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On duality inducing automorphisms and sources of simple modules in classical groups

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1 Introduction

Let p be a prime number, k an algebraically closed field of characteristic p, P a finite p-group and W an indecomposable kP-module. Given a finite group G and an indecomposable kG-module V, we say that (P, W) is a vertex–source pair for (G, V) if there is an inclusion $P \hookrightarrow G$ of groups, under which P is a vertex of V and W is a source of V.

Endo-permutation modules occur frequently as sources of simple modules of finite groups. For instance, every simple module of a p-soluble group has endo-permutation source; if V is a simple kG-module, where G is a finite group, lying in a nilpotent block of kG, then V has endo-permutation source, and any 2-block of a finite group whose defect groups are isomorphic to the Klein 4-group possesses a simple module with endo-permutation source.

By results of Berger and Feit, and independently Puig, the proof of which invokes the classification of finite simple groups, if W is an endo-permutation kP-module which occurs as a source of a simple kG-module, for a p-soluble group G, then the class of W is a torsion element in the Dade group of P. Further, Mazza [8] has shown that any endo-permutation kP-module whose isomorphism class is a torsion element of the Dade group of P and which satisfies certain structural constraints identified by Puig does occur as source of some simple module of some p-nilpotent group. By contrast, the question of which endo-permutation modules occur as sources of simple modules in simple, quasi-simple or almost simple groups has not been extensively studied.

The smallest interesting case is the case where P is elementary abelian of rank 2, since if P is a finite cyclic group, there are no non-torsion endo-permutation kP-modules. In this paper, we study two situations in which simple modules of groups related to the finite classical groups having vertex–source pairs (P, W), with P elementary abelian of order p^2 and W endo-permutation, occur and we prove that in both cases, W must be a self-dual module and hence corresponds to an element of order at most 2 in the corresponding Dade group.

Notation. Let H be a finite group. For a finite-dimensional kH-module M, we denote by $M^\vee$ the contragradient dual of M; that is, $M^\vee$ is the kH-module given by $M^\vee := \mathrm{Hom}_k(M, k)$ as k-vector space, and $h.\alpha(m) = \alpha(h^{-1}m)$, for $\alpha \in M^\vee$, $h \in H$ and $m \in M$.

If $\phi : G \to H$ is an isomorphism of groups, and M is a kG-module, we will denote by ${}^{\phi}M$ the kH-module $\mathrm{Res}_{\phi^{-1}} M$.

For a finite p-group, we denote by $D_k(P)$ the Dade group of P over k. For the definition and basic properties of the Dade group, we refer to [12]. Here, we merely recall that if W is an indecomposable kP-module, with vertex P, then the (isomorphism class) of W defines a unique element $[W]$ of $D_k(P)$, the identity of $D_k(P)$ is $[k]$, and that the inverse of the element $[W]$ in $D_k(P)$ is the element $[W^\vee]$. In particular, if W is self-dual, then $[W]$ has order at most 2 in $D_k(P)$.

Theorem 1.1. *Let L be a finite symmetric group S_n or one of the finite classical groups $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{O}_{2n+1}(q)$, $\mathrm{O}^+_{2n}(q)$, $\mathrm{O}^-_{2n}(q)$, or $\mathrm{Sp}_{2n}(q)$, where q is a prime power. Let G be a subgroup of L containing the derived subgroup of L and let Z be a subgroup of $Z(G)$. Let P be an elementary abelian group of order p^2, and let W be an indecomposable endo-permutation kP-module. Suppose that there exists a simple kG/Z module with (P, W) as vertex–source pair. Then $W \cong W^\vee$, and if $p = 2$, then $W \cong k$.*

Before stating our next result, we recall some standard facts from modular representation theory. Let G be a finite group and N a normal subgroup of G such that G/N is a p-group. Suppose that U is a simple projective kN-module whose isomorphism class is stable under the conjugation action of G on N. Then the kN-module structure on U extends uniquely to a kG-module structure. The kG module U is simple, and if (P, W) is a vertex–source pair of (G, V), then P is isomorphic to G/N and W is an endo-permutation kP-module.

The question of whether any W that appears in the above context is torsion in the Dade group of P has been investigated by Salminen in [10], [11] for odd primes p. He has reduced the problem to the case where P is elementary abelian of order p^2 and where N is a central p'-extension of a projective special linear or unitary group, or $p = 3$ and N is a central extension of the simple group $D_4(q)$.

In this paper, we prove that groups of type A do not pose a problem.

Theorem 1.2. *Let p be an odd prime. Let G be a finite group with a normal subgroup N such that $[G : N]$ is elementary abelian of order p^2. Suppose that N is a quasi-simple group, with $Z(N)$ a p'-group and such that $N/Z(N)$ is isomorphic to $\mathrm{PSL}_n(q)$ or to $\mathrm{PSU}_n(q)$ where q is a prime power not divisible by p. Suppose that U is a simple projective kN-module which is G-stable and let (P, W) be a vertex–source pair for the kG-module U. Then $W \cong W^\vee$.*

Combining the above with Salminen's work thus proves that if $p \geqslant 5$, then for a finite group G, and a simple kG-module U with vertex–source pair (P, W) such that $\mathrm{Res}^G_N(U)$ is a simple projective kN-module for some normal subgroup N of p-power order, $[W]$ is a torsion element in $D_k(P)$. We remark that this result is a special case of a long-standing conjecture on the finiteness of the number of source algebra equivalences of nilpotent blocks of finite groups, a proof of which has been recently announced by Puig.

