s lie in Brauer corresponding blocks.

In the case where $Q = \mathrm{C}_2 \times \mathrm{C}_2$ the characters $\mathrm{IBr}(X \mid Q)$ are defined to be the Brauer characters of the principal block. The group $\mathrm{N}_X(Q)$ has only one 2-block, the principal one. This block contains all characters of $\mathrm{dz}(\mathrm{N}_X(Q)/Q)$ and hence $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$ for every $\chi \in \mathrm{IBr}(X \mid Q)$, whenever $\chi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ is the lift of some $\Omega_Q^X(\chi)$. □

Finally we consider the simple groups of Lie type which are defined over a field of characteristic p, and prove hereby Theorem C.

Proof of Theorem C. Let X be the universal p'-covering group of S. According to [5] the inductive BAW condition holds for the groups $\mathrm{Sp}_4(2)'$, ${}^2\mathsf{G}_2(2)'$, and ${}^2\mathsf{F}_4(2)'$ and every prime.

For every other simple group S of Lie type the exceptional part of the Schur multiplier is a p-group, see [11, Table 6.1.3]. Hence the group X coincides with $\mathbf{G}^F$ for some simply-connected simple algebraic group $\mathbf{G}$ and some Steinberg map $F : \mathbf{G} \to \mathbf{G}$, where $\mathbf{G}$ is defined over a field of characteristic p.

The simple group S is AWC-good for p according to [27, Theorem C]. Let $\mathrm{IBr}^{(NT)}(X \mid Q) \subset \mathrm{IBr}(X)$ be the subset and let $(\Omega_Q^X)^{(NT)}$ be the bijection for $Q \in \mathrm{Rad}(X)$ as introduced in the proof of this statement.

The Steinberg character $\mathrm{St} \in \mathrm{Irr}(X)$ is the unique defect zero character of X and hence $\mathrm{dz}(X) = \{\mathrm{St}\}$. Straight-forward computations show that the trivial Brauer character is contained in the set $\mathrm{IBr}^{(NT)}(X \mid 1)$. The trivial character is mapped via $(\Omega_1^X)^{(NT)}$ to St. According to [4] the character St^0 is contained in the set $\mathrm{IBr}^{(NT)}(X \mid U)$ for every $U \in \mathrm{Syl}_p(X)$ and $(\Omega_U^X)^{(NT)}(\mathrm{St}^0) = 1^0_{\mathrm{N}_X(U)/U}$.

For this proof we define the subsets of $\mathrm{IBr}(X)$ in the following way:

$$\mathrm{IBr}(X \mid Q) = \begin{cases} \{1_X\} \cup \mathrm{IBr}^{(NT)}(X \mid Q) \setminus \{\mathrm{St}^0\} & \text{if } Q \in \mathrm{Syl}_p(X), \\ \{\mathrm{St}^0\} & \text{if } Q = 1, \\ \mathrm{IBr}^{(NT)}(X \mid Q) & \text{otherwise.} \end{cases}$$

This partitioning has the properties required in Definition 4.1 (i) by definition, since the sets of $\mathrm{IBr}^{(NT)}(X \mid Q)$ have the analogous properties.

We define the map $\Omega_Q^X : \mathrm{IBr}(X \mid Q) \to \mathrm{dz}(\mathrm{N}_X(Q)/Q)$ for $Q \in \mathrm{Rad}(X)$ and $\chi \in \mathrm{IBr}(X \mid Q)$ by

$$\Omega_Q^X(\chi) = \begin{cases} \mathrm{St} & \text{if } \chi = \mathrm{St}^0, \\ 1_{\mathrm{N}_X(Q)/Q} & \text{if } \chi = 1_X^0, \\ (\Omega_Q^X)^{(NT)}(\chi) & \text{otherwise.} \end{cases}$$

For every $Q \in \mathrm{Rad}(X)$ the map Ω_Q^X is a bijection as $(\Omega_Q^X)^{(NT)}$ is one.

