Idempotents of the Norton–Sakuma algebras

Alonso Castillo-Ramirez

Communicated by Christopher W. Parker

Abstract. The concept of Majorana representation was introduced by A. A. Ivanov (2009), as a tool to identify and study subalgebras of the Conway–Griess–Norton Monster algebra $V_M$. Sakuma’s theorem in “6-transposition property of $\tau$-involutions of vertex operator algebras”, Int. Math. Res. Not. IMRN 2007 (2007), Article ID rnm030, states that there are eight possibilities for the isomorphism type of an algebra with scalar product generated by a pair of distinct Majorana axes. These algebras, now known as the Norton–Sakuma algebras, were described by S. P. Norton (1982) as 2-generated subalgebras of $V_M$ and labelled by types $2A$, $2B$, $3A$, $3C$, $4A$, $4B$, $5A$ and $6A$. In the present paper, we contribute to the understanding of the Norton–Sakuma algebras by finding all their idempotent elements and their automorphism groups. In particular, we find that an algebra of type $4A$ has infinitely many idempotents of length 2, and that an algebra of type $6A$ has exactly 208 idempotents.

1 Introduction

Let $M$ be the Monster group, the largest of the sporadic simple groups. The group $M$ contains two conjugacy classes of involutions (elements of order 2), denoted by $2A$ and $2B$. These conjugacy classes are labelled according to the isomorphism type of the centraliser of their elements. If $t$ is an involution in the class $2A$, it is called a Monster transposition and

$$C_M(t) \cong 2 \cdot BM$$

where $C_M(t)$ is the centraliser of $t$ in $M$ and $BM$ is the Baby Monster sporadic simple group. If $z$ is an involution in the class $2B$, then

$$C_M(z) \cong 2_{+}^{1+24} Co_1$$

where $Co_1$ is the largest Conway sporadic simple group and $2_{+}^{1+24}$ is an extraspecial group.

This research was supported by the Universidad de Guadalajara and an Imperial College International Scholarship.
Robert Griess proved the existence of $M$ in 1982, constructing a 196,883-dimensional commutative non-associative algebra, commonly known as the *Griess algebra.* Conway simplified the construction of the Griess algebra in [1] and he adjoined a 1-dimensional trivial direct summand. Hence, he defined a 196,884-dimensional algebra $V_M$ that is known as the *Conway–Griess–Norton algebra.*

The theory of Majorana representations is an important tool in studying subalgebras of $V_M$. Inspired by the results of Sakuma in [13], Ivanov defined the concept of a Majorana representation in his book [2]. Roughly speaking, given a finite group $G$, a Majorana representation of $G$ is a commutative non-associative algebra with inner product that satisfies certain axioms. If $G$ is a subgroup of $M$ generated by the $2A$ involutions contained in $G$ and if a Majorana representation of $G$ is isomorphic to a subalgebra of $V_M$, then we say that the obtained representation of $G$ is based on an embedding in $M$.

The Majorana representations of many groups have already been obtained. Sakuma’s theorem implies that the Majorana representations of the dihedral groups $D_n$ ($n \leq 6$) are the so-called Norton–Sakuma algebras. In [5], Ivanov, Pasechnik, Seress and Shpectorov showed that the symmetric group $S_4$ has exactly four Majorana representations, all of them based on an embedding in $M$. After this, Majorana representations of several groups were obtained: the projective special linear group $L_3(2)$ in [7], the alternating group $A_5$ in [6], and $A_6$ and $A_7$ in [3,4]. Other groups with 2-closed Majorana representations were discussed [14].

The Norton–Sakuma algebras play a pivotal role in the construction and study of Majorana representations. A deep understanding of these algebras is an important factor for the further development of Majorana theory. The present paper determines and organizes all the idempotent elements of the Norton–Sakuma algebras within the framework of Majorana theory. The results are then used to calculate the automorphism group of each algebra. The treatment in this paper is purely elementary and does not require any further knowledge of $M$ or $V_M$.

### 2 Majorana representations

Let $G$ be a finite group. We say that a subset $T$ of $G$ is $G$-stable if $g^{-1}Tg = T$ for any $g \in G$. It is straightforward to prove that $T$ is $G$-stable if and only if $T$ is the union of some conjugacy classes of $G$. If $g, h \in G$, denote the image of $g$ under the conjugation by $h$ as $g^h$.

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1 Results about these idempotents for the low-dimensional algebras were obtained in [8, Appendix A], in the context of vertex operator algebras. The results in the present paper were obtained independently, using a different context and algorithm for the solution of the systems of equations.
Let $V$ be a real vector space. Recall that a positive definite inner product on $V$ is a symmetric bilinear map $(\cdot, \cdot): V \times V \to \mathbb{R}$ such that $(v, v) \geq 0$ for any $v \in V$, with equality if and only if $v = 0$. In this situation, the non-negative real number $l(v) := (v, v)$ is called the length of $v$. Recall also that a commutative algebra product on $V$ is a bilinear map of the form $\cdot : V \times V \to V$ such that $u \cdot v = v \cdot u$ for any $u, v \in V$. Say that an element $x \in V$ is idempotent if $x \cdot x = x$. Observe that any $v \in V$ induces a linear transformation $\text{ad}_v : V \to V$ defined by

$$\text{ad}_v(u) = v \cdot u \quad \text{for } u \in V.$$ 

We call $\text{ad}_v$ the adjoint transformation of $v$. If $X$ is a subset of $V$, we denote by $\langle X \rangle$ the subalgebra generated by $X$, i.e., the smallest subalgebra in $V$ containing $X$.

Now we are ready to present the definition of a Majorana representation of a finite group.

**Definition 2.1.** Let $G$ be a finite group. Let $V$ be a real vector space equipped with a positive definite inner product and a non-associative commutative algebra product. Let $T$ be a $G$-stable set of generating involutions of $G$. Let $\varphi : G \to \text{GL}(V)$ be a representation of $G$ on $V$ whose image preserves both products $\cdot$ and $(\cdot, \cdot)$ (this means that $v^{\varphi(g)} \cdot u^{\varphi(g)} = (v \cdot u)^{\varphi(g)}$ and $(v^{\varphi(g)}, u^{\varphi(g)}) = (v, u)$ for any $u, v \in V, g \in G$). Finally, let $\psi : T \to V \setminus \{0\}$ be an injective mapping such that

$$\psi(t^g) = \psi(t)^{\varphi(g)}.$$

The tuple

$$(G, T, V, (\cdot, \cdot), \varphi, \psi)$$

is called a Majorana representation of $G$ if the following eight conditions are satisfied:

(M1) The inner product $(\cdot, \cdot)$ associates with $\cdot$ in the sense that

$$(u, v \cdot w) = (u \cdot v, w)$$

for all $u, v, w \in V$.

(M2) The Norton inequality holds; this means that

$$(u \cdot u, v \cdot v) \geq (u \cdot v, u \cdot v)$$

for all $u, v \in V$.

(M3) The elements of $\psi(T)$ are idempotents of length 1.

(M4) The adjoint transformation of any $a \in \psi(T)$ has eigenvalues $1, 0, \frac{1}{2^2}$ and $\frac{1}{2^5}$. Furthermore, if $V^{(a)}_\mu$ is the set of $\mu$-eigenvectors, then

$$V = V_{1}^{(a)} \oplus V_{0}^{(a)} \oplus V_{\frac{1}{2^2}}^{(a)} \oplus V_{\frac{1}{2^5}}^{(a)}.$$
(M5) For any \( a \in \psi(T) \), we have that \( V_1^{(a)} = \{ \lambda a : \lambda \in \mathbb{R} \} \).

(M6) For any \( a \in \psi(T) \), the linear transformation \( \tau(a) \) of \( V \) defined via

\[
\tau(a) : u \mapsto (-1)^{2^5 \mu} u
\]

for \( \mu \in \{ 1, 0, \frac{1}{2^2}, \frac{1}{2^3} \} \) and \( u \in V^{(a)}_\mu \) preserves the algebra product in \( V \).

(M7) If \( a \in \psi(T) \) and

\[
V_+^{(a)} := V_1^{(a)} \oplus V_0^{(a)} \oplus V_\frac{1}{2^2}^{(a)}
\]

is the centraliser of \( \tau(a) \) in \( V \), then the linear transformation \( \sigma(a) \) of \( V_+^{(a)} \) defined via

\[
\sigma(a) : u \mapsto (-1)^{2^2 \mu} u
\]

for \( \mu \in \{ 1, 0, \frac{1}{2^2} \} \) and \( u \in V^{(a)}_\mu \) preserves the restriction of the algebra product to \( V_+^{(a)} \).

(M8) For any \( t \in T \),

\[
\tau(\psi(t)) = \varphi(t).
\]

The elements of \( \psi(T) \) are called Majorana axes while the automorphisms in \( \varphi(T) \) are called Majorana involutions.\(^2\) In general, if \( x \in T \), we denote the vector \( \psi(x) \) by \( a_x \).