The proof of both Theorem 1.1 and Theorem 1.2 is based on the following elementary proposition.

Proposition 1.3. *Suppose that G is a finite group, and M is an indecomposable kG module with vertex–source pair (P, W) such that P is elementary abelian of order p^2 and W is an endo-permutation kP-module. If there exists an automorphism $\phi : G \to G$ such that ${}^{\phi}M \cong M^{\vee}$ as kG-modules, then W is self-dual and if $p = 2$, $W = k$.*

Theorem 1.1 is an easy consequence of the above proposition, once one observes that the symmetric and classical groups have automorphisms which invert conjugacy classes. However, as will be explained in more detail later on, the proof of Theorem 1.2 is somewhat more subtle and it relies on a rather curious fact which we state below.

Notation. For our next result, G will denote either the general linear group $\mathrm{GL}_n(q)$ or the general unitary group $\mathrm{GU}_n(q)$ for some prime power q.

If $G = \mathrm{GL}_n(q)$ we set $\varepsilon = 1$ and if $G = \mathrm{GU}_n(q)$, we set $\varepsilon = -1$. If $G = \mathrm{GL}_n(q)$, set $N := \mathrm{SL}_n(q)$ and if $G = \mathrm{GU}_n(q)$, set $N = \mathrm{SU}_n(q)$. If q is a prime power not divisible by p such that $q = q'^p$ for some q', then if $G = \mathrm{GL}_n(q)$, we let $\phi : G \to G$ be the automorphism $(a_{ij}) \mapsto (a_{ij}^{q'})$ and if $G = \mathrm{GU}_n(q)$, we let $\phi : G \to G$ be the automorphism $(a_{ij}) \mapsto (a_{ij}^{q'})^{t^{-1}}$. Note that since N is ϕ-stable and G/N is cyclic, any subgroup I of G containing N is ϕ-stable, and we have a natural inclusion of groups $I \rtimes \langle \phi \rangle \leqslant G \rtimes \langle \phi \rangle$. This is not, in general, a normal inclusion.

Theorem 1.4. *With the notation above suppose that p is an odd prime such that $p \mid q - \varepsilon$ and $p \mid n$. Suppose further that $q = q'^p$ for some prime power q'. Let c be a block of kN with central defect group and let $I(c) = \mathrm{Stab}_G(c)$. If c is stable under ϕ, then $I(c) \rtimes \langle \phi \rangle$ is a normal subgroup of $G \rtimes \langle \phi \rangle$.*

The paper is organized into six sections. In Section 2, we prove Proposition 1.3 as well as some other general results which are needed for the proofs of the main theorems. In Section 3, we prove Theorem 1.1. Section 4 contains some block-theoretic results needed for the proof of Theorem 1.2. In Section 5, we recall some facts from the representation theory of the finite general linear and unitary groups and their commutator subgroups and prove Theorem 1.4. Theorem 1.2 is proved in the final section.

2 A criterion for torsion

Notation. We keep the notation of the introduction. In addition, let $(K, \mathcal{O}, k)$ be a p-modular system.

Definition 2.1. Let H be a finite group and let V be a kH-module. Then V is *automorphically dual* if there exists an automorphism $\phi : H \to H$ such that ${}^{\phi}V \cong V^{\vee} := \mathrm{Hom}_k(V, k)$.

Clearly, any self-dual module is automorphically dual. Our results are based on the observation that the converse holds for endo-permutation modules for elementary abelian p-groups of order p^2.

Lemma 2.2. *Let P be an elementary abelian p-group of order p^2 and let W be an indecomposable endo-permutation kP-module with vertex P. If W is automorphically dual, then W is self-dual.*

Proof. Let $\phi : P \to P$ be an automorphism of P such that ${}^{\phi}W \cong W^{\vee}$. The result is an easy consequence of Dade's classification of of endo-permutation modules for abelian p-groups [2, (10.1) and (12.5)]. By this classification, if $p = 2$, then W is isomorphic to $\Omega^n(P)$ for some uniquely determined $n \in \mathbb{Z}$, whence ${}^{\phi}W$ is isomorphic to $\Omega^n(k)$ and $W^{\vee}$ is isomorphic to $\Omega^{-n}(P)$. But we are given that ${}^{\phi}W \cong W^{\vee}$. Hence $n = 0$. Now suppose that p is odd. Then

$$[W] = [\Omega^n(k) \otimes_k V]$$

for some $n \in \mathbb{Z}$ and some self-dual kP-module V, with n and $[V]$ uniquely determined by W, whence

$$[{}^{\phi}W] = [\Omega^n(k) \otimes_k {}^{\phi}V]$$

and

$$[W^{\vee}] = [\Omega^{-n}(k) \otimes_k V^{\vee}] = [\Omega^{-n}(k) \otimes_k V].$$