Let $Q \in \mathrm{Rad}(X)$ and $a \in \mathrm{Aut}(X)$. We observe that the bijection $(\Omega_Q^X)^{(NT)}$ satisfies

$$(\Omega_Q^X)^{(NT)}(\chi)^a = (\Omega_{Q^a}^X)^{(NT)}(\chi^a) \quad \text{for every } \chi \in \mathrm{IBr}^{(NT)}(X \mid Q).$$

As the characters St and $1_X \in \mathrm{Irr}(X)$ are $\mathrm{Aut}(X)$-invariant, this implies

$$\Omega_Q^X(\chi)^a = \Omega_{Q^a}^X(\chi^a) \quad \text{for every } Q \in \mathrm{Rad}(X) \text{ and } \chi \in \mathrm{IBr}^{(NT)}(X \mid Q).$$

In the next step we verify that for every $Q \in \mathrm{Rad}(X)$ and $\chi \in \mathrm{IBr}(X \mid Q)$ the equation

$$\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$$

holds, where $\chi' \in \mathrm{Irr}(\mathrm{N}_X(Q))$ is the lift of $\Omega_Q^X(\chi)$. Clearly for $Q = 1$ the characters St and St^0 belong to the unique block of X with defect zero. Any irreducible Brauer character $\chi \in \mathrm{IBr}(X)$ with $\chi \neq \mathrm{St}^0$ belongs to a block, whose defect group is a Sylow p-subgroup by [12] and $\chi \in \mathrm{IBr}(X \mid Q)$ for some non-trivial group Q. The block $\mathrm{bl}(\chi')^X$ also has positive defect. The central functions of $\mathrm{bl}(\chi')$ and $\mathrm{bl}(\chi)$ coincide on the center of X, since $\mathrm{IBr}(\chi_{\mathrm{Z}(X)}) = \mathrm{IBr}((\chi'^0)_{\mathrm{Z}(X)})$. For any $U \in \mathrm{Syl}_p(X)$ every p'-element in $\mathrm{C}_X(U)$ is central, according to [12, Remark 5]. By [25, Exercise (4.5)] this implies

$$\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X,$$

and we have $\mathrm{bl}(\chi) = \mathrm{bl}(\chi')^X$ for every $\chi \in \mathrm{IBr}(X \mid Q)$ with $1 \neq Q \in \mathrm{Rad}(X)$.

It remains to check the condition from Definition 4.1 (iii). If $\chi = \mathrm{St}^0$, the group $\mathrm{Z}(X)$ is contained in $\ker(\chi)$ and according to the proof of [27, Theorem C] there exists a group $A(\chi)$ with $S \lhd A(\chi)$ such that $A(\chi)/\mathrm{C}_{A(\chi)}(S) \cong \mathrm{Aut}(X)$ on S and St^0 extends to some character $\Upsilon(\mathrm{St}^0) \in \mathrm{IBr}(A(\chi))$. With $\Upsilon(\Omega_1^X(\mathrm{St}^0)) := \Upsilon(\mathrm{St}^0)$, Definition 4.1 (iii) holds for St^0.

Clearly, Definition 4.1 (iii) holds for the trivial Brauer character, in particular the trivial characters 1^0_{SH} and 1_H satisfy $\mathrm{bl}(1^0_{SH}) = \mathrm{bl}(1_H)^{SH}$ for every $\mathrm{N}_S(U) \leq H \leq \mathrm{N}_{\mathrm{Aut}(S)}(U)$, where U denotes a Sylow p-subgroup of S.

For all other characters χ we have to check that the extensions constructed in [27] also satisfy Definition 4.1 (iii) (4). Because of Lemma 6.1 we may assume that $\mathrm{Aut}(X)/S$ is non-cyclic. Hence X does not have a Suzuki or Ree automorphism. Let $\overline{X} := X/\ker(\chi_{\mathrm{Z}(X)})$.

In [27] for every $Q \in \mathrm{Rad}(X)$ and $\chi \in \mathrm{IBr}(X \mid Q)$ a group $A(\chi)$ and extensions $\Upsilon(\chi)$ and $\Upsilon(\Omega_Q^X(\chi))$ have been constructed. We have to show that for every $\mathrm{N}_{\overline{X}}(Q) \leq H \leq \mathrm{N}_{A(\chi)}(Q)$ we have

$$\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$$

where $\overline{X} := X/\ker(\chi_{\mathrm{Z}(X)})$. Because $\chi \neq \mathrm{St}^0$, some defect group of $\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H})$ and one of $\mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$ contains a Sylow p-subgroup U of X. By [25, Exercise (4.5)] it suffices to compare the central functions associated with the blocks $\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H})$ and $\mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}$ on the p'-conjugacy classes centralizing U. According to the description of $\mathrm{Aut}(X)$ given in [11, Theorem 2.5.1], the only p'-automorphisms centralizing U lie in $\mathrm{C}_{A(\chi)}(\overline{X})$. By the construction given in [27] we know

