A characterization of
hypercyclically embedded subgroups
using cover-avoidance property

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Abstract. A normal subgroup $K$ of a finite group $G$ is said to be hypercyclically embedded in $G$ if every chief factor of $G$ below $K$ is cyclic. A subgroup $H$ has the cover-avoidance property in $G$ if $H$ either covers or avoids every chief factor of $G$. In this paper we connect these two concepts and give a new characterization of normal hypercyclically embedded subgroups. Our main result is that a normal subgroup $K$ is hypercyclically embedded in $G$ if and only if the members of a certain class of subgroups of $K$ have the cover-avoidance property in $G$.

1 Introduction

All groups considered in this paper are finite. We use standard notions and notation, as in [3]. The letter $G$ always denotes a finite group, $|G|$ the order of $G$, $\pi(G)$ the set of all primes dividing $|G|$, $G_p$ a Sylow $p$-subgroup of $G$ for any prime $p \in \pi(G)$. The symbol $\mathcal{U}$ denotes the class of all supersoluble groups.

A normal subgroup $K$ is said to be hypercyclically embedded in $G$ if every chief factor of $G$ below $K$ is cyclic, see [15, p.217]. The product of all normal hypercyclically embedded subgroups of $G$ is denoted by $Z_{\mathcal{U}}(G)$ and is called the $\mathcal{U}$-hypercentre of $G$, see [3]. It is easy to see that $Z_{\mathcal{U}}(G)$ itself is hypercyclically embedded in $G$ and any normal subgroup $N$ of $G$ is hypercyclically embedded in $G$ if and only if $N \leq Z_{\mathcal{U}}(G)$. The concept of hypercyclical embedding is strongly connected to the formation of supersoluble groups and it has very good properties. For instance, $G$ is supersoluble if and only if $G$ is hypercyclically embedded in $G$ itself. More generally, if $G/N$ is a supersoluble group, then $G$ is supersoluble if and only if $N$ is hypercyclically embedded in $G$. In 2011, Skiba [16] gave a characterization of normal hypercyclically embedded subgroups related to

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S-quasinormal subgroups. In this paper, we will give a new characterization by using the well-known cover-avoidance property.

**Definition 1.1.** Let $L/K$ be a chief factor of $G$ and $H$ a subgroup of $G$. We say that

(i) $H$ covers $L/K$ if $L \leq HK$,

(ii) $H$ avoids $L/K$ if $L \cap H \leq K$,

(iii) $H$ has the cover-avoidance property in $G$ or $H$ is a CAP-subgroup of $G$ for short, if $H$ either covers or avoids every chief factor of $G$.

As a generalization of the cover-avoidance property, Fan, Guo and Shum [5] defined semi-cover-avoidance as follows:

**Definition 1.2 ([5]).** A subgroup $H$ of $G$ is said to have the semi-cover-avoidance property in $G$ if there exists a chief series of $G$,

$$1 = G_0 \leq G_1 \leq \cdots \leq G_n = G,$$

such that $H$ either covers or avoids the chief factor $G_{i+1}/G_i$ for each index $i \in \{0, 1, \ldots, n-1\}$. In this case $H$ is called an SCAP-subgroup ([13]) of $G$ or a partial CAP-subgroup ([2]) of $G$.

**Lemma 1.3 ([14, Proposition 13]).** A subgroup $U$ is a CAP-subgroup of $G$ if and only if $U$ either covers or avoids each chief factor of $G$ in the section $U^G/U_G$.

The cover-avoidance property of subgroups was first studied by Gaschütz in [7] to study solvable groups and later by Gillam [8] and Tomkinson [17].

Recently, some authors have used the cover-avoidance property to characterize the class of all supersoluble groups. For instances, Ezquerro [4] shows that $G$ is supersoluble if and only if every maximal subgroup of every Sylow-subgroup has the cover-avoidance property. Later, from [10] we can see that if every minimal subgroup of $G$ and every cyclic subgroup of order 4 have the semi-cover-avoidance property, then $G$ is also supersoluble.