### 2.1 Norton–Sakuma algebras

It is known that the Monster \( \mathbf{M} \) is generated by its transpositions. Furthermore, if \( t \) and \( g \) are transpositions in \( \mathbf{M} \), the product \( tg \) has order at most 6 and is contained in one of the following conjugacy classes of \( \mathbf{M} \): \( 1A, 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A \). The number in the label of each class denotes the order of \( tg \). Conway showed in [1] that there exists an injective map \( \psi' \) between the set of transpositions of \( \mathbf{M} \) and a set of idempotents in \( V_{\mathbf{M}} \). The elements in the image of \( \psi' \) are called axes. Norton gave in [10] an explicit description of the subalgebras of \( V_{\mathbf{M}} \) generated by a pair of axes. We shall call these subalgebras the Norton–Sakuma algebras. The following proposition, deduced from [10], gives the description of the Norton–Sakuma algebras.

**Proposition 2.2.** Let \( t, g \in \mathbf{M} \) be distinct transpositions and suppose that \( \rho = tg \) is in the conjugacy class \( NX \), where \( NX \in \{ 2A, 2B, 3A, 3C, 4A, 4B, 5A, 6A \} \). Set \( a_t = \psi'(t) \) and \( a_{g_i} = \psi'(t\rho^i) \), where \( i \in \mathbb{N} \). Then the subalgebra \( \langle a_t, a_{g_i} \rangle \) of \( V_{\mathbf{M}} \) is isomorphic to the Norton–Sakuma algebra of type \( NX \) described in Table 1.

\(^2\) The Majorana involutions in \( V_{\mathbf{M}} \) can be seen as restrictions of the Miyamoto involutions of the Moonshine module [9].
<table>
<thead>
<tr>
<th>Type</th>
<th>Basis</th>
<th>Products</th>
</tr>
</thead>
</table>
| 2A   | $a_t, a_g, a_{\rho}$ | $a_t \cdot a_g = \frac{1}{2^7} (a_t + a_g - a_{\rho})$,
$a_t \cdot a_{\rho} = \frac{1}{2^3} (a_t + a_{\rho} - a_g)$,
$(a_t, a_g) = (a_t, a_{\rho}) = (a_g, a_{\rho}) = \frac{1}{2^3}$ |
| 2B   | $a_t, a_g$ | $a_t \cdot a_g = 0$, $(a_t, a_g) = 0$ |
| 3A   | $a_t, a_g, a_{g-1}, u_{\rho}$ | $a_t \cdot a_g = \frac{1}{2^5} (2a_t + 2a_g + a_{g-1}) - \frac{3^3 5}{217} u_{\rho}$,
$a_t \cdot u_{\rho} = \frac{1}{2^7} (2a_t - a_g - a_{g-1}) + \frac{5}{2^7} u_{\rho}$,
$u_{\rho} \cdot u_{\rho} = u_{\rho}$,
$(a_t, a_g) = \frac{13}{2^5}, (a_t, u_{\rho}) = \frac{1}{2^7}, (u_{\rho}, u_{\rho}) = \frac{3^3 5}{2}$ |
| 3C   | $a_t, a_g, a_{g-1}$ | $a_t \cdot a_g = \frac{1}{2^6} (a_t + a_g - a_{g-1})$, $(a_t, a_g) = \frac{1}{2^6}$ |
| 4A   | $a_t, a_g, a_{g-1}, a_{g_2}, v_{\rho}$ | $a_t \cdot a_g = \frac{1}{2^6} (3a_t + 3a_g + a_{g_2} + a_{g-1} - 3v_{\rho})$
$a_t \cdot v_{\rho} = \frac{1}{2^7} (5a_t - 2a_g - a_{g_2} - 2a_{g-1} + 3v_{\rho})$
$v_{\rho} \cdot v_{\rho} = v_{\rho}$,
$(a_t, a_g) = 0$, $(a_t, v_{\rho}) = \frac{1}{2^7}, (v_{\rho}, v_{\rho}) = 2$ |
| 4B   | $a_t, a_g, a_{g-1}, a_{g_2}, a_{\rho^2}$ | $a_t \cdot a_g = \frac{1}{2^6} (a_t + a_g - a_{g-1} - a_{g_2} + a_{\rho^2})$
$a_t \cdot a_{g_2} = \frac{1}{2^3} (a_t + a_{g_2} - a_{\rho^2})$
$(a_t, a_g) = \frac{1}{2^5}, (a_t, a_{g_2}) = (a_t, a_{\rho^2}) = \frac{1}{2^3}$ |
| 5A   | $a_t, a_g, a_{g-1}, a_{g_2}$, $a_{g-2}, w_{\rho}$ | $a_t \cdot a_g = \frac{1}{2^7} (3a_t + 3a_g - a_{g_2} - a_{g-1} - a_{g_2}) + w_{\rho}$
$a_t \cdot a_{g_2} = \frac{1}{2^7} (3a_t + 3a_{g_2} - a_{g_1} - a_{g-1} - a_{g_2}) - w_{\rho}$
$a_t \cdot w_{\rho} = \frac{7}{219} (a_g + a_{g-1} - a_{g_2} - a_{g-2}) + \frac{7}{2^7} w_{\rho}$
$w_{\rho} \cdot w_{\rho} = \frac{5^2 7}{219} (a_{g_2} + a_{g-1} + a_t + a_g + a_{g_2})$
$(a_t, a_g) = \frac{3}{2^7}, (a_t, w_{\rho}) = 0$, $(w_{\rho}, w_{\rho}) = \frac{5^2 7}{219}$ |
| 6A   | $a_t, a_g, a_{g-1}, a_{g_2}$, $a_{g-2}, a_{g_3}, a_{\rho^3}, u_{\rho^2}$ | $a_t \cdot a_g = \frac{1}{2^6} (a_t + a_g - a_{g-2} - a_{g_1} - a_{g_2} - a_{g_3} + a_{\rho^3}) + \frac{3^2 5}{217} u_{\rho^2}$
$a_t \cdot a_{g_2} = \frac{1}{2^8} (2a_t + 2a_{g_2} + a_{g-2}) - \frac{3^3 5}{217} u_{\rho^2}$
$a_t \cdot u_{\rho^2} = \frac{1}{2^6} (2a_t - a_{g_2} - a_{g_2}) + \frac{5}{2^7} u_{\rho^2}$
$a_t \cdot a_{g_3} = \frac{1}{2^5} (a_t + a_{g_3} - a_{\rho^3}), a_{\rho^3} \cdot u_{\rho^2} = 0$
$(a_t, a_g) = \frac{5}{2^8}, (a_t, a_{g_2}) = \frac{13}{2^8}, (a_t, a_{g_3}) = \frac{1}{2^7},$
$(a_{\rho^3}, u_{\rho^2}) = 0$ |

Table 1. Norton–Sakuma algebras.
Observe that Table 1 does not contain all pairwise inner and algebra products of the basis vectors. The missing products can be obtained using the symmetries of the algebras and their mutual inclusions. These inclusions are implicit in [10], and we describe them in the following proposition.

**Proposition 2.3.** Regarding the Norton–Sakuma algebras of Table 1, we have:

1. An algebra of type $4A$ has subalgebras of type $2B$ with bases $\{a_t, a_g\}$ and $\{a_g, a_{g-1}\}$.
2. An algebra of type $4B$ has subalgebras of type $2A$ with bases $\{a_t, a_{g_2}, a_{\rho^2}\}$ and $\{a_g, a_{g-1}, a_{\rho^2}\}$.
3. An algebra of type $6A$ has subalgebras of type $2A$ with bases $\{a_t, a_{g_3}, a_{f^3}\}$, $\{a_g, a_{g-2}, a_{f^3}\}$ and $\{a_{g_2}, a_{g-1}, a_{f^3}\}$, and subalgebras of type $3A$ with bases $\{a_t, a_{g_2}, a_{g-2}, a_{f^2}\}$ and $\{a_g, a_{g-1}, a_{g_3}, a_{f^2}\}$.

The scaling of the products in Table 1 is different from the scaling used by Norton in [10]. The inner product of Norton is $16$ times the inner product in Table 1. Moreover, Norton’s vectors $t_0, u, v$ and $w$ are $64$, $90$, $192$ and $8192$ times the vectors $a_0, u_\rho, v_\rho$ and $w_\rho$ in Table 1 respectively.

Sakuma’s theorem in [13] implies the following proposition.

**Proposition 2.4.** Let $(G, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)$ be a Majorana representation of a finite group $G$ and $t, g \in T$ be distinct involutions. Let $\rho = tg$ and $g_i = t^p i$, $i \in \mathbb{N}$. Let $D$ be the dihedral subgroup of $G$ of order $2N$ generated by $t$ and $g$. Then the subalgebra $\langle a_t, a_g \rangle$ of $V = (V, (\cdot, \cdot), \cdot)$ is isomorphic to a Norton–Sakuma algebra of type $NX$, with $X \in \{A, B, C\}$.