$$\mathrm{IBr}(\Upsilon(\chi)_{\mathrm{C}_{A(\chi)}(\overline{X})}) = \mathrm{IBr}(\Upsilon(\Omega_Q^X(\chi))_{\mathrm{C}_{A(\chi)}(\overline{X})})$$

and this implies

$$\mathrm{bl}(\Upsilon(\chi)_{\overline{X}H}) = \mathrm{bl}(\Upsilon(\Omega_Q^X(\chi))_H)^{\overline{X}H}.$$

Hence the condition described in Definition 4.1 (iii) holds for every character $\chi \in \mathrm{IBr}(X \mid Q)$. □

Together with a computational result of Breuer about the first Janko group this implies the following.

Corollary 6.6. *Let S be a non-abelian simple group with an abelian Sylow 2-subgroup. Then S satisfies the inductive BAW condition for any p.*

Proof. According to Walter's result in [3, 35] the group S has to be the Janko group J_1, a Ree group ${}^2\mathsf{G}_2(3^{2n+1})$ or a special linear group $\mathrm{PSL}_2(q)$ with $q = 2^n$ or $q \equiv \pm 3 \mod 8$. The inductive BAW condition holds for $S = \mathrm{PSL}_2(q)$ and

$p \mid q$ by Theorem C. The first Janko group satisfies the inductive BAW condition according to [5]. For the remaining simple groups the inductive BAW condition has been verified in Corollary 6.3, Proposition 6.4 and Proposition 6.5. □

Accordingly we can generalize [27, Theorem 8.7], which states that the non-blockwise version of the Alperin weight conjecture holds for G whenever its Sylow 2-subgroup is abelian. We prove that Conjecture 1.1 holds for G and p whenever G is a finite group whose non-abelian composition factors have abelian Sylow 2-subgroups.

Proof of Theorem D. By assumption every non-abelian simple group S involved in G has an abelian Sylow 2-subgroup and satisfies the inductive BAW condition for p, by Corollary 6.6. Then Conjecture 1.1 holds for G according to Theorem 5.3. □

Acknowledgments. The author would like to thank Burkhard Külshammer and Masafumi Murai for fruitful discussions, Thomas Breuer and Gunter Malle for reading a first version of this manuscript, and the referee for numerous suggestions improving the readability of the paper.

References

[1] J. L. Alperin, Weights for finite groups, in: *The Arcata Conference on Representations of Finite Groups* (Arcata 1986), Proceedings of Symposia in Pure Mathematics 47, American Mathematical Society, Providence (1987), 369–379.

[2] J. An and H. Dietrich, The AWC-goodness and essential rank of sporadic simple groups, *J. Algebra* **356** (2012), 325–354.

[3] H. Bender, On groups with abelian Sylow 2-subgroups, *Math. Z.* **117** (1970), 164–176.

[4] A. Borel, R. Carter, C. Curtis, N. Iwahori, T. Springer, R. Steinberg and C. Curtis, Modular representations of finite groups with split (B, N)-pairs, in: *Seminar on Algebraic Groups and Related Finite Groups*, Lecture Notes in Mathematics 131, Springer-Verlag, Berlin (1970), 57–95.

[5] T. Breuer, Ordinary character tables of normalizers of radical p-subgroups in central extensions of sporadic simple groups in GAP, in preparation, 2012.

[6] M. Cabanes, Two remarks on the reduction of Alperin's weight conjecture, *Bull. Lond. Math. Soc.*, to appear.

[7] C. Curtis and I. Reiner, *Methods of Representation Theory with Applications to Finite Groups and Orders*, Volume 1, John Wiley & Sons, New York, 1990.

[8] E. C. Dade, A correspondence of characters, in: *Finite Groups* (Santa Cruz 1979), Proceedings of Symposia in Pure Mathematics 37, American Mathematical Society, Providence (1980), 401–403.

[9] C. Eaton, B. Külshammer and B. Sambale, On 2-blocks with minimal nonabelian defect groups II, *J. Group Theory*, to appear.