In this paper, we will focus on the relation between the cover-avoidance property and normal hypercyclically embedded subgroups. To reveal this relationship, we first present the following proposition:

**Proposition 1.4.** Suppose that $N$ is a normal subgroup of $G$. If $N$ is hypercyclically embedded in $G$, then every subgroup of $N$ has the cover-avoidance property in $G$.

**Proof.** Let $L$ be a subgroup of $N$ and let $H/K$ be a chief factor of $G$ below $N$. First we show that $L$ must either cover or avoid $H/K$. From the definition of $N$ be-
ing hypercyclically embedded in $G$, we know that $|H/K| = p$ for some prime $P$. Since $H/K$ is a minimal subgroup of $G/K$, it is easy to see that either

$$H/K \leq LK/K \quad \text{or} \quad H/K \cap LK/K = 1,$$

that is, $L$ either covers or avoids $H/K$. Hence $L$ either avoids or covers every chief factor in the section $N/1$. It is obvious that $L^G \leq N$ and $L_G \geq 1$. By applying Lemma 1.3 to the subgroup $L$, we see that $L$ is a CAP-subgroup of $G$.

It is natural to ask whether the converse of Proposition 1.4 is true. That is, if every subgroup of a normal subgroup $N$ has the cover-avoidance property in $G$, is $N$ necessarily hypercyclically embedded in $G$? We will see that the answer is yes. In fact, the main object of this paper is to show that instead of requiring that all subgroups are CAP-subgroups of $G$, one only needs to require that certain classes of subgroups of $N$ enjoy the cover-avoidance property in order to obtain that $N$ is hypercyclically embedded in $G$.

We will prove the following main theorem and give some applications.

**Main Theorem.** Let $L$ be a normal subgroup of $G$. Then $L \leq Z_u(G)$ if and only if there exists a normal subgroup $E$ of $G$ contained in $L$ such that $F^*(L) \leq E$ and $E$ satisfies the following properties: for every non-cyclic Sylow $p$-subgroup $E_p$ of $E$, all subgroups of $E_p$ with a fixed order $d_p$ ($1 < d_p < E_p$, $d_p \mid |E_p|$) and all cyclic groups of $E_p$ with order 4 (if $d_p = 2$ and $E_2$ is non-abelian) have the cover-avoidance property in $G$.

## 2 Preliminaries

The following lemma is evident.

**Lemma 2.1.** Let $A$, $B$ and $N$ be normal subgroups of $G$ with $N \leq A$.

1. If $A$ is hypercyclically embedded in $G$, then $N$ is hypercyclically embedded in $G$.

2. If $A/N$ is hypercyclically embedded in $G/N$ and $N$ is hypercyclically embedded in $G$, then $A$ is hypercyclically embedded in $G$.

3. If $A$ is a cyclic group, then $A$ is hypercyclically embedded in $G$.

Denote by $\mathcal{A}(p-1)$ the formation of all abelian groups of exponent divisible by $p-1$.

**Lemma 2.2** ([18, Theorem 1.4]). Let $H/K$ be a chief factor of $G$ and $p$ be a prime divisor of $|H/K|$. Then $|H/K| = p$ if and only if $G/C_G(H/K) \in \mathcal{A}(p-1)$. 

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Let \( f \) be a formation function and \( N \) be a normal subgroup of \( G \). We say that \( G \) acts \( f \)-centrally on \( N \) if \( G/C_G(H/K) \in f(p) \) for every chief factor \( H/K \) of \( G \) below \( N \) and every prime \( p \) dividing \(|H/K|\), see [3, p. 387, Definitions 6.2]. Define a formation function \( f_U(p) \) by setting \( f_U(p) = A(p-1) \) for all prime \( p \). From Lemma 2.2 we can prove without difficulty that \( N \) is hypercyclically embedded in \( G \) if and only if \( G \) acts \( f_U \)-centrally on \( N \). By applying [3, p. 388, Theorem 6.7], we get the following useful result:

**Lemma 2.3.** Let \( K \) be a normal subgroup of \( G \). Then \( K \leq Z_U(G) \) if and only if \( K/\Phi(K) \leq Z_U(G/\Phi(K)) \).

**Lemma 2.4** ([14, Proposition 10]). Let \( S \) be a CAP-subgroup of \( G \) and \( N \) be a normal subgroup of \( G \). Then \( SN/N \) is a CAP-subgroup of \( G/N \).