A proof of Proposition 2.4 using the language of Majorana representations may be found in [5]. From this point, we work with the Norton–Sakuma algebras in the context of Majorana theory. Proposition 2.4 implies that $(D, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)$ defines a Majorana representation, where $D$ is the dihedral group of order $2N$ and $(V, (\cdot, \cdot), \cdot)$ is a Norton–Sakuma algebra of type $NX$. For the cases $2A$, $3A$, $3C$ and $5A$, the set $T$ contains all the involutions of $D$. For the case $2B$, the set $T$ contains only two involutions. For the cases $4A$, $4B$ and $6A$, we found it convenient to define $T$ as the set of all involutions but the central involution. This implies that for an algebra of type $4B$ (or $6A$) the basis element $a_{\rho^2}$ (or $a_{\rho^3}$) is not considered a Majorana axis, although it satisfies the properties (M3)–(M8) of Definition 2.1.

In general, if $V$ is a commutative algebra, we say that $V$ has an identity if there is $\text{id} \in V$ such that $\text{id} \cdot v = v$ for all $v \in V$. Clearly, any commutative algebra may have at most one identity. Conway showed in [1, Section 17] that any subalgebra of $V_M$ has an identity, which is the orthogonal projection of the identity of $V_M$. We
calculate explicitly the identities of the Norton–Sakuma algebras in Table 2, where \( a \) denotes the sum of the Majorana axes of the algebra.

<table>
<thead>
<tr>
<th>Type</th>
<th>Identity</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>2A</td>
<td>( \frac{2^2}{5}a )</td>
<td>( \frac{2^3}{5} )</td>
</tr>
<tr>
<td>2B</td>
<td>( a )</td>
<td>2</td>
</tr>
<tr>
<td>3A</td>
<td>( \frac{2^4}{3 \cdot 7}a + \frac{3^2}{7}u_\rho )</td>
<td>( \frac{2^2}{5 \cdot 7} )</td>
</tr>
<tr>
<td>3C</td>
<td>( \frac{2^5}{3 \cdot 11}a )</td>
<td>( \frac{2^5}{11} )</td>
</tr>
<tr>
<td>4A</td>
<td>( \frac{2^2}{5}a + \frac{2}{5}v_\rho )</td>
<td>4</td>
</tr>
<tr>
<td>4B</td>
<td>( \frac{2^5}{5}a + \frac{3}{5}a_\rho^2 )</td>
<td>( \frac{19}{5} )</td>
</tr>
<tr>
<td>5A</td>
<td>( \frac{2^5}{5 \cdot 7}a )</td>
<td>( \frac{2^5}{7} )</td>
</tr>
<tr>
<td>6A</td>
<td>( \frac{2}{3}a + \frac{1}{2}a_\rho^3 + \frac{3}{8}u_\rho^2 )</td>
<td>( \frac{3 \cdot 17}{2 \cdot 5} )</td>
</tr>
</tbody>
</table>

Table 2. Identities of the Norton–Sakuma algebras.

It is clear that the identity of any algebra is idempotent. For the Norton–Sakuma algebras, we also have the following useful result.

**Proposition 2.5.** Let \( x \) be an idempotent of a Norton–Sakuma algebra of type \( NX \) with identity \( \text{id} \). Then

\[
l(x) = (\text{id}, x).
\]

Moreover, \( \text{id} - x \) is an idempotent with \( l(\text{id} - x) = l(\text{id}) - l(x) \).

**Proof.** By (M1) of Definition 2.1,

\[
l(x) = (x, x) = (\text{id} \cdot x, x) = (\text{id}, x \cdot x) = (\text{id}, x).
\]

The rest of the proof is straightforward.

\[\square\]

### 2.2 Automorphism groups

In the present paper we are also interested in the automorphism groups of the Norton–Sakuma algebras. Denote by \( \text{Aut}(NX) \) the automorphism group of the algebra \( V \) of type \( NX \), so \( \text{Aut}(NX) \) is the set of all invertible linear transformations of \( V \) that preserve the inner and algebra product. In this section, we are going to
show the existence of certain subgroups $G_{NX}$ of Aut($NX$) which play an important role in the rest of the paper.

Let $(D,T,V,(\cdot,\cdot),\cdot,\varphi,\psi)$ be a Majorana representation of the dihedral group of order $2N$, so $V$ is a Norton–Sakuma algebra of type $NX$. Let $\Omega$ be the set of Majorana axes in $V$. When $N = 2$, it is easy to check from Table 1 that any permutation of $\Omega$ induces an automorphism of $V$, so define

$$G_{2X} = \text{Sym}(\Omega) \quad \text{with } X \in \{A,B\}.$$  

Since $\varphi$ is an representation of $D \cong D_{2N}$ on $V$ that preserves both products, we have that $\varphi(D) \leq \text{Aut}(NX)$. Moreover, because of the relation $\psi(tg) = \psi(t)\varphi(g)$, the kernel of $\varphi$ coincides with the centre of $D$. For $N = 3$, the kernel of $\varphi$ is trivial, so $\varphi(D) \cong D \cong S_3$. Define

$$G_{3X} = \varphi(D) \quad \text{with } X \in \{A,C\}.$$  

When $N = 4$, we have that $\ker \varphi \cong C_2$, so $\varphi(D) \cong C_2 \times C_2$. The group $\varphi(D)$ is generated by the Majorana involutions

$$\varphi(t) = (a_g,a_{g-1}) \in \text{Sym}(\Omega) \quad \text{and} \quad \varphi(g) = (a_t,a_{g_2}) \in \text{Sym}(\Omega).$$

Now, using Table 1 it is possible to show that

$$\phi = (a_t,a_g)(a_{g-1},a_{g_2}) \in \text{Sym}(\Omega)$$

induces an automorphism of $V$ not in $\varphi(D)$. Hence, define

$$G_{4X} = \langle \varphi(D), \phi \rangle \quad \text{with } X \in \{A,B\}.$$  

Note that $\varphi(g) = \varphi(t)^{\phi}$, so $G_{4X} = \langle \varphi(t), \phi \rangle \cong D_8$.

For $N = 5$, the group $\varphi(D) \cong D_{10}$ is generated by the Majorana involutions

$$\varphi(t) = (a_g,a_{g-1})(a_{g_2},a_{g_2}) \quad \text{and} \quad \varphi(g) = (a_t,a_{g_2})(a_{g-1},a_{g-2}).$$

Again, with Table 1, we verify that there is an automorphism $\xi$ of $V$ which acts on $\Omega$ as $(a_g,a_{g_2},a_{g-1},a_{g-2})$ and permutes $w_\rho$ and $-w_\rho$. Clearly, $\xi$ is not in $\varphi(D)$. Define

$$G_{5A} = \langle \varphi(D), \xi \rangle.$$  

For this group we have the following result.

**Proposition 2.6.** The group $G_{5A}$ defined above is isomorphic to the Frobenius group of order 20.
Proof. Recall that the Frobenius group of order 20 has presentation
\[ \langle c, f \mid c^5 = f^4 = 1, cf = fc^2 \rangle. \]
Writing the automorphisms in \( G_{5A} \) as elements in \( \text{GL}_6(\mathbb{R}) \), we can see that
\[ c = \varphi(t)\varphi(g) \]
is an element of order 5 and
\[ f = \varphi(g)\xi \]
is an element of order 4 such that \( cf = fc^2 \). Moreover, the elements \( c \) and \( f \)
generate the whole of \( G_{5A} \) because \( \varphi(t) = f^2c, \varphi(g) = f^2c^2 \) and \( \xi = fcf^2 \). The result follows.

Finally, if \( N = 6 \), we have that \( \ker \varphi \cong C_2 \) and \( \varphi(D) \cong D_6 \). In this case, \( \varphi(D) \)
is generated by the Majorana involutions
\[ \varphi(t) = (a_g, a_{g-1})(a_{g_2}, a_{g-2}) \quad \text{and} \quad \varphi(g) = (a_t, a_{g_2})(a_{g-1}, a_{g_3}) \]
so it is possible to show that
\[ \xi = (a_t, a_g)(a_{g-1}, a_{g_2})(a_{g-2}, a_{g_3}) \in \text{Sym}(\Omega) \]
induces an automorphism of \( V \) not in \( \varphi(D) \). Define
\[ G_{6A} = \langle \varphi(D), \xi \rangle. \]
It is clear that \( G_{6A} \cong D_{12} \) because \( \varphi(t) \) and \( \xi \) are two involutions that generate \( G_{6A} \) (since \( \varphi(g) = \varphi(t)^\xi \)) such that \( \varphi(t)\xi \) has order 6.

The following lemma is a key result that we use in order to determine the automorphism groups of the Norton–Sakuma algebras.

**Lemma 2.7.** Let \((D, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)\) be a Majorana representation of the dihedral group of order \(2N\), so \( V \) is a Norton–Sakuma algebra of type \( NX \), with \( N \neq 5 \). Let \( \Omega = \psi(T) \) and suppose that \( H_{NX} \) is a subgroup of \( \text{Aut}(NX) \) which contains \( G_{NX} \) and permutes \( \Omega \). Then \( H_{NX} = G_{NX} \).