Thus ${}^{\phi}W \cong W^{\vee}$ implies that $n = 0$, thanks to the uniqueness of n and $[V]$. □

Lemma 2.3. *Let M be an indecomposable kG module and let (S, V) be a vertex–source pair for M. Then $(S, V^{\vee})$ is a vertex–source pair for $M^{\vee}$.*

Proof. Let H be a subgroup of G and let U be a kH-module such that M is a summand of $\mathrm{Ind}_H^G(U)$. Then $M^{\vee}$ is a summand of $(\mathrm{Ind}_H^G(U))^{\vee} \cong \mathrm{Ind}_H^G(U^{\vee})$. The result follows. □

Lemma 2.4. *Suppose that G is a finite group and M is an indecomposable kG module with vertex–source pair (P, W). If M is automorphically dual, then so is W.*

Proof. Let $\phi : G \to G$ be such that ${}^{\phi}M \cong M^{\vee}$. By Lemma 2.3, $(P, W^{\vee})$ is a vertex–source pair for the kG-module $M^{\vee}$. On the other hand, by transport of structure, $({}^{\phi}P, {}^{\phi}W)$ is a vertex–source pair for the kG-module ${}^{\phi}M$. Thus, by hypothesis, both $(P, W^{\vee})$ and $({}^{\phi}P, {}^{\phi}W)$ are vertex–source pairs for $M^{\vee}$. So there exists $g \in G$ such that $P = g\phi_P$ and $W^\vee \cong g\phi_W$. The result follows by considering the automorphism $y \mapsto g\phi(y)g^{-1}$ of P. □

Proof of Proposition 1.3. By hypothesis, M is automorphically dual. Hence by Lemma 2.4, W is automorphically dual. The result follows by Lemma 2.2. □

3 Proof of Theorem 1.1

We keep all the notation of the previous sections.

Lemma 3.1. *Let G be a finite group and $\tau : G \to G$ an automorphism such that $\tau(g)$ is conjugate to g^{-1} for all $g \in G_{p'}$. If N is a normal subgroup of G which is τ-stable, then every simple kN-module is automorphically dual.*

Proof. First, let M be a simple kG-module and let ϕ_M be the Brauer character of M. Then by hypothesis, the Brauer character $\phi_{{}^\tau M}$ of ${}^\tau M$ satisfies

$$\phi_{{}^\tau M}(g) = \phi_M(g^{-1}) = \phi_{M^\vee}(g) \quad \text{for all } g \in G_{p'}.$$

Hence M is automorphically dual. Now let U be a simple kN-module and let M be a simple kG-module such that U is a summand of $\mathrm{Res}_N\, M$. Then $U^\vee$ is a summand of $\mathrm{Res}_N\, M^\vee$ and ${}^\tau U$ is a summand of $\mathrm{Res}_N\, {}^\tau M$. But as shown above, ${}^\tau M \cong M^\vee$, hence ${}^\tau U$ and $U^\vee$ are covered by the same simple kG-module. Thus there exists $g \in G$ such that ${}^{g\tau}U \cong U^\vee$. The result follows by considering the automorphism $y \mapsto g\tau(y)g^{-1}$ of N. □

Lemma 3.2. *Let L be a finite symmetric group S_n or one of the finite classical groups $\mathrm{GL}_n(q)$, $\mathrm{GU}_n(q)$, $\mathrm{O}_{2n+1}(q)$, $\mathrm{O}^+_{2n}(q)$, $\mathrm{O}^-_{2n}(q)$, or $\mathrm{Sp}_{2n}(q)$, where q is a prime power. Let G be a subgroup of L such that $[L, L] \leqslant G$ and let Z be a central subgroup of G. Every simple module $k(G/Z)$-module is automorphically dual.*

Proof. First, note that since $Z(G)$ is cyclic, it suffices to show that every simple kG-module is automorphically dual. In each case above, there is an automorphism $\tau : L \to L$ such that $\tau(x)$ is conjugate to x^{-1} for all $x \in L$: If L is a symmetric group, then take τ to be the identity map. If L is an orthogonal group, or L is a symplectic group $\mathrm{Sp}_{2n}(q)$, where q is a power of 2, then by results of Gow ([6, Lemmas 2.1 and 2.2]) and Wonenburger [13], every element of L is conjugate to its inverse, hence again τ may be taken to be the identity map. If $L = \mathrm{Sp}_{2n}(q)$ where q is odd, then by [6, Lemma 2.1 (b)], every element of L is the product of two skew symplectic involutions; hence, we may take τ to be conjugation by any fixed skew-symplectic involution. If $L = \mathrm{GL}_n(q)$ we take τ to be the transpose inverse map. Finally, if $L = \mathrm{GU}_n(q)$ we let τ be the map which raises every entry of every matrix to its qth power (regarding $\mathrm{GU}_n(q)$ as the fixed point subgroup of $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ under the map which sends a matrix (a_{ij}) to the matrix $((a_{ij}^q))^{-t})$. It is easy to check that in each case G is τ-stable. Now the result follows from Lemma 3.1. □

Proof of Theorem 1.1. This is now immediate from Lemma 3.2 and Proposition 1.3. □

4 Some results from block theory

Notation. We keep the notation of the previous sections. Let G be a finite group. By a block of kG (or of $\mathcal{O}G$), we mean a primitive idempotent of the center $Z(kG)$ of the group algebra kG (or of the center $Z(\mathcal{O}G)$ of the group algebra $\mathcal{O}G$).