**Lemma 2.5** ([12, Lemma 2.3]). Every minimal normal subgroup of \( G \) is a minimal CAP-subgroup of \( G \).

**Lemma 2.6** ([10, Lemma 2.5]). If \( S \) be a SCAP subgroup of \( G \) and \( S \leq K \leq G \), then \( S \) is a SCAP-subgroup of \( K \).

**Lemma 2.7** ([10, Theorem 3.8]). Let \( G \) be a finite group and \( p \) be the minimal prime divisor of \(|G|\). If every cyclic subgroup with order \( p \) (or 4 if \( p = 2 \)) is a SCAP-subgroup of \( G \), then \( G \) is \( p \)-nilpotent.

**Lemma 2.8.** Let \( G \) be a group and \( P \) be a normal \( p \)-subgroup of \( G \). Suppose that every cyclic subgroup of \( P \) of order \( p \) (or 4 if \( p = 2 \) and \( P \) is non-abelian) is a CAP-subgroup of \( G \). Then \( P \leq Z_U(G) \).

**Proof.** Suppose that this is false and \((G, P)\) is a counter-example with \(|G||P|\) minimal. Let \( P/K \) be a chief factor of \( G \). Obviously \((G, K)\) satisfies the hypothesis. By the minimal choice of \((G, P)\), we know that \( K \leq Z_U(G) \) and \( P/K \) is not a cyclic group. If there is some element \( x \in P\setminus K \) with \( o(x) = p \) or \( o(x) = 4 \) (if \( p = 2 \) and \( P \) is non-abelian), then \( \langle x \rangle \) is a CAP-subgroup of \( G \) by the hypotheses. By Lemma 2.4 and Lemma 2.5, we have \( P/K = \langle x \rangle K/K \), a contradiction. Therefore all cyclic subgroups of order \( p \) or order 4 (if \( p = 2 \) and \( P \) is non-abelian) of \( P \) are contained in \( K \). Suppose that we have a chief series of \( G \) below \( P \) as follows:

\[
1 = U_0 \leq U_1 \leq \cdots \leq U_n = K \leq P.
\]

Then \(|U_i/U_{i-1}| = p \) for \( i \in \{1, 2, \ldots, n\} \) since \( K \leq Z_U(G) \). Thus

\[
G/C_G(U_i/U_{i-1}) \in A(p-1)
\]
by Lemma 2.2. Write

\[ X = \bigcap_{i=1}^{n} C_G(U_i/U_{i-1}). \]

Then \( G/X \in \mathcal{A}(p - 1) \) since \( \mathcal{A}(p - 1) \) is a formation. Let \( Q \) be a \( q \)-subgroup of \( X \) (\( q \neq p \)). Then \( Q \) centralizes \( K \) by [9, p. 178, Theorem 3.2] and thus \( Q \) centralizes all elements of \( P \) of order \( p \) and order 4 (if \( p = 2 \) and \( P \) is non-abelian). Then \( Q \) centralizes \( P \) by [6, Theorem 2.4]. Thus \( X/C_X(P/K) \) is a \( p \)-group. Then \( XC_G(P/K)/C_G(P/K) \) is a normal \( p \)-subgroup of \( G/C_G(P/K) \). Hence

\[ XC_G(P/K)/C_G(P/K) = 1 \]

by [18, Corollary 6.4, p. 221]. So \( X \leq C_G(P/K) \). We have

\[ G/C_G(P/K) \in \mathcal{A}(p - 1) \]

since \( G/X \in \mathcal{A}(p - 1) \). Again by Lemma 2.2, we know that \( |P/K| = p \), a final contradiction. 

\[ \square \]

**Lemma 2.9** ([11, IV, Satz 2.8]). Let \( p \) be the minimal prime divisor of \( |G| \). If the Sylow \( p \)-subgroups of \( G \) are cyclic, then the \( G \) is a \( p \)-nilpotent group.

**Lemma 2.10** ([16, Theorem C]). Let \( E \) be a normal subgroup of \( G \). If \( F^*(E) \) is hypercyclically embedded in \( G \), then \( E \) is hypercyclically embedded in \( G \).