Proof. Consider the graph \( \Gamma_{NX} \) with vertices \( \Omega \) and edges
\[ \{\{a, b\} : a, b \in \Omega, V = \langle a, b \rangle\}. \]
Since \( \langle \Omega \rangle \) is always equal to \( V \), the group \( H_{NX} \) acts faithfully over \( \Gamma_{NX} \), so \( H_{NX} \leq \text{Aut}(\Gamma_{NX}) \). For an algebra of type \( NX \in \{3A, 3C, 4A, 4B, 6A\} \), the graph \( \Gamma_{NX} \) is a cycle with \( N \) vertices. This implies that \( \text{Aut}(\Gamma_{NX}) \cong D_{2N} \) for \( N \neq 2, 5 \), so the result follows for these cases. Similarly, we can see that \( \text{Aut}(\Gamma_{2B}) \cong C_2 \) and \( \text{Aut}(\Gamma_{2A}) \cong S_3 \).
3 Idempotents of the Norton–Sakuma algebras

In this section we obtain and classify all the idempotents of the Norton–Sakuma algebras of type $NX$. Then we use these results to determine the automorphism group of each algebra.

Let $(D, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)$ be a Majorana representation of the dihedral group of order $2N$, as defined in Section 2.1. If $v \in V$, denote by $[v]$ the $G_{NX}$-orbit of $v$, with $G_{NX}$ as defined in Section 2.2. Observe that if $x$ is an idempotent of length $l(x)$, all the elements of $[x]$ are also idempotents of the same length. For this reason, we are going to describe the idempotents in terms of their $G_{NX}$-orbits.

It is clear that $0 \in V$ is always a $G$-invariant idempotent of length 0. By (M3) of Definition 2.1, the Majorana axes of each Norton–Sakuma algebra are idempotents of length 1. We also have that $a_{\rho^2}$ and $a_{\rho^3}$ are idempotents of length 1 in the algebras of type 4B and 6A respectively. However, it is not clear if these are all the idempotents of length 1.

The strategy to find all the idempotents is the following. If we write a general element of the algebra in terms of the basis given in Table 1 and assume this element is idempotent, we obtain a system of $n \times n$ non-linear equations, where $n$ is the dimension of the algebra. Then, the real solutions of this system are in bijection with the idempotents.

In order to obtain the solutions of the system, we use two commands in Maple 14. The most important command that we use is RootFinding[Isolate], which implements the algorithm that obtains the Rational Univariate Representation of the solutions of the system developed in [11] and [12, Section 4.2]. The output of the algorithm is an isolating interval for each root of the system. It was shown in [11, Section 5.1] that the algorithm does not lose geometrical information, so in particular no real roots are ever lost. If the system has an infinite number of solutions, an error message is displayed in Maple 14.

The second command that we use is SolveTools[PolynomialSystem]. This implements an algorithm that combines the methods of triangular decomposition and Gröbner basis calculation in order to solve the system. The solutions found are given in radical expressions, so we use this command to explicitly find some idempotents.

In the following sections, we describe the non-linear systems using the natural action of $S_n$ over $\mathbb{Q}[\lambda_1, \ldots, \lambda_n]$. If $p = p(\lambda_1, \ldots, \lambda_n)$ is any polynomial with rational coefficients and $\sigma \in S_n$, we define the action of $\sigma$ on $p$ by

$$p(x_1, \ldots, x_n)^{\sigma} = p(x_{\sigma(1)}, \ldots, x_{\sigma(n)}).$$

---

3.1 Algebras of type $2A$ and $2B$

Let $(D, T, V, (\cdot, \cdot), (\cdot, \varphi, \psi))$ be a Majorana representation of the dihedral group of order 4, where $V = \langle a_t, a_g \rangle$, $t, g \in T$, is of type $2A$.

**Proposition 3.1** ($2A$). The algebra $V$ of type $2A$ has exactly eight idempotents. The Majorana axes of $V$ are the only idempotents of length 1.

**Proof.** By Table 1, an arbitrary element of $V$ must have the form

$$\lambda_1 a_t + \lambda_2 a_g + \lambda_3 a_{tg}, \quad \lambda_i \in \mathbb{R}.$$  

Assuming this element is idempotent, we obtain the following system of equations:

$$0 = P(\lambda_1, \lambda_2, \lambda_3),$$

$$0 = P(\lambda_1, \lambda_2, \lambda_3)^{(1,2)},$$

$$0 = P(\lambda_1, \lambda_2, \lambda_3)^{(1,3)},$$

where $P \in \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3]$ is defined by

$$P = \lambda_1(\lambda_1 - 1) + \frac{1}{22} (\lambda_1(\lambda_2 + \lambda_3) - \lambda_2\lambda_3).$$

The command `RootFinding[Isolate]` finds that the system has precisely eight solutions, so $V$ must have precisely eight idempotents. With 0 and the Majorana axes, we already have found four idempotents. The identity of the algebra $id$ is another idempotent with $l(id) = \frac{2^3}{5}$. Finally, by Proposition 2.5 we know that $id - a_t, id - a_g$ and $id - a_{tg}$ are idempotents of length $l(id) - 1 = \frac{7}{5}$.  

**Corollary 3.2.** We have $\text{Aut}(2A) = G_{2A} \cong S_3$.

**Proof.** By Proposition 3.1, $\text{Aut}(2A)$ preserves the set of Majorana axes, so the result follows by Lemma 2.7.  

Suppose now that $V = \langle a_t, a_g \rangle$ is of type $2B$.

**Proposition 3.3** ($2B$). The algebra $V$ of type $2B$ has exactly four idempotents. The Majorana axes of $V$ are the only idempotents of length 1. Furthermore, we have $\text{Aut}(2B) = G_{2B} \cong C_2$.

**Proof.** The four idempotents are 0, $a_t$, $a_g$ and the identity

$$id = a_t + a_g$$

of length 2.  

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3.2 Algebras of type 3A and 3C

Let \((D, T, V, \cdot, \cdot, \varphi, \psi)\) be a Majorana representation of the dihedral group of order 6, where \(V = \langle a_t, a_g \rangle\), \(t, g \in T\), is of type 3A, and let \(G = G_{3A} = \varphi(D)\).

**Proposition 3.4 (3A).** The algebra \(V\) of type 3A has exactly sixteen idempotents. The Majorana axes of \(V\) are the only idempotents of length 1.

**Proof.** Let \(x = \lambda_1 a_t + \lambda_2 a_g + \lambda_3 a_{g-1} + \lambda_4 u_{\rho}, \lambda_i \in \mathbb{R}\), be an element of \(V\). Observe that any automorphism in \(G\) induces a permutation of the coefficients of \(x\). For example, \(\varphi(t)\) induces the permutation \((\lambda_2, \lambda_3)\), which can be identified with \((2,3)\) in \(S_3\).

Supposing that \(x\) is idempotent, we obtain the following system:

\[
\begin{align*}
0 &= P(\lambda_1, \lambda_2, \lambda_3, \lambda_4), \\
0 &= P(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\varphi(g^{-1}), \\
0 &= P(\lambda_1, \lambda_2, \lambda_3, \lambda_4)\varphi(g), \\
0 &= S(\lambda_1, \lambda_2, \lambda_3, \lambda_4),
\end{align*}
\]

where \(P, S \in \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \lambda_4]\) are defined by

\[
\begin{align*}
P &= \lambda_1(\lambda_1 - 1) + \frac{1}{24}(2\lambda_1(\lambda_2 + \lambda_3) + \lambda_3\lambda_2) + \frac{2}{32}\lambda_4(2\lambda_1 - \lambda_2 - \lambda_3), \\
S &= \lambda_4(\lambda_4 - 1) - \frac{3^{35}}{2^{10}}(\lambda_1(\lambda_2 + \lambda_3) + \lambda_3\lambda_2) + \frac{5}{24}\lambda_4(\lambda_1 + \lambda_2 + \lambda_3).
\end{align*}
\]

The command `RootFinding[Isolate]` tells us that there are precisely sixteen solutions of this system. If \(id\) is the identity of \(V\), the \(G\)-invariant idempotents of \(V\) are listed in the left table. The right table contains the remaining twelve idempotents organized by \(G\)-orbits (each one of size 3), where

\[
y = \frac{2^3}{32}(a_t + a_g) + \frac{2}{32}a_{g-1} - \frac{1}{2^2}u_{\rho}.
\]

<table>
<thead>
<tr>
<th>(G)-orbit</th>
<th>Length</th>
<th>(G)-orbit</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>0</td>
<td>([a_t])</td>
<td>1</td>
</tr>
<tr>
<td>id</td>
<td>(\frac{2^2 29}{5^7})</td>
<td>([id - a_t])</td>
<td>(\frac{3^4}{5^7})</td>
</tr>
<tr>
<td>(u_\rho)</td>
<td>(\frac{8}{5})</td>
<td>([y])</td>
<td>(\frac{8}{5})</td>
</tr>
<tr>
<td>(id - u_\rho)</td>
<td>(\frac{2^2 3}{7})</td>
<td>([id - y])</td>
<td>(\frac{2^2 3}{7})</td>
</tr>
</tbody>
</table>

Only the idempotents in \([a_t]\) have length 1, so the proposition follows. \(\square\)
Corollary 3.5. We have \( \text{Aut}(3A) = G_{3A} \cong S_3 \).