Lemma 4.1. *Let H be a finite group, and let N, J be normal subgroups of H such that $N \leqslant J$. Suppose that H/J is a p-group and that J/N is a cyclic p'-group. Suppose further that b is an H-stable block of kN of defect* 0. *Then there is a block f of kJ which is H-stable and such that $bf = f$. Further, if J/N is central in H/N, then any block f of kJ such that $bf = f$ is H-stable.*

Proof. Let W be the unique simple kNb-module. Then W is H-stable and hence W is J-stable. Now since J/N is a cyclic p'-group it follows that W extends in $[J:N]$ ways to a kJ-module and these are all the simple kJ-modules covering W. Let S be the set of isomorphism classes of these extensions. Since W is H-stable, H acts on S by conjugation. Since the normal subgroup J of H is clearly in the kernel of this action, H/J acts on S. But H/J is a p-group and $|S| = [J:N]$ is prime to p, so this action must have a fixed point. In other words, there exists a kJ-module V such that $\mathrm{Res}^J_N(V) \cong W$ and such that ${}^xV \cong V$ for all $x \in H$. Let f be the block of kJ containing V. Then f has the required properties.

Now suppose that J/N is central in H/N, let V be an H-stable extension of W to J as above, and let U be any extension of W to J. Then there is a 1-dimensional kJ/N-module Z such that $U \cong V \otimes_k \mathrm{Inf}^J_{J/N}(Z)$, where $\mathrm{Inf}_{J/N} Z$ is the inflation to J via the canonical map $J \to J/N$, and where $V \otimes_k \mathrm{Inf}^J_{J/N}(Z)$ has the kJ-module structure given by

$$x.(v \otimes z) = xv \otimes xz, \quad \text{for } x \in J,\ v \in V,\ z \in \mathrm{Inf}^J_{J/N}(Z).$$

Since J/N is central in H/N, we have ${}^{gJ}Z \cong {}^gJZ$, and hence ${}^gU \cong U$ for any $g \in H$. The result now follows as above. □

In the sequel, we will use the following fact about block idempotents without comment: if N is a normal subgroup of a finite group G such that G/N is a p-group, then any block of kG is a central idempotent of kN, and consequently, if b is a block of kN which is stable under the conjugation action of G on N, then b is a block of kG.

Lemma 4.2. *Let L be a finite group, and let G be a subgroup of L and N, J be normal subgroups of L. Suppose that $L = JG$, $J \cap G = N$, that G/N is a p-group and that J/N is a cyclic p'-group. Suppose further that b is an L-stable block of kN of defect* 0 *and that f is an L-stable block of kJ such that $bf = f$. Then*

(i) *The blocks kGb and kLf are nilpotent.*

Furthermore, letting U be the unique (up to isomorphism) kGb-module and V the unique (up to isomorphism) simple kLf-module, and letting P be a defect group of kGb, we have the following assertions.

(ii) $\mathrm{Res}^L_G(V) \cong U$.

(iii) *P is a defect group of kLf.*

(iv) *The map*

$$kGb \to kLf, \quad a \mapsto af$$

is an interior P-algebra isomorphism, and induces an isomorphism of interior P-algebras between a source algebra of kGb and that of kLf.

(v) *There is a pair (P, W) which is a vertex–source pair of the kG-module U and which is also a vertex–source pair for the kL-module V.*

Proof. Since G/N is a p-group and b is a G-stable block of kN, it is clear that kGb is a nilpotent block and that $P \cong G/N$. Since J/N is a p' group, and f covers b, kJf is a block of defect 0. Also $L/J = GJ/J \cong G/N$, and G/N is a p-group, so it follows that kLf is nilpotent with defect groups isomorphic to $G/N \cong P$. Now $b = \mathrm{Tr}^G_P(x)$, for some $x \in (kG)^P$. Since $f \in Z(kL)$ this yields

$$f = bf = \mathrm{Tr}^G_P(xf).$$

The fact that $[L : G]$ is prime to p then gives

$$f = \mathrm{Tr}^L_G\left(\frac{1}{[L:G]} f\right) = \mathrm{Tr}^L_P\left(\frac{1}{[L:G]} xf\right),$$

from which it follows that P is contained in a defect group of kLf. But P is isomorphic to the defect groups of kLf, proving (iii).