### 3 Proof of the main result

In order to present the proof of our main result in a more compact way, we first prove the following lemmas. They are used in the inductive arguments of the proof of the main theorem.

**Lemma 3.1.** Let \( E \) be a normal subgroup of \( G \). Let \( E_p \) be a Sylow \( p \)-subgroup of \( E \). Let \( N \) be a minimal normal subgroup of \( G \) contained in \( E \) such that \( p \) divides \( |N| \). If every subgroup of \( E_p \) with order \( d_p \) (\( 1 < d_p < |E_p|, d_p \mid |E_p| \)) is a CAP-subgroup in \( G \), then \( N \) is a \( p \)-group and either \( |N| = d_p = p \) or \( d_p > |N| \).

**Proof.** Suppose that \( N \) is not a \( p \)-group. Take \( x \in N \cap E_p \) with \( o(x) = p \), and choose a subgroup \( K \) contained in \( E_p \) with order \( d_p \) and \( \langle x \rangle \subseteq K \). From the hypothesis we know that \( K \) is a CAP-subgroup of \( G \). It is clear that \( \langle x \rangle \in K \cap N \neq 1 \), and \( N \not\subseteq K \) because \( N \) is not a \( p \)-group. Therefore \( K \) can neither cover nor avoid the chief factor \( N/1 \) of \( G \). This contradiction shows that \( N \) must be a \( p \)-group.

By Lemma 2.5, it is easy to see that \( d_p \geq |N| \). To prove Lemma 3.1, we only need to show that if \( d_p = |N| \), then \( d_p = p \). Suppose that this is not true and \( |N| = d_p > p \). Since \( |E_p| > |d_p| = |N| \), we can choose a subgroup \( M \) of \( E_p \).
such that $N$ is a maximal subgroup of $M$. Since $N$ is not cyclic, $M$ is not cyclic either, hence we can choose a maximal subgroup $K$ of $M$ other than $N$. It is clear that $M = NK$ and $|K| = |N| = d_p$, thus $K$ is a CAP-subgroup of $G$ by hypothesis. If $N \cap K = 1$, then $|M| = |N|^2$ and $|N| = |N|^2/|N| = |M|/|N| = p$, a contradiction to our assumption; therefore $N \cap K \neq 1$. But $N \notin K$ since $N$ and $K$ are two different maximal subgroups of $M$. Consequently $K$ can neither cover nor avoid the chief factor $N/1$ of $G$. This contradiction shows $|N| = d_p = p$. □

**Lemma 3.2.** Let $E$ be a normal subgroup of $G$ and $E_p$ be a Sylow $p$-subgroup of $E$. Let $N$ be a minimal normal subgroup of $G$ contained in $E$ such that $p \mid |N|$. If every subgroup of $E_p$ with order $d_p$ ($1 < d_p < |E_p|$, $d_p \mid |E_p|$) is a CAP-subgroup of $G$ and $N \notin \Phi(E)$, then $|N| = p$.

**Proof.** From Lemma 3.1 we already know that $N$ is a $p$-group. Suppose that this lemma is not true and assume $|N| > p$. Since $N \notin \Phi(E)$, we can choose a maximal subgroup $M$ of $E$ such that $E = NM$. Hence

$$E_p = E_p \cap NM = N(E_p \cap M).$$

Clearly $(E_p \cap M) < E_p$ since $N$ is not contained in $M$, so we can choose a maximal subgroup $K$ of $E_p$ such that $E_p \cap M \leq K$. Note that now $E_p = NK$. If $N \cap K = 1$, then by a simple calculation we know that $|N| = p$, which contradicts our assumption. Hence $N \cap K > 1$. But $N \cap K < N$ since $N$ is not contained in $K$. Clearly $|N| \leq d_p \leq |K|$, hence we can choose a subgroup $H$ with order $d_p$ such that $1 < N \cap K < H \leq K$. As in Lemma 3.1, it is easy to verify that $H$ can neither cover nor avoid the chief factor $N/1$, a contradiction to the hypothesis. □

**Lemma 3.3.** Let $E$ be a normal subgroup of $G$ and $E_2$ be a Sylow 2-group of $E$. Let $N$ be a minimal normal subgroup of $G$ contained in $E$ such that $2 \mid |N|$. If every subgroup of $E_2$ of order 4 is a CAP-subgroup of $G$, then every subgroup of $E_2$ of order 2 is also a CAP-subgroup of $G$.