Proof. By Proposition 3.4 and Lemma 2.7. \( \square \)

Using Proposition 3.4, it is easy to show that any automorphism in \( \text{Aut}(3A) \) acts trivially on the basis element \( u_\rho \).

Suppose now that \( V = \langle a_t, a_g \rangle \) is of type 3C.

Proposition 3.6 (3C). The algebra \( V \) of type 3C has exactly eight idempotents. The Majorana axes of \( V \) are the only idempotents of length 1.

Proof. Assuming that \( \lambda_1 a_t + \lambda_2 a_g + \lambda_3 a_{g-1}, \lambda_i \in \mathbb{R} \), is idempotent, we obtain the following system:

\[
0 = P(\lambda_1, \lambda_2, \lambda_3), \\
0 = P(\lambda_1, \lambda_2, \lambda_3)^{\phi(g-1)}, \\
0 = P(\lambda_1, \lambda_2, \lambda_3)^{\phi(g)},
\]

where \( P \in \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3] \) is defined by

\[
P = \lambda_1(\lambda_1 - 1) + \frac{1}{25}(\lambda_1(\lambda_2 + \lambda_3) - \lambda_2\lambda_3).
\]

The command RootFinding[Isolate] tells us that there are precisely eight solutions of the system. Thus, all the idempotents of \( V \) are 0, the Majorana axes \([a_t]\), the identity id of length \( \frac{25}{11} \) and the three idempotents in the \( G_{3C} \)-orbit \([\text{id} - a_t]\) of length \( l(\text{id}) - 1 = \frac{37}{11} \). \( \square \)

Corollary 3.7. We have \( \text{Aut}(3C) = G_{3C} \cong S_3 \).

3.3 Algebras of type 4A and 4B

Let \( (D, T, V, (\cdot, \cdot), \varphi, \psi) \) be a Majorana representation of the dihedral group of order 8, where \( V = \langle a_t, a_g \rangle, t, g \in T \), is of type 4A. Let \( G = G_{4A} = \langle \psi(D), \phi \rangle \) with \( \phi = (a_t, a_g)(a_{g-1}, a_{g_2}) \).

Supposing that \( x = \lambda_1 a_t + \lambda_2 a_g + \lambda_3 a_{g-1} + \lambda_4 a_{g_2} + \lambda_5 v_\rho, \lambda_i \in \mathbb{R} \), is idempotent, we obtain the following system:

\[
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5), \\
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^{\phi}, \\
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^{\phi^{(t)}}, \\
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)^{\phi(g)}, \\
0 = S(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5),
\]
where $P, S \in \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$ are defined by

$$P = \lambda_1(\lambda_1 - 1) + \frac{1}{25}(3\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3) + \frac{1}{23}\lambda_5(5\lambda_1 - 2\lambda_2 - 2\lambda_3 - \lambda_4),$$

$$S = \lambda_5(\lambda_5 - 1) - \frac{3}{25}(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3) + \frac{3}{23}\lambda_5(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4).$$

The result on the number of idempotents in $V$ in this case is somehow unexpected.

**Proposition 3.8.** The algebra $V$ of type 4A has infinitely many $D$-invariant idempotents.

**Proof.** Let $W = \langle v, b, c \rangle$ denote the subalgebra of $V$ generated by the elements $v = v_\rho$, $b = a_t + a_{g_2}$ and $c = a_g + a_{g^{-1}}$. Observe that $v$, $b$ and $c$ are idempotents, and

$$b \cdot c = \frac{1}{24}(2b + 2c - 3v),$$

$$b \cdot v = \frac{1}{23}(2b - 2c + 3v),$$

$$c \cdot v = \frac{1}{23}(2c - 2b + 3v).$$

Hence $W$ is actually equal to the linear span of $v$, $b$ and $c$. Note that $v \in V$ is $D$-invariant (not necessarily $G$-invariant) if and only if $v \in W$.

Say that $v \in V$ is quasi-idempotent if $v \cdot v = \alpha v$ for some $\alpha \in \mathbb{R}$. Observe that in this case $\frac{1}{\alpha}v$ is idempotent. Suppose now that $w = v + \beta_1 b + \beta_2 c \in W$, $\beta_i \in \mathbb{R}$, is quasi-idempotent. Using the algebra product of $W$ described above, we obtain the following system of $2 \times 2$ quadratic equations:

$$0 = (\beta_1 - 2)(2\beta_1 + 2\beta_2 + 3\beta_1\beta_2),$$

$$0 = (\beta_2 - 2)(2\beta_1 + 2\beta_2 + 3\beta_1\beta_2).$$

Clearly, this system has infinitely many solutions, so there are infinitely many linearly independent quasi-idempotents in $W$. Therefore, $W$ has infinitely many idempotents. \qed

With the previous result, we can see that there are infinitely many $G$-orbits of size 2 of idempotents in $V$. Explicitly, these orbits are

$$[y(\lambda)] = [f(\lambda)(a_t + a_{g_2}) + \bar{f}(\lambda)(a_g + a_{g^{-1}}) + \lambda v_\rho],$$

where $\lambda \in [-\frac{3}{5}, 1]$,

$$f(\lambda) = \frac{1}{2}(1 - \lambda) + \frac{1}{6}\sqrt{-15\lambda^2 + 6\lambda + 9}.$$
and \( \overline{f(\lambda)} \) is the conjugate of \( f(\lambda) \) in the field \( \mathbb{Q}(\sqrt{-15\lambda^2 + 6\lambda + 9}) \). We have the property that for any \( \lambda \in [-\frac{3}{2}, 1] \),

\[
id = y(\lambda)\phi + y\left(\frac{2}{5} - \lambda\right)
\]

so no new idempotents can be found only using Proposition 2.5. The previous families in fact contain all \( D \)-invariant idempotents of \( V \), except for the identity id and 0 (which are actually \( G \)-invariant). As we already know, the length of id is 4, while the length of all other non-zero \( D \)-invariant idempotents is 2.

By Proposition 2.3, \( \{a_t, a_{g_2}\} \) and \( \{a_g, a_{g-1}\} \) are bases for subalgebras \( U_1 \) and \( U_2 \) of type 2B respectively. Observe that \( (U_1)^\phi = U_2 \), and that the identities \( id_i \) of \( U_i, i = 1, 2 \), are already contained in the previous infinite family:

\[
y(0) = id_1 \quad \text{and} \quad y(0)^\phi = id_2.
\]

Now we focus on non-invariant idempotents, so we work again with the first non-linear system obtained. In SolveTools[PolynomialSystem] it is possible to specify that we want solutions with \( \lambda_1 \neq \lambda_4 \) or \( \lambda_2 \neq \lambda_3 \). With this, we obtain the list of all non-\( D \)-invariant idempotents. In the table, each \( G \)-orbit has size 4:

<table>
<thead>
<tr>
<th>G-orbit</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>([a_t])</td>
<td>1</td>
</tr>
<tr>
<td>([id - a_t])</td>
<td>3</td>
</tr>
<tr>
<td>([y_3])</td>
<td>(\frac{12}{7})</td>
</tr>
<tr>
<td>([id - y_3])</td>
<td>(\frac{16}{7})</td>
</tr>
</tbody>
</table>

where

\[
y_3 = \frac{2}{7}(2 - \sqrt{2})(a_t + a_g) + \frac{2}{7}(2 + \sqrt{2})(a_{g-1} + a_{g_2}) - \frac{2}{7}v_\rho.
\]

Hence we have proved the following.

**Proposition 3.9 (4A).** The algebra \( V \) of type 4A has an infinite family of \( D \)-invariant idempotents plus eighteen extra idempotents. The Majorana axes of \( V \) are the only idempotents of length 1.

**Corollary 3.10.** We have \( \text{Aut}(4A) = G_{4A} \cong D_8 \).

Using Proposition 3.9, it is easy to show that any automorphism in \( \text{Aut}(4A) \) acts trivially on the basis element \( v_\rho \).

Now assume that \( V = \langle a_t, a_g \rangle \) is of type 4B and let \( G = G_{4B} \).
Proposition 3.11 (4B). The algebra $V$ of type 4B has exactly 32 idempotents. The Majorana axes of $V$ and $a_{\rho^2}$ are the only idempotents of length 1.

Proof. Supposing that $\lambda_1 a_t + \lambda_2 a_g + \lambda_3 a_{g-1} + \lambda_4 a_{g_2} + \lambda_5 a_{\rho^2}, \lambda_i \in \mathbb{R}$, is idempotent, we obtain the following system:

\[
\begin{align*}
0 &= P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5), \\
0 &= P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\phi, \\
0 &= P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\phi\varphi(t), \\
0 &= P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)\varphi(g), \\
0 &= S(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5).
\end{align*}
\]

where $P, S \in \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5]$ are defined by

\[
P = \lambda_1(\lambda_1 - 1) + \frac{1}{2^5}(\lambda_1 - \lambda_4)(\lambda_2 + \lambda_3) + \frac{1}{2^2}(\lambda_1(\lambda_4 + \lambda_5) - \lambda_4\lambda_5),
\]

\[
S = \lambda_5(\lambda_5 - 1) + \frac{1}{2^5}(\lambda_1 + \lambda_4)(\lambda_2 + \lambda_3) + \frac{1}{2^2}(\lambda_5(\lambda_1 + \lambda_2 + \lambda_3 + \lambda_4) - \lambda_1\lambda_4 - \lambda_2\lambda_3).
\]

The command RootFinding[Isolate] tells us that there are precisely 32 solutions of the system.