Now $\mathrm{Res}^G_N(U)$ is the unique (up to isomorphism) simple kNb-module, so since b is L-stable it follows that $\mathrm{Res}^G_N(U)$ is J-stable. Since J/N is a cyclic p'-group, $\mathrm{Res}^G_N(U)$ extends in precisely $[J : N]$ ways to a kJ-module, each of which has the same dimension as U, and furthermore, these $[J : N]$ extensions are the only simple kJ-modules whose restriction to N contain direct summands isomorphic to $\mathrm{Res}^G_N(U)$. On the other hand, $\mathrm{Res}^L_J(V)$ is a simple kNf-module, so the fact that $fb = f$ means that $\mathrm{Res}^L_J(V)$ is one of these $[J : N]$ extensions. Thus

$$\mathrm{Res}^G_N(\mathrm{Res}^L_G(V)) = \mathrm{Res}^J_N(\mathrm{Res}^L_J(V)) = \mathrm{Res}^G_N(U).$$

But U is the unique extension of $\mathrm{Res}^G_N(U)$ to a kG-module, hence $\mathrm{Res}^L_G(V) \cong U$, proving (ii).

We now prove (iv). The second assertion of (iv) is an immediate consequence of the first assertion of (iv) and (iii). Thus it suffices to prove that the map $a \mapsto af$ is an isomorphism of interior P-algebras between kGb and kLf. Since f is a central idempotent of kL, the map is clearly a P-algebra homomorphism. Furthermore, since by (i), (ii) and (iii) both kLf and kGb are nilpotent, with defect groups of the same order and with simple modules of the same k-dimension, it follows that kLf and kGb have the same k-dimension; thus it suffices to prove that the map $a \mapsto af$ is injective. Let $a \in kGb$ be such that $af = 0$. Let P be the projective cover of the simple kLf-module V. Then $af = 0$ implies that $aP = afP = 0$. But it is easy to see that $\mathrm{Res}^L_G(P)$ is a projective cover of U. Thus a annihilates the unique projective kGb-module, and hence $akGb = 0$, that is, $a = 0$. Thus the map $a \mapsto af$ is injective, and this finishes the proof of (iv). Statement (v) is a consequence of (iv). □

Lemma 4.3. *Let G be a finite group with $O_p(G) = 1$. Let N be a normal subgroup of G. If G/N is a p-group, then $C_G(N) = Z(N)$ is a p'-group. Consequently, the canonical homomorphism of G/N into* $\mathrm{Out}(N)$ *is injective.*

Proof. Let P be a Sylow p-subgroup of G. Then $G = NP$, and hence $C_P(N)$ is a normal subgroup of G (as $C_P(N)$ is normalized by both N and P). By hypothesis, $C_P(N) = 1$. Thus $C_G(N)$ is a p'-group and it also follows from this that $C_G(N) \leqslant Z(N)$. □

Lemma 4.4. *Let N, G, G' be finite groups such that both G and G' contain N as a normal subgroup. Let $\gamma_G : G \to \mathrm{Aut}(N)$ and $\gamma_{G'} : G' \to \mathrm{Aut}(N)$ be the canonical homomorphisms and let $\pi : \mathrm{Aut}(N) \to \mathrm{Out}(N)$ be the canonical surjection. Suppose that G/N and G'/N are both p-groups, that $O_p(G) = 1 = O_p(G')$, and that kN has a G-stable block of defect 0. If $\pi \circ \gamma_G(G)$ and $\pi \circ \gamma_{G'}(G')$ are conjugate subgroups of* $\mathrm{Out}(N)$, *then G and G' are isomorphic through an isomorphism which stabilizes N. Furthermore, if $\pi \circ \gamma_G(G) = \pi \circ \gamma_{G'}(G')$, then the above isomorphism may be chosen to be the identity on N.*

Proof. Since G is a p extension of N and b is G-stable, b is a block of kG. Let P be a defect group of kGb. Then $G = NP$, a semidirect product. Set $P_0 := \gamma_G(P)$ and let

$$\phi : G \to N \rtimes P_0$$

be defined by

$$\phi(nx) = n\gamma_G(x), \quad \text{for } n \in N,\ x \in P.$$

By Lemma 4.3, it follows that γ_G induces an isomorphism between P and P_0, that ϕ is an isomorphism and that $\gamma_G(G) = \mathrm{Inn}(N)P_0$, a semidirect product.

Now G and G' both contain N. Hence, by hypothesis, there exists $\alpha \in \mathrm{Aut}(N)$ such that

$$\gamma_{G'}(G') = {}^{\alpha}\gamma_G(G) = \mathrm{Inn}(N)^{\alpha}P_0.$$

If $\pi \circ \gamma_G(G) = \pi \circ \gamma_{G'}(G')$, then choose α to be the identity.