**Proof.** By Lemma 3.1, $N$ is a 2-subgroup. As $d_2 = 4 > 2$, we know that $d_2 > |N|$ by Lemma 3.1 and thus $|N| = 2$. Take a subgroup $A$ of $E_2$ of order 2. We are now going to show that $A$ is a CAP-subgroup of $G$. If $A = N$, then $A$ is obviously a CAP-subgroup of $G$. Hence we may assume that $A \cap N = 1$. Write $B = A \times N$. Then $B$ is a subgroup of order 4, thus $B$ is a CAP-subgroup of $G$ by hypothesis. Let $H/K$ be a chief factor of $G$. If $B$ avoids $H/K$, say, $B \cap H = B \cap K$, then $A \cap H = A \cap (B \cap H) = A \cap (B \cap K) = (A \cap B) \cap K = A \cap K$, which means that $A$ avoids $H/K$.

If $B$ covers $H/K$, we show that in this case we will have $|H/K| = 2$. Suppose that $|H/K| > 2$, by the fact that $B$ covers $H/K$ we have that $H/K \leq BK/K$ and
thus $|H/K|$ is a divisor of $|BK/K|$. But $|B| = 4$ and $|H/K| > 2$, so

$$|H/K| = |BK/K| = 4$$

and hence $H/K = BK/K$. If $N \leq K$, then

$$|H/K| = |BK/K| = |(A \times N)K/K| = |AK/K| \leq 2,$$

a contradiction. Thus $N \cap K = 1$ and $NK/K$ is a normal subgroup of order 2 strictly contained in $H/K$, contrary to the fact that $H/K$ is a chief factor of $G$. This contradiction shows that $|H/K| = 2$.

Using similar arguments to the ones in the proof of Proposition 1.4, we know that in this case every subgroup of $G$ must either cover or avoid $H/K$ and so does $A$. The arbitrary choice of $H/K$ shows that $A$ is a CAP-subgroup of $G$. 

The following proposition is useful in the proof of our main theorem. Also, the proposition itself is of independent interest. Suppose that $p$ is the smallest prime divisor of $|G|$. If we take the normal subgroup $E$ in the following proposition to be $G$ itself, then we conclude that $G$ is $p$-nilpotent if and only if the members of a certain class of subgroups contained in $G_p$ have the cover-avoidance property in $G$.

**Proposition 3.4.** Let $E$ be a normal subgroup of $G$, $p$ be the minimal prime dividing $E$, and $E_p$ be a Sylow $p$-subgroup of $E$. If either $E$ is cyclic or every subgroup of $E_p$ of order $d_p$ ($1 < d_p < |E_p|$, $d_p \mid |E_p|$) and every cyclic subgroup of $E_p$ with order 4 (if $d_p = 2$ and $E_2$ is non-abelian) has the cover-avoidance property, then $E$ is $p$-nilpotent.

**Proof.** Suppose this proposition is not true and let $(G, E)$ be a counter-example for which $|G||E|$ is minimal. If $E_p$ is cyclic, then $E$ is $p$-nilpotent by Lemma 2.9, hence we may assume that $E_p$ is not cyclic. Let $N$ be a minimal normal subgroup of $G$ contained in $E$; then by our hypothesis and Lemma 2.5, we know that $N < E$.

**Step 1.** $O_{p'}(E) = 1$.

If $O_{p'}(E) \neq 1$, then obviously $(G/O_{p'}(E), E/O_{p'}(E))$ still satisfies the hypothesis of this proposition, hence $E/O_{p'}(E)$ is $p$-nilpotent by the minimal of $|G||E|$, which implies $E$ is $p$-nilpotent, a contradiction to the choice of $(G, E)$.

**Step 2.** $N$ is a 2-group.

This follows from Step 1 and Lemma 3.1.