By Proposition 2.3, \{\text{id}, a_{t}, a_{g_{2}}, a_{\rho^2}\} and \{\text{id}, a_{g_{-1}}, a_{\rho^2}\} are bases for subalgebras $U_1$ and $U_2$ of type 2A respectively. Hence, the idempotents of $U_1$ and $U_2$ are also idempotents of $V$. We obtain fourteen idempotents like this, including all the Majorana axes. Denote by $\text{id}_i$ the identity of $U_i$, $i = 1, 2$, and note that $(U_1)^\phi = U_2$. We know that the identity $\text{id}$ of $V$ is another idempotent of length $\frac{19}{5}$.

The list of the rest of the idempotents of $V$, classified by $G$-orbits, is:

<table>
<thead>
<tr>
<th>$G$-orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\text{id} - a_{\rho^2}]$</td>
<td>1</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>$[\text{id} - a_t]$</td>
<td>4</td>
<td>$\frac{2}{5}$</td>
</tr>
<tr>
<td>$[\text{id} - \text{id}_1 + a_t]$</td>
<td>4</td>
<td>$\frac{2^3}{5}$</td>
</tr>
<tr>
<td>$[\text{id}]$</td>
<td>4</td>
<td>$\frac{3\cdot7}{11}$</td>
</tr>
<tr>
<td>$[\text{id} - \text{id}]$</td>
<td>4</td>
<td>$\frac{2\cdot13}{5\cdot11}$</td>
</tr>
</tbody>
</table>

where \( y = \frac{2^2}{11}(1 + \sqrt{2})(a_t + a_g) + \frac{2^2}{11}(1 - \sqrt{2})(a_{g_{-1}} + a_{g_2}) + \frac{5}{11}a_{\rho^2}. \)
Corollary 3.12. We have $\text{Aut}(4B) = G_{4B} \cong D_8$.

Using Proposition 3.9, it is easy to show that $\text{Aut}(4B)$ acts trivially on the basis element $a_{\rho}^2$.

3.4 Algebra of type $5A$

Let $(D, T, V, (\cdot, \cdot), \varphi, \psi)$ be the Majorana representation of the dihedral group of order 10, so $V = \langle a_t, a_g \rangle$, $t, g \in T$, is a Norton–Sakuma algebra of type $5A$. Let $G = G_{5A} = \langle \varphi(D), \xi \rangle$ with $\xi = (a_g, a_g^2, a_{g-1}, a_{g-2})$.

Proposition 3.13 ($5A$). The algebra $V$ of type $5A$ has exactly 44 idempotents. The Majorana axes of $V$ are the only idempotents of length 1.

Proof. Supposing that $\lambda_1 a_t + \lambda_2 a_g + \lambda_3 a_{g-1} + \lambda_4 a_{g+2} + \lambda_5 a_{g-2} + \lambda_6 w_\rho$, $\lambda_i \in \mathbb{R}$, is idempotent, we obtain the following system:

$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6),
$$
$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\varphi(g^{-2}),
$$
$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\varphi(g^2),
$$
$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\varphi(g),
$$
$$
0 = P(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6)\varphi(g^{-1}),
$$
$$
0 = \frac{7}{24}\lambda_6 S + 2(\lambda_1 R - Q_2),
$$

where $P \in \mathbb{Q}[\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6]$ is defined by

$$
P = \lambda_1(\lambda_1 - 1) + \frac{7}{24}\lambda_6 \left(\frac{5}{2}\lambda_6 + R\right) + \frac{1}{26}(3\lambda_1(S - \lambda_1) - Q_1)
$$

and $R, S, Q_1$ and $Q_2$ are defined by

$$
R = \lambda_2 + \lambda_3 - \lambda_4 - \lambda_5,
$$
$$
S = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5,
$$
$$
Q_1 = \lambda_2(\lambda_3 + \lambda_4 + \lambda_5) + \lambda_3(\lambda_4 + \lambda_5) + \lambda_4\lambda_5,
$$
$$
Q_2 = \lambda_2(\lambda_3 - \lambda_4 + \lambda_5) + \lambda_3(\lambda_4 - \lambda_5) - \lambda_5\lambda_4.
$$

The command RootFinding[Isolate] tells us that there are precisely 44 distinct real solutions of the system.
The only $G$-invariant idempotents of $V$ are 0 and the identity id of length $\frac{2^5}{7}$. Observe that

$$y_1 = \frac{2^4}{5 \cdot 7} (a_t + a_g + a_{g-1} + a_{g_2} + a_{g_2} - \frac{2^7}{7} \sqrt{5} w_\rho)$$

is a $D$-invariant idempotent of length $l(y_1) = \frac{2^4}{7}$, and id $- y_1$ is another $D$-invariant idempotent\(^4\) of length $l(id - y_1) = \frac{2^4}{7}$. Note that $(y_1)^\xi = id - y_1$, so these idempotents actually form a $G$-orbit of size 2.

The following table contains the rest of the idempotents of $V$:

<table>
<thead>
<tr>
<th>$G$-orbit</th>
<th>Size</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[a_t]$</td>
<td>5</td>
<td>$\frac{5^2}{7}$</td>
</tr>
<tr>
<td>$[id - a_t]$</td>
<td>5</td>
<td>$\frac{5^2}{7}$</td>
</tr>
<tr>
<td>$[y_2]$</td>
<td>10</td>
<td>$\frac{3 \cdot 13}{2 \cdot 7}$</td>
</tr>
<tr>
<td>$[id - y_2]$</td>
<td>10</td>
<td>$\frac{2^4}{7}$</td>
</tr>
<tr>
<td>$[y_3]$</td>
<td>10</td>
<td>$\frac{2^4}{7}$</td>
</tr>
</tbody>
</table>

where

$$y_2 = \frac{1}{5} \left( \frac{-3}{2 \cdot 7} a_t + \alpha (a_g + a_{g-1}) + \alpha (a_{g_2} + a_{g_2} - \frac{2^7}{7} \sqrt{5} w_\rho) \right),$$

$$y_3 = \frac{2^2}{5} \left( \frac{2^2}{7} a_t + \beta (a_g + a_{g-1}) + \beta (a_{g_2} + a_{g_2} - \frac{2^7}{7} \sqrt{5} w_\rho) \right)$$

with $\alpha = \frac{2^4}{7} + \sqrt{5}$, $\beta = \frac{2^2}{7} - \frac{1}{5} \sqrt{5}$ and $\alpha, \beta$ are their conjugates in $\mathbb{Q}(\sqrt{5})$.

Since the only orbit of length 1 is $[a_t]$, the proposition follows.

Before calculating the automorphism group of the algebra, we need the next result.

**Corollary 3.14.** Let $\gamma \in \text{Aut}(5A)$. Then $(w_\rho)^\gamma = w_\rho$ or $(w_\rho)^\gamma = -w_\rho$.

**Proof.** Let $\Omega$ be the set of Majorana axes in $V$. Observe that by Proposition 3.13, $\text{Aut}(5A)$ permutes $\Omega$. Since $(y_1)^\gamma$ must be an idempotent element of length $\frac{2^4}{7}$, it must be $y_1$, id $- y_1$ or an element of the $G_{5A}$-orbit $[y_3]$. If $(y_1)^\gamma = y_1$, we have that

$$(w_\rho)^\gamma = w_\rho.$$
If \((y_1)^\gamma = \text{id} - y_1\), we have that
\[(w_\rho)^\gamma = -w_\rho.\]

Now if \((y_1)^\gamma = y_3\), it is easy to see that
\[(a, y_1) \neq (b, y_3)\]
for any \(a, b \in \Omega\), which is a contradiction. \(\square\)

**Corollary 3.15.** We have \(\text{Aut}(5A) = G_{5A}\), which is isomorphic to the Frobenius group of order 20.

**Proof.** Let \(\Omega\) be the set of Majorana axes in \(V\). By Proposition 3.13, the group \(\text{Aut}(5A)\) acts faithfully over \(\Omega\), so \(\text{Aut}(5A) \leq \text{Sym}(\Omega) \cong S_5\). We know, by Proposition 2.6, that \(\text{Aut}(5A)\) contains the group \(G_{5A}\) isomorphic to the Frobenius group of order 20. Recall the Frobenius group of order 20 is defined as a transitive permutation group on five points, such that no non-trivial element fixes more than one point and some non-trivial element fixes a point. To prove the corollary, it is enough to show that no non-trivial element of \(\text{Aut}(5A)\) fixes more than one point.