Let U be the full inverse image of ${}^{\alpha}P_0$ in G' through $\gamma_{G'}$ and let S be a Sylow p-subgroup of U. By Lemma 4.3, applied to G', the map $\gamma_{G'}$ induces an isomorphism between S and ${}^{\alpha}P_0$. Now

$$\gamma_{G'}(NS) = \mathrm{Inn}(N)\,{}^{\alpha}P_0 = \gamma_{G'}(G')$$

and by Lemma 4.3, $\mathrm{Ker}(\gamma_{G'}) = Z(N) \leqslant N$, hence $G' = NS$. On the other hand,

$$|N|\,|S| = |N|\,|{}^{\alpha}P_0| = |Z(N)|\,|\mathrm{Inn}(N)|\,|{}^{\alpha}P_0| = |\mathrm{Ker}(\gamma_{G'})|\,|\gamma_{G'}(G')| = |G'|.$$

Thus $G = NS$, a semidirect product, and the map

$$\varphi : G' \to N \rtimes {}^{\alpha}P_0$$

defined by

$$\varphi(nx) = n\gamma_{G'}(x), \quad \text{for } n \in N,\ x \in S,$$

is an isomorphism.

Let

$$\psi : N \rtimes P_0 \to N \rtimes {}^{\alpha}P_0$$

be defined by

$$\psi(nx) = \alpha(n)\,{}^{\alpha}x, \quad \text{for } n \in N,\ x \in P_0.$$

Then $\varphi^{-1} \circ \psi \circ \phi : G \to G'$ is an isomorphism with the required properties. □

Remarks. (i) The condition that kN has a G-stable block of defect 0 in the above may be replaced by the weaker condition that N has a complement in G.

(ii) The above proposition may be also understood more structurally as a consequence of the fact that, as $Z(N)$ is an abelian p'-group, and as G/N (respectively G'/N) is a p-group, restriction from G/N (respectively G'/N) induces an injective map from $H^2(G/Z(N), Z(N))$ (respectively G'/N) to $H^2(N/Z(N), Z(N))$, and of the fact that $G/Z(N)$ and $G'/Z(N)$ are isomorphic groups.

5 On characters and blocks of finite general linear and unitary groups and their commutator subgroups

The aim of this section is to prove Theorem 1.4. For this, we recall some facts from the Lusztig parametrization of characters of finite groups of Lie type in our special situation. We will follow [1] and [3].

Notation. We keep the notation of the previous sections and that introduced for the statement of Theorem 1.4. In particular, G will denote either a finite general linear group or a finite unitary group. For a semisimple element s of G, we let $[s]$ denote the G-conjugacy class of s.

5.A The ordinary characters of *G*. For a semisimple element s of G, we let $\mathscr{E}(G,[s])$ denote the rational Lusztig series corresponding to $[s]$. The subsets $\mathscr{E}(G,[s])$ as $[s]$ runs over the semisimple classes of G partition the set of ordinary irreducible characters of G; if an irreducible character χ belongs to the subset $\mathscr{E}(G,[s])$, we will say that χ has semisimple label $[s]$. Now let ϕ be the automorphism of G which appears in the statement of Theorem 1.4 (so ϕ is only defined when q is a power of p).

The above labelling of characters is compatible with the action of ϕ in the following sense.

Lemma 5.1. *Let χ be an irreducible character of G. Let $[s]$ be the semisimple label of χ and $[t]$ be the semisimple label of ${}^{\phi}\chi$. Then $[s]=[\phi(t)]$.*

Proof. This is immediate from [9, Corollary 2.4]. □

5.B The blocks of *kG*. For the rest of this section, we will assume that $p \mid q-\varepsilon$. Then by the Fong–Srinivasan classification of the blocks of the finite general linear and unitary groups [4], two ordinary irreducible characters χ and χ' of G lie in the same p-block of G if and only if the p'-parts of s and s' are conjugate in G, where $[s]$ and $[s']$ are the semisimple labels of χ and χ' respectively. Thus the p-blocks of G are partitioned by G-conjugacy classes of p'-semisimple elements of G. If b is a block of G, the p'-part of the semisimple label of whose irreducible characters is the conjugacy class $[t]$, we say that b has semisimple label $[t]$. If b has label $[t]$, then any Sylow p-subgroup of $C_G(t)$ is a defect group of b.

5.C Characters of *N*. Let ψ be an ordinary irreducible character of N, let χ be an irreducible character of G covering ψ and let $[s]$ be the semisimple label of χ. Let $I(\psi)$ be the stabilizer of ψ in G. If χ' is another character of G covering ψ, the semisimple label of χ' is $[sz]$ for some $z\in Z(G)$, and conversely, for any $z\in Z(G)$, there exists some irreducible character χ' of G with label $[sz]$ covering ψ (see for instance [1, Proposition 11.7]). If we set $d_{[s]}$ to be the number of distinct conjugacy classes of G of the form $[sz]$, where $z\in Z(G)$, we get that $[I(\psi):N]$ is divisible by $d_{[s]}$ (see [1, Corollary 11.13]).

Now let s be as above. Multiplication yields a transitive action of $Z(G)$ on the set of conjugacy classes of G of the form $[sz]$ with $z\in Z(G)$. Denote by $Z(s)$ the stabilizer of $[s]$ under this action, so that $d_{[s]}=(q-\varepsilon)/|Z(s)|$ and the number of G-conjugates of ψ divides $|Z(s)|$.