[10] W. Feit, *The Representation Theory of Finite Groups*, North-Holland Mathematical Library 25, North-Holland, Amsterdam, 1982.

[11] D. Gorenstein, R. Lyons and R. Solomon, *The Classification of the Finite Simple Groups*, Number 3, Part I, Chapter A, American Mathematical Society, Providence, 1998.

[12] J. E. Humphreys, Defect groups for finite groups of Lie type, *Math. Z.* **119** (1971), 149–152.

[13] I. M. Isaacs, *Character Theory of Finite Groups*, Pure and Applied Mathematics 69, Academic Press, New York, 1976.

[14] I. M. Isaacs, *Finite Group Theory*, Graduate Studies in Mathematics 92, American Mathematical Society, Providence, 2008.

[15] I. M. Isaacs, G. Malle and G. Navarro, A reduction theorem for the McKay conjecture, *Invent. Math.* **170** (2007), 33–101.

[16] R. Kessar, S. Koshitani and M. Linckelmann, Conjectures of Alperin and Broué for 2-blocks with elementary abelian defect groups of order 8, *J. Reine Angew. Math.* **671** (2012), 85–130.

[17] R. Kessar and G. Malle, Quasi-isolated blocks and Brauer's height zero conjecture, preprint (2011), `http://arxiv.org/abs/1112.2642`.

[18] R. Knörr and G. Robinson, Some remarks on a conjecture of Alperin, *J. Lond. Math. Soc. (2)* **39** (1989), 48–60.

[19] B. Külshammer and L. Puig, Extensions of nilpotent blocks, *Invent. Math.* **102** (1990), 17–71.

[20] F. Ladisch, Character correspondences induced by magic representations, preprint (2010), `http://arxiv.org/abs/1004.4538`.

[21] G. Malle, Inductive Alperin–McKay and inductive Alperin weight condition, in preparation.

[22] M. Murai, Normal subgroups and heights of characters, *J. Math. Kyoto Univ.* **36** (1996), 31–43.

[23] M. Murai, Blocks of factor groups and heights of characters, *Osaka J. Math.* **35** (1998), 835–854.

[24] H. Nagao and Y. Tsushima, *Representations of Finite Groups*, Academic Press, Boston, 1989.

[25] G. Navarro, *Characters and Blocks of Finite Groups*, London Mathematical Society Lecture Note Series, Cambridge University Press, Cambridge, 1998.

[26] G. Navarro, Actions and characters in blocks, *J. Algebra* **275** (2004), 471–480.

[27] G. Navarro and P. Tiep, A reduction theorem for the Alperin weight conjecture, *Invent. Math.* **184** (2011), 529–565.

[28] L. Puig, Local extensions in endo-permutation modules split: A proof of Dade's theorem, in: *Séminaire sur les groupes finis*, Tome III, Publications Mathématiques de l'Université Paris VII 25, Paris (1987), 199–205.

[29] L. Puig, *Frobenius Categories Versus Brauer Blocks. The Grothendieck Group of the Frobenius Category of a Brauer Block*, Progress in Mathematics 274, Birkhäuser-Verlag, Basel, 2009.

[30] L. Puig, On the reduction of Alperin's conjecture to the quasi-simple groups, *J. Algebra* **328** (2011), 372–398.

[31] L. Puig, Note on the reduction of Alperin's conjecture, preprint (2011), `http://arxiv.org/abs/1109.4249`.

[32] L. Puig, Equivariant Alperin–Robinson's conjecture reduces to almost-simple k^*-groups, preprint (2012), `http://arxiv.org/abs/1203.3851`.

[33] B. Späth, Inductive McKay condition in defining characteristic, *Bull. Lond. Math. Soc.* **44** (2012), 426–438.

[34] B. Späth, A reduction theorem for the Alperin–McKay conjecture, *J. Reine Angew. Math.*, to appear.

[35] J. H. Walter, The characterization of finite groups with abelian Sylow 2-subgroups, *Ann. of Math. (2)* **89** (1969), 405–514.

Received October 10, 2011; revised September 7, 2012.

Author information

Britta Späth, FB Mathematik, TU Kaiserslautern,
Postfach 3049, 67653 Kaiserslautern, Germany.
E-mail: `spaeth@mathematik.uni-kl.de`