Let \(\gamma \in \text{Aut}(5A)\) be a non-trivial automorphism that fixes two points of \(\Omega\). Without loss of generality, assume that \((a_t)^\gamma = a_t\) and \((a_g)^\gamma = a_g\). As \(\text{Aut}(5A)\) acts faithfully, \(\gamma\) has to move at least one point of \(\Omega\), say \((a_{g_2})^\gamma = a_{-g_2}\). Hence \(\gamma\) must be \((a_{g_2}, a_{-g_2})\) or \((a_{g_2}, a_{g_2}, a_{-g_2})\). However, using Corollary 3.14 we can show that in both cases,
\[(a_g \cdot w_f)^\gamma \neq a_g^\gamma \cdot w_f^\gamma,\]
which is a contradiction. \(\square\)

### 3.5 Algebra of type 6A

Let \((D, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)\) be the Majorana representation of the dihedral group of order 12, so \(V = \langle a_t, a_g \rangle\), \(t, g \in T\), is a Norton–Sakuma algebra of type 6A. Let \(G = G_{6A} = \langle \varphi(t), \zeta \rangle\) with \(\zeta = (a_t, a_g)(a_{g-1}, a_{g_2})(a_{g-2}, a_{g_3})\).

Let
\[x = \lambda_1 a_t + \lambda_2 a_g + \lambda_3 a_{g-1} + \lambda_4 a_{g_2} + \lambda_5 a_{g-2} + \lambda_6 a_{g_3} + \lambda_7 a_{g_2} + \lambda_8 a_{g_2}^2, \quad \lambda_i \in \mathbb{R},\]
be an arbitrary element of \(V\). Let
\[\zeta_1 = \zeta \varphi(gt),\]
\[\zeta_2 = \zeta \varphi(tg)\]
be automorphisms in \(G\). If we suppose that \(x\) is idempotent, we obtain the follow-
ing system:

\[
0 = K(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8),
\]

\[
0 = K(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)\phi,
\]

\[
0 = K(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)\xi_1,
\]

\[
0 = K(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)\varphi(g),
\]

\[
0 = K(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)\varphi(g-1),
\]

\[
0 = K(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5, \lambda_6, \lambda_7, \lambda_8)\xi_2,
\]

\[
0 = \lambda_7(\lambda_7 + \frac{1}{22}S - 1) - \frac{1}{25}P_2 - \frac{1}{25}(\lambda_1\lambda_6 + \lambda_2\lambda_5 + \lambda_3\lambda_4),
\]

\[
0 = \lambda_8(\lambda_8 + \frac{5}{24}S - 1) + \frac{325}{210}(P_1 - 3R),
\]

where

\[
K = \lambda_1(\lambda_1 - 1) + \frac{1}{25}(P_1 + 2Q) + \frac{2}{32}\lambda_8(2\lambda_1 - \lambda_4 - \lambda_5)
\]

and

\[
P_1 = \lambda_1(\lambda_2 + \lambda_3) - \lambda_4(\lambda_2 + \lambda_6) - \lambda_5(\lambda_3 + \lambda_6),
\]

\[
P_2 = \lambda_1(\lambda_2 + \lambda_3) + \lambda_4(\lambda_2 + \lambda_6) + \lambda_5(\lambda_3 + \lambda_6),
\]

\[
Q = 2\lambda_1(\lambda_4 + \lambda_5 + 2\lambda_6 + 2\lambda_7) + \lambda_4\lambda_5 - 4\lambda_6\lambda_7,
\]

\[
R = \lambda_1(\lambda_4 + \lambda_5) + \lambda_4\lambda_5 + \lambda_2(\lambda_3 + \lambda_6) + \lambda_3\lambda_6,
\]

\[
S = \lambda_1 + \lambda_2 + \lambda_3 + \lambda_4 + \lambda_5 + \lambda_6.
\]

The command RootFinding[Isolate] tells us that there are precisely 208 real solutions of the system.

By Proposition 2.3, the sets \(\{a_t, a_{g_3}, a_{p_3}\}, \{a_g, a_{g-2}, a_{p_3}\}\) and \(\{a_{g_2}, a_{g-1}, a_{p_3}\}\) are bases for subalgebras \(U_1, U_2\) and \(U_3\) of type \(2A\) respectively, so the idempotents in these subalgebras are also idempotents in \(V\). We obtain twenty idempotents like this, including 0 and the Majorana axes. Let \(id_t^{2A}\) be the identity of \(U_i\), for \(i = 1, 2, 3\).

We also know that the sets \(\{a_t, a_{g_2}, a_{g-2}, u_{p_2}\}\) and \(\{a_g, a_{g-1}, a_{g_3}, u_{p_2}\}\) are bases for subalgebras \(W_1\) and \(W_2\) of type \(3A\) respectively. We obtain 23 new idempotents contained in these subalgebras, including \(u_{p_2}\). Let \(id_j^{3A}\) be the identity of \(W_j\), \(j = 1, 2\). Observe that \((W_1)^\xi = W_2\). Let

\[
y_1 = \frac{3^3}{32}(a_t + a_{g_2}) + \frac{2}{32}a_{g-2} - \frac{1}{22}u_{p_2} \in W_1.
\]
By the proof of Proposition 3.4, $y_1$ is an idempotent of length $\frac{8}{5}$ in $W_1$, so $y_1^\xi$ is an idempotent of length $\frac{8}{5}$ in $W_2$.

Let id be the identity of $V$, which we know has length $\frac{3\cdot17}{2\cdot5}$ and is not contained in any of the Norton–Sakuma subalgebras of $V$. Besides the identity, the following is the table of all non-zero $G$-invariant idempotents.

<table>
<thead>
<tr>
<th>Idempotent</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>id $- a_{\rho^3}$</td>
<td>$\frac{41}{2\cdot5}$</td>
</tr>
<tr>
<td>id $- u_{\rho^2}$</td>
<td>$\frac{7}{2}$</td>
</tr>
<tr>
<td>$a_{f^3} + u_{\rho^2}$</td>
<td>$\frac{13}{5}$</td>
</tr>
<tr>
<td>id $- a_{\rho^3} - u_{\rho^2}$</td>
<td>$\frac{5}{2}$</td>
</tr>
</tbody>
</table>

The following is the table of the $D$-invariant idempotents which are not $G$-invariant and are not contained in any subalgebra of type $2A$ or $3A$. Each $G$-orbit has size 2.

<table>
<thead>
<tr>
<th>$G$-orbit</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[id $- id_{1}^{3A}$]</td>
<td>$\frac{5^2}{2\cdot7}$</td>
</tr>
<tr>
<td>[id $- id_{1}^{3A} + u_{\rho^2}$]</td>
<td>$\frac{3\cdot79}{2\cdot5\cdot7}$</td>
</tr>
</tbody>
</table>

Now we are going to describe the non-invariant idempotents with rational coefficients that are not contained in any subalgebra of type $2A$ or $3A$. First we have the following table in which the $D$-orbits and $G$-orbits coincide and have size 3:

<table>
<thead>
<tr>
<th>$G$-orbit</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>[id $- id_{1}^{2A}$]</td>
<td>$\frac{3^3}{2\cdot5}$</td>
</tr>
<tr>
<td>[id $- id_{1}^{2A} + a_{\rho^3}$]</td>
<td>$\frac{37}{2\cdot5}$</td>
</tr>
<tr>
<td>[y$_2$]</td>
<td>$\frac{2^2\cdot29}{5\cdot7}$</td>
</tr>
<tr>
<td>[id $- y_2$]</td>
<td>$\frac{5^2}{2\cdot7}$</td>
</tr>
<tr>
<td>[y$<em>2$ $- a</em>{\rho^3}$]</td>
<td>$\frac{3^4}{5\cdot7}$</td>
</tr>
<tr>
<td>[id $- y_2 + a_{\rho^3}$]</td>
<td>$\frac{3\cdot13}{2\cdot7}$</td>
</tr>
</tbody>
</table>

where

$$y_2 = \frac{1}{7}\left(\frac{2^4}{3}(a_g + a_{g-1} + a_{g2} + a_{g-2}) + \frac{2^2}{3}(a_f + a_{g3}) + 2^2 a_{\rho^3} - 3u_{\rho^2}\right).$$
In the next table, each entry forms a $G$-orbit of size 6 which is the disjoint union of two $D$-orbits of size 3:

<table>
<thead>
<tr>
<th>$G$-orbit</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[\text{id} - a_t]$</td>
<td>$\frac{41}{2.5}$</td>
</tr>
<tr>
<td>$[\text{id} - \text{id}_1^3A + a_t]$</td>
<td>$\frac{3.13}{2.7}$</td>
</tr>
<tr>
<td>$[\text{id} - y_1]$</td>
<td>$\frac{7}{2}$</td>
</tr>
<tr>
<td>$[\text{id} - \text{id}_1^3A + y_1]$</td>
<td>$\frac{3.79}{2.5:7}$</td>
</tr>
<tr>
<td>$[y_3]$</td>
<td>$\frac{13}{5}$</td>
</tr>
<tr>
<td>$[\text{id} - y_3]$</td>
<td>$\frac{5}{2}$</td>
</tr>
<tr>
<td>$[y_4]$</td>
<td>$\frac{22.3}{7}$</td>
</tr>
<tr>
<td>$[\text{id} - y_4]$</td>
<td>$\frac{3.79}{2.5:7}$</td>
</tr>
<tr>
<td>$[\text{id} - \text{id}_1^2A + a_t]$</td>
<td>$\frac{37}{2.5}$</td>
</tr>
</tbody>
</table>

where

$$y_3 = a_t + \frac{2^3}{3^2} (a_g + a_{g-1}) + \frac{2}{3^2} a_{g_3} - \frac{1}{2^2 u \rho^2},$$

$$y_4 = \frac{1}{7} \left[ \frac{2^2}{3} (a_t + 2^2 a_{g_2} + 2^2 a_{g-2}) - \frac{2}{3^2} (a_{g_3} + 2^2 a_g + 2^2 a_{g-1}) 
+ 2^2 a_\rho^3 - \frac{5}{2^2 u \rho^2} \right].$$