**Step 3.** If $d_p > p$ (if $p \neq 2$) or $d_p > 4$ (if $p = 2$), then $E/N$ is $p$-nilpotent.

We show that in this case $(G/N, E/N)$ still satisfies the hypothesis of the proposition, thus $E/N$ is $p$-nilpotent by the minimal choice of $(G, E)$. Since $d_p > p$, by Lemma 3.1, we know that $d_p > |N|$. 

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If \( p \neq 2 \) or \( p = 2 \) and \( d_p/|N| \geq 4 \) or \( p = 2 \) and \( E_p/N \) is abelian, let
\[
d'_p = \frac{d_p}{|N|};
\]
then it is easy to verify that every subgroup of \( E_p/N \) of order \( d'_p \) (note that in this case if \( p = 2 \) and \( E_p/N \) is non-abelian, then \( d'_p \geq 4 \)) is a CAP-subgroup of \( G/N \).

Hence the hypothesis still holds for \((G/N, E/N)\) in this case.

\begin{itemize}
\item If \( p = 2 \), \( d_p/|N| = 2 \) and \( E_p/N \) is non-abelian, take a cyclic subgroup \( K = N \) of \( E_p/N \) of order 4. Since \( d_p > 4 \) and \( d_p/|N| = 2 \), we have \( |N| > 2 \). Hence \( N \) is not a cyclic subgroup and neither is \( K \). If \( N \leq \Phi(K) \), then \( K \) is a cyclic group since \( K/N \) is a cyclic group, a contradiction. Thus \( N \not\subset \Phi(K) \), and we can find a maximal subgroup \( M \) of \( K \) such that \( K = MN \).
\end{itemize}

Hence the hypothesis still holds for \((G/N, E/N)\) in this case.

**Step 4.** Final contradiction.

First we will show that \( d_p = p \) or \( d_p = 4 \). If not, then from Step 3 we know that \( E/N \) is \( p \)-nilpotent and thus \( N \not\subset \Phi(E) \) since \( E \) is not \( p \)-nilpotent. But then by Lemma 3.2, we know \( |N| = p \) at this time, which implies \( N \leq Z(E) \) since \( p \) is the smallest divisor of \( |E| \). Again, this will lead to \( E \) being \( p \)-nilpotent, contradicting the choice of \((G, E)\).

By Lemma 3.3, we know that if \( d_p = 4 \), then every subgroup of \( E_2 \) of order 2 or 4 is a CAP-subgroup of \( G \). Hence every minimal subgroup of \( E \) and every cyclic subgroup of \( E \) with order 4 (if \( d_p = 2 \) and \( E_2 \) is non-abelian) of \( E_p \) is a CAP-subgroup of \( G \), and so it is an SCAP-subgroup of \( E \) by Lemma 2.3. From Lemma 2.7, \( E \) is \( p \)-nilpotent, the final contradiction.

\[ \square \]

**Proof of the Main Theorem.** The necessity has already been proved in Proposition 1.4. In order to prove that \( L \leq Z_{\mathcal{U}}(G) \), we will first show that \( E \leq Z_{\mathcal{U}}(G) \) under the hypothesis of the Main Theorem. Suppose that this is not true and consider a counter-example \((G, E)\) for which \( |G||E| \) is smallest. Let \( p \) be the minimal prime divisor of \( E \). Let \( N \) be a minimal normal subgroup of \( G \) contained in \( E \). We will get a contradiction in the following steps:

**Step 1.** \( E \) is a \( p \)-group.

By Proposition 3.4, we know that \( E \) is \( p \)-nilpotent. Hence \( O_{p'}(E) \) is the Hall \( p' \)-subgroup of \( E \). Therefore the hypothesis holds for \((G, O_{p'}(E))\). Then, by Lemma 2.4, the hypothesis also holds for \((G/O_{p'}(E), E/O_{p'}(E))\). Consequently, if \( O_{p'}(E) \neq 1 \),
then, by the minimal choice of \((G, E)\), we have that

\[ E/O_p'(E) \leq Z_\mathcal{U}(G/O_p'(E)) \quad \text{and} \quad O_p'(E) \leq Z_\mathcal{U}(G). \]

By Lemma 2.1 (2), we have that \(E \leq Z_\mathcal{U}(G)\), a contradiction.