Suppose that $z\in Z(s)$. If $G=\mathrm{GL}_n(q)$, then since s is semisimple there is a diagonal matrix $\mathrm{diag}(\alpha_1,\dots,\alpha_n)\in\mathrm{GL}_n(\overline{\mathbb{F}}_q)$ such that s is conjugate to $\mathrm{diag}(\alpha_1,\dots,\alpha_n)$ in $\mathrm{GL}_n(\overline{\mathbb{F}}_q)$. Thus sz is conjugate to $\mathrm{diag}(\zeta\alpha_1,\zeta\alpha_2,\dots,\zeta\alpha_n)$, where $z=\mathrm{diag}(\zeta,\dots,\zeta)$ and $\zeta\in\mathbb{F}_q^{\times}$. Hence $\mathrm{diag}(\alpha_1,\alpha_2,\dots,\alpha_n)$ is conjugate to $\mathrm{diag}(\zeta\alpha_1,\dots,\zeta\alpha_n)$. Taking determinants, one sees that $\zeta^n=1$. Furthermore, letting $o(s)$ denote the order of s, we have that $1=(\mathrm{diag}(\alpha_1,\alpha_2,\dots,\alpha_n))^{o(s)}$ is conjugate to $z^{o(s)}$, hence $z^{o(s)}=1$. Of course

$z^{q-\varepsilon} = 1$. Thus $Z(s)$ is a cyclic group of order dividing $\gcd(n, q-1, o(s))$. Arguing similarly when $G = \mathrm{GU}_n(q)$ and noting that $Z(\mathrm{GU}_n(q))$ consists of scalar matrices of order dividing $q+1$, we obtain that $Z(s)$ is a cyclic group of order dividing $\gcd(n, q+1, o(s))$.

Summarizing the above discussion we get the following result.

Proposition 5.2. *Let χ be an ordinary irreducible character G and let $[s]$ be the semisimple label of χ. Let ψ be an irreducible character of N covered by χ.*

(i) *The number of G-conjugates of ψ is a divisor of $|Z(s)|$, and $I(\psi)/N$ has order divisible by $(q-\varepsilon)/|Z(s)|$.*

(ii) *$|Z(s)|$ is a divisor of* $\gcd(n, q-\varepsilon, o(s))$.

We also get an analogous result for blocks:

Proposition 5.3. *Let b be a block of G and let $[t]$ be the semisimple label of b. Let c be a block of N covered by b. If b' is a block of G covering c, then the semisimple label of b' is of the form $[tz]$ for some $z \in Z(G)$; the number of elements in the G-orbit of c is a divisor of $|Z(t)|$ and in particular, this number is prime to p.*

Proof. Since b and b' cover the same blocks of N, there is a sequence $b =: b_1, b_2, \ldots, b_r := b'$ of blocks of G and ordinary characters χ_i in b_i for $1 \leqslant i \leqslant r$ such that for each i with $1 \leqslant i \leqslant r-1$, χ_i and χ_{i+1} cover a common character of N. Let $[s_i]$ be the semisimple label of χ_i for $1 \leqslant i \leqslant r$. Then from the above discussion, it follows that for $1 \leqslant i \leqslant r-1$ there is some $z_i \in Z(G)$ such that $[s_{i+1}] = [s_1 z_i]$. Setting t_i to be the p'-part of s_i and v_i to be the p'-part of z_i, it follows that $[t_{i+1}] = [t_i v_i]$ for $1 \leqslant i \leqslant r-1$. This proves the first assertion.

The block b contains an irreducible character, say χ, with semisimple label $[t]$ and c contains a character, say ψ, covered by χ. Since $I(b)$ contains $I(\psi)$, the second statement follows from Proposition 5.2(i). Since t is a p'-element, the last assertion is immediate from Proposition 5.2(ii). □

In what follows we will identify $\mathrm{GL}_n(q)$ with the group of invertible linear transformations of an n-dimensional vector space over a field of q elements, and identify $\mathrm{GU}_n(q)$ with a subgroup of $\mathrm{GL}_n(q^2)$ in the natural way. As before, we assume that p divides $q-\varepsilon$. In addition, we assume from now on that $q = q'^p$, and ϕ is the automorphism of G appearing in the statement of Theorem 1.4.

Lemma 5.4. *Let t be a semisimple element of* $\mathrm{GL}_n(q)$ *and suppose that the minimal polynomial of t over $\mathbb{F}_q$ has an irreducible factor whose degree is distinct from the degrees of all other irreducible factors of the minimal polynomial of t. Suppose that $[\phi(t)] = [tz]$ for some $z = \mathrm{diag}(\eta, \eta, \ldots, \eta) \in Z(\mathrm{GL}_n(q))$. Then $|Z(t)|$ divides $q'-1$.*