We are going to describe the idempotents with irrational coefficients. In the next table, all $D$-orbits coincide with the $G$-orbits and have size 6:

<table>
<thead>
<tr>
<th>$G$-orbit</th>
<th>Length</th>
</tr>
</thead>
<tbody>
<tr>
<td>$[y_5]$</td>
<td>$\frac{11}{2.3}$</td>
</tr>
<tr>
<td>$[\text{id} - y_5]$</td>
<td>$\frac{7.2}{3.5}$</td>
</tr>
<tr>
<td>$[y_6]$</td>
<td>$\frac{9.7}{2.3:5}$</td>
</tr>
<tr>
<td>$[\text{id} - y_6]$</td>
<td>$\frac{2.7}{3.5}$</td>
</tr>
<tr>
<td>$[y_7]$</td>
<td>$\frac{3.7}{11}$</td>
</tr>
<tr>
<td>$[\text{id} - y_7]$</td>
<td>$\frac{3^{3/13}}{2.5:11}$</td>
</tr>
</tbody>
</table>
where

\[ y_5 = \frac{1}{3} \left( \frac{2}{3^2} \left[ \alpha(a_t + a_g) + \alpha(a_{g-2} + a_{g_3}) - (a_{g-1} + a_{g_2}) \right] + \frac{1}{2} a_{\rho^3} + \frac{5}{2^3} u_{\rho^2} \right). \]

\[ y_6 = \frac{1}{3} \left( \frac{2}{3^2} \left[ \alpha(a_t + a_g) + \alpha(a_{g-2} + a_{g_3}) \right] + \frac{2 \cdot 7^2}{3^2} (a_{g-1} + a_{g_2}) - \frac{1}{2} a_{\rho^3} + \frac{5}{2^3} u_{\rho^2} \right). \]

\[ y_7 = \frac{1}{11} \left( \frac{2^3}{3} \left[ \beta(a_t + a_g) + \beta(a_{g-2} + a_{g_3}) - \frac{2^2}{3} (a_{g-1} + a_{g_2}) + a_{\rho^3} + \frac{3 \cdot 5}{2} u_{\rho^2} \right). \]

with \( \alpha = 5 + 2^2 \sqrt{3}, \beta = 1 + \sqrt{3} \), and \( \alpha, \beta \) are their conjugates in \( \mathbb{Q}(\sqrt{3}) \).

Only the idempotents with trivial \( G \)-stabiliser remain. We will denote by \( B_r(x) \) the open interval in \( \mathbb{R} \) centered on \( x \in \mathbb{R} \) with radius \( r > 0 \). The algorithm in \textit{RootFinding[Isolate]} guarantees that there exists an idempotent

\[ y_8 = x_1a_t + x_2a_g + x_3a_{g-1} + x_4a_{g_2} + x_5a_{g-2} + x_6a_{g_3} + x_7a_{\rho^3} + x_8u_{\rho^2} \]

with

\[ x_1 \in B_r(0.118600343195), \]
\[ x_2 \in B_r(0.116899056660), \]
\[ x_3 \in B_r(0.672945208716), \]
\[ x_4 \in B_r(0.891963849266), \]
\[ x_5 \in B_r(0.034809133018), \]
\[ x_6 \in B_r(0.960846592395), \]
\[ x_7 \in B_r(-0.258738375363), \]
\[ x_8 \in B_r(-0.226937866453), \]

where \( r = 10^{-10} \).

Since all the intervals are disjoint, \([y_8]\) and \([\text{id} - y_8]\) are both \( G \)-orbits of size 12. With the functions \textit{minimize} and \textit{maximize} of \textit{Maple 14}, it is easy to show that the length of \( y_8 \) must be in the open interval \((2, 2.2)\). Therefore, neither \( y_8 \) nor \( \text{id} - y_8 \) can be idempotents of length 1.

Similarly, the algorithm in \textit{RootFinding[Isolate]} guarantees that there exists an idempotent

\[ y_9 = z_1a_t + z_2a_g + z_3a_{g-1} + z_4a_{g_2} + z_5a_{g-2} + z_6a_{g_3} + z_7a_{\rho^3} + z_8u_{\rho^2} \]
with

\[ z_1 \in B_r(0.753\,376\,146\,443), \]
\[ z_2 \in B_r(-0.031\,896\,831\,434), \]
\[ z_3 \in B_r(-0.153\,112\,021\,089), \]
\[ z_4 \in B_r(0.729\,547\,069\,626), \]
\[ z_5 \in B_r(0.110\,690\,245\,253), \]
\[ z_6 \in B_r(0.844\,782\,757\,936), \]
\[ z_7 \in B_r(0.620\,071\,135\,272), \]
\[ z_8 \in B_r(-0.121\,749\,860\,276), \]

where \( r = 10^{-10} \).

Again we have that \([y_9]\) and \([\text{id} - y_9]\) are \(G\)-orbits of size 12, and this time the length of \(y_9\) is in the open interval \((2.63, 2.73)\). Hence neither \(y_9\) nor \(\text{id} - y_9\) can be idempotents of length 1.

With this we have shown the following.

**Proposition 3.16** (6A). The algebra \(V\) of type 6A has exactly 208 idempotents. The Majorana axes of \(V\) and \(a_{\rho^3}\) are the only idempotents of length 1.

**Corollary 3.17.** We have \(\text{Aut}(6A) = G_{6A} \cong D_{12}\).

Using Proposition 3.16, it is easy to show that any automorphism in \(\text{Aut}(6A)\) acts trivially on the basis elements \(a_{\rho^3}\) and \(u_{\rho^2}\).

### 4 Conclusions

We have found and calculated the lengths of all the idempotents in the Norton–Sakuma algebras, and we used these results to obtain the automorphism groups of the algebras. The results are stated in the following theorem.

**Theorem 4.1.** Let \((D, T, V, (\cdot, \cdot), \cdot, \varphi, \psi)\) be a Majorana representation of the dihedral group of order \(2N\), so \(V = \langle a_t, a_g \rangle, t, g \in T, \) is a Norton–Sakuma algebra of type \(NX, X \in \{A, B, C\}\). Then the following statements hold:

1. An algebra of type 2A has exactly eight idempotents and
   \[ \text{Aut}(2A) = \text{Sym}\{a_t, a_g, a_{tg}\} \cong S_3. \]

2. An algebra of type 2B has exactly four idempotents and
   \[ \text{Aut}(2B) = \text{Sym}\{a_t, a_g\} \cong C_2. \]
(3) An algebra of type $3A$ has exactly sixteen idempotents and
\[
\text{Aut}(3A) = \varphi(D) \cong S_3.
\]

(4) An algebra of type $3C$ has exactly eight idempotents and
\[
\text{Aut}(3C) = \varphi(D) \cong S_3.
\]

(5) An algebra of type $4A$ an infinite family of idempotents plus eighteen extra idempotents. In this case,
\[
\text{Aut}(4C) = \langle \varphi(t), \phi \rangle \cong D_8
\]
with \( \phi = (a_t, a_g)(a_{g-1}, a_{g_2}). \)

(6) An algebra of type $4B$ has exactly 32 idempotents and
\[
\text{Aut}(4B) = \langle \varphi(t), \phi \rangle \cong D_8
\]
with \( \phi = (a_t, a_g)(a_{g-1}, a_{g_2}). \)

(7) An algebra of type $5A$ has exactly 44 idempotents and
\[
\text{Aut}(5A) = \langle \varphi(tg), \varphi(g)\xi \rangle
\]
with \( \xi = (a_g, a_{g_2}, a_{g-1}, a_{g-2}) \) is isomorphic to the Frobenius group of order 20.

(8) An algebra of type $6A$ has exactly 208 idempotents and
\[
\text{Aut}(6A) = \langle \varphi(t), \xi \rangle \cong D_{12}
\]
with \( \xi = (a_t, a_g)(a_{g-1}, a_{g_2})(a_{g-2}, a_{g_3}). \)

**Acknowledgments.** I am very grateful to my PhD supervisor Professor A. A. Ivanov for all his valuable comments and support. My thanks also to the referee of this paper, for all his precise observations.

**Bibliography**


Received August 1, 2011; revised November 6, 2012.

**Author information**

Alonso Castillo-Ramirez, Department of Mathematics, Imperial College, 180 Queen’s Gt., SW7 2AZ, London, United Kingdom. E-mail: alonso.castillo-ramirez09@imperial.ac.uk