**Step 2.** \(E\) is not cyclic and \(N < E\).

If \(E\) is cyclic, then \(E \leq Z_\mathcal{U}(G)\) by Lemma 2.1 (3), contrary to the choice of \((G, E)\). Thus \(E\) is not cyclic by the hypothesis. Therefore some subgroup strictly contained in \(E\) is a CAP-subgroup in \(G\). By Lemma 2.5, \(E\) is not a minimal normal subgroup of \(G\), so \(N < E\).

**Step 3.** If \(d_p > p\) (if \(p \neq 2\)) or \(d_p > 4\) (if \(p = 2\)), then \(E/N \leq Z_\mathcal{U}(G/N)\).

Using the same argument as in the proof of Step 3 of Proposition 3.4, we know that \((G/N, E/N)\) satisfies our hypothesis under the given circumstances, hence \(E/N \leq Z_\mathcal{U}(G/N)\) by the minimal choice of \((G, E)\).

**Step 4.** Final contradiction.

First suppose that \(d_p > p\) (if \(p \neq 2\)) or \(d_p > 4\) (if \(p = 2\)). By Step 3, we have that

\[ E/N \leq Z_\mathcal{U}(G/N). \]

If \(N \leq \Phi(E)\), then

\[ E \leq Z_\mathcal{U}(G) \]

by Lemma 2.3. If \(N \not\leq \Phi(E)\), then \(|N| = p\) by Lemma 3.2; thus \(N \leq Z_\mathcal{U}(G)\) by Lemma 2.1 (3) and \(E \leq Z_\mathcal{U}(G)\) by Lemma 2.1 (2). These contradictions show that \(d_p = p\) or \(d_p = 4\).

By Lemma 3.3, every minimal subgroup and every cyclic subgroup with order 4 (if \(d_p = 2\) and \(E_2\) is non-abelian) of \(E\) is a CAP-subgroup of \(G\). By Step 1 and Lemma 2.8, we know that \(E \leq Z_\mathcal{U}(G)\), a final contradiction.

Now we will show that \(L \leq Z_\mathcal{U}(G)\). Note that \(F^*(L) \leq E\), by Lemma 2.1 (1) and \(E \leq Z_\mathcal{U}(G)\) we have

\[ F^*(L) \leq Z_\mathcal{U}(G). \]

Hence \(L \leq Z_\mathcal{U}(G)\) by Lemma 2.10.

**Remark.** In the Main Theorem, the cover-avoidance property cannot be replaced by the semi-cover-avoidance property. For instance, \(G\) is not necessarily a supersoluble group even if for every non-cyclic Sylow \(p\)-subgroup \(G_p\) of \(G\), all subgroups of \(G_p\) with a fixed order \(d_p\) (\(1 < d_p < |G_p|\), \(d_p \mid |G_p|\)) and all cyclic subgroups of \(G_p\) with order 4 (if \(d_p = 2\) and \(G_2\) is non-abelian) have the semi-cover-avoidance property in \(G\). For an example, we refer the reader to [1, Example 1.2].
4 Some applications

The first three corollaries below follow directly from our main theorem.

Corollary 4.1. A group $G$ is supersoluble if and only if for every non-cyclic Sylow $p$-subgroup $E_p$ of $G$, all subgroups of $G_p$ with a fixed order $d_p$ ($1 < d_p < |G_p|$, $d_p \mid |G_p|$) and all cyclic subgroups of $G_p$ with order 4 (if $d_p = 2$ and $G_2$ is non-abelian) have the cover-avoidance property in $G$.

Corollary 4.2. Let $L$ be a normal subgroup of $G$. Then $L \leq Z\mathcal{U}(G)$ if and only if for every non-cyclic Sylow $p$-subgroup $L_p$ of $L$, all subgroups of $L_p$ with a fixed order $d_p$ ($1 < d_p < |L_p|$, $d_p \mid |L_p|$) and all cyclic subgroups of $L_p$ with order 4 (if $d_p = 2$ and $L_2$ is non-abelian) have the cover-avoidance property.

Corollary 4.3. Let $L$ be a normal subgroup of $G$. Then $L \leq Z\mathcal{U}(G)$ if and only if for every non-cyclic Sylow $p$-subgroup $F^*(L)_p$ of $F^*(L)$, all subgroups of $F^*(L)_p$ with a fixed order $d_p$ ($1 < d_p < |F^*(L)_p|$, $d_p \mid |F^*(L)_p|$) and all cyclic subgroups of $F^*(L)_p$ with order 4 (if $d_p = 2$ and $F^*(L)_2$ is non-abelian) have the cover-avoidance property.

Corollary 4.4. Let $\mathcal{F}$ be a saturated formation containing the class of all supersoluble groups $\mathcal{U}$ and let $L$ be a normal subgroup of $G$ such that $G/L \in \mathcal{F}$. If for every non-cyclic Sylow $p$-subgroup $L_p$ of $L$, all subgroups of $L_p$ with a fixed order $d_p$ ($1 < d_p < |L_p|$, $d_p \mid |L_p|$) and all cyclic subgroups of $L_p$ with order 4 (if $d_p = 2$ and $L_2$ is non-abelian) have the cover-avoidance property, then $G \in \mathcal{F}$. If $\mathcal{F} = \mathcal{U}$, then the converse is also true.

Proof. Since $\mathcal{F}$ is a saturated formation, there exits a unique full and integrated formation function $h$ such that $\mathcal{F} = LF(h)$ (see [3, IV, Theorem 3.7]). Since $\mathcal{U} \subseteq \mathcal{F}$, we have $f_\mathcal{U}(p) \subseteq h(p)$ for all primes $p$ ([3, IV, Proposition 3.11]). Hence $Z\mathcal{U}(G) \leq Z\mathcal{F}(G)$. We already know that $L \leq Z\mathcal{U}(G)$ in our Main Theorem, thus

$$L \leq Z\mathcal{U}(G) \leq Z\mathcal{F}(G).$$

But $G/L \in \mathcal{F}$ by our assumption and consequently $G \in \mathcal{F}$.

If $G \in \mathcal{U}$, then every subgroup of $G$ has the cover-avoidance property. Hence the converse is also true. $\square$

Corollary 4.5. Let $\mathcal{F}$ be a saturated formation containing the class of all supersoluble groups $\mathcal{U}$ and let $L$ be a normal subgroup of $G$ such that $G/L \in \mathcal{F}$. If for every non-cyclic Sylow $p$-subgroup $F^*(L)_p$ of $F^*(L)$, all subgroups of $F^*(L)_p$ with a fixed order $d_p$ ($1 < d_p < |F^*(L)_p|$, $d_p \mid |F^*(L)_p|$) and all cyclic sub-
groups of $F^*(L)_p$ with order 4 (if $d_p = 2$ and $F^*(L)_2$ is non-abelian) have the cover-avoidance property, then $G \in \mathcal{F}$. Furthermore, if $\mathcal{F} = \mathcal{U}$, then the converse is also true.

Proof. The proof is similar to that of Corollary 4.4.

Corollary 4.6. Let $E$ be a normal subgroup of $G$. Let $p$ be the smallest prime divisor of $|E|$ and assume that $G/E$ is $p$-supersoluble. Let $E_p$ be a Sylow $p$-subgroup of $E$. If either $E_p$ is cyclic or every subgroup of $E_p$ of order $d_p$ and every cyclic subgroup of $E_p$ with order 4 (if $p = 2$ and $d_p = 2$) have the cover-avoidance property, then $G$ is $p$-supersoluble. In particular, if $p$ is also the smallest prime divisor of $|G|$, then $G$ is $p$-nilpotent.

Proof. Suppose that this corollary is not true and let $(G, E)$ be a counter-example for which $|G||E|$ is minimal. Using a similar argument to the one in the proof of Step 1 of Proposition 3.4, we have $O_p^*(E) = 1$. This implies that $E$ is a $p$-group since $E$ is $p$-nilpotent by Proposition 3.4. Because $E$ is a $p$-group, it follows that $E$ satisfies the hypothesis of our Main Theorem. As a consequence, every chief factor of $G$ below $E$ has order $p$. But $G/E$ is $p$-supersoluble, thus $G$ is $p$-supersoluble, a contradiction.

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