Proof. Let $p(x)$ be an irreducible factor of the minimal polynomial of t with degree distinct from that of every other irreducible factor. Let λ be a root of $p(x)$ in $\overline{\mathbb{F}}_q$. and let r be the order of λ. We claim that there exists a positive integer v such that v is relatively prime to p and such that the order r is a factor of $(q'^v - 1)(q - 1)$. Indeed, by hypothesis, $\lambda^{q'}$ is an eigenvalue of tz, i.e. $\lambda^{q'} = \lambda'\eta$ for some eigenvalue λ' of t. The minimal polynomials of λ and $\lambda^{q'}$ over $\mathbb{F}_q$ have the same degree and the minimal polynomials of $\lambda'\eta$ and λ' over $\mathbb{F}_q$ also have the same degree. It follows that λ' is a root of $p(x)$, that is, $\lambda^{q'} = \lambda^{q^u}\eta$ for some u. This gives $\lambda^{q'(q'^{pu-1}-1)} = \eta \in \mathbb{F}_q$, whence $\lambda^{(q'^{pu-1}-1)} \in \mathbb{F}_q$. The claim follows by setting $v = pu - 1$.

Now let $y = \mathrm{diag}(\zeta, \zeta, \ldots, \zeta) \in Z(t)$, and let λ be a root of $p(x)$. We claim that $\zeta = \lambda^{q^m - 1}$, for some integer m. Indeed, the eigenvalues of yt are of the form $\zeta\alpha$, where α is an eigenvalue of t. Hence $[t] = [yt]$ implies that $\lambda\zeta$ is also an eigenvalue of t. Again, since $\zeta \in \mathbb{F}_q$, the minimal polynomial of $\lambda\zeta$ over $\mathbb{F}_q$ has the same degree as the minimal polynomial of λ over $\mathbb{F}_q$, which means that $\lambda\zeta$ is a root of $p(x)$, so $\lambda\zeta = \lambda^{q^m}$ for some m, proving the claim.

Since $\zeta^{q-1} = 1$, it follows from the claim that r is a divisor of

$$(q^m - 1)(q - 1) = (q'^{pm} - 1)(q - 1).$$

Combining this with what we showed previously, we conclude that r is a factor of $\gcd((q'^{pm} - 1)(q - 1), (q'^v - 1)(q - 1)) = (q'^{\gcd(m,v)} - 1)(q - 1)$, the last equality holding because v is relatively prime to p. Since $q^m - 1$ is divisible by $q'^{\gcd(m,v)} - 1$, and by $q - 1$ and since $\gcd(q'^{\gcd(m,v)} - 1, q - 1) = q' - 1$, this gives that r is a factor of $(q^m - 1)(q' - 1)$. Thus

$$\zeta^{(q'-1)} = \lambda^{(q^m-1)(q'-1)} = 1.$$

It follows that $Z(t)$ has order dividing $q' - 1$. □

We need an analogous result for the unitary groups.

Lemma 5.5. *Let t be a semisimple element of* $\mathrm{GU}_n(q)$, *and suppose that t has an eigenvalue λ satisfying the following*:

for any eigenvalue λ' and any element $\mathrm{diag}(\zeta, \zeta, \ldots, \zeta) \in Z(\mathrm{GU}_n(q))$ *such that either $\lambda^{-q'}$ or λ equals $\zeta\lambda'$, we have $\lambda' = \lambda^{(-q)^m}$ for some non-negative integer m.*

If $[\phi(t)] = [tz]$ for some $z = (\eta, \eta, \ldots, \eta) \in Z(\mathrm{GU}_n(q))$, then $|Z(t)| \mid q' + 1$.

Proof. Let r be the order of λ. We claim that there exists a positive integer v such that v is relatively prime to p and such that the order r is a factor of $((-q')^v - 1)(q + 1)$. Indeed, by hypothesis, $\lambda^{-q'} = \lambda'\eta$ for some eigenvalue λ' of t, and some $\eta \in \overline{\mathbb{F}}$ such that $\eta^{q+1} = 1$. Thus $\lambda^{-q'} = \lambda^{(-q)^u}\eta$ for some u. This gives $\lambda^{q'((-q)'^{pu-1}-1)} = \eta$, and $\eta^{q+1} = 1$. The claim follows by setting $v = pu - 1$.

Now let $y = \mathrm{diag}(\zeta, \zeta, \ldots, \zeta) \in Z(t)$. So $[t] = [yt]$ implies $\lambda\zeta$ is also an eigenvalue of t. Again, the nature of λ gives that $\lambda\zeta = \lambda^{(-q)^m}$ for some m. Since $\zeta^{q+1} = 1$, it follows from the above that r is a divisor of $((-q)^m - 1)(q + 1) = ((-q')^{pm} - 1)(q + 1)$. Combining this with what we showed previously, we conclude that r is a factor of $\gcd(((-q')^{pm} - 1)(q + 1), ((-q')^v - 1)(q + 1))$.

We claim that $d := \gcd(q+1, (-q')^v - 1) = q' + 1$. Indeed, suppose first that v is even, say $v = 2w$. Then $(-q')^v = q'^v$ and d is a factor of

$$\gcd(q'^{2p} - 1, q'^{2w} - 1) = q'^2 - 1,$$

the last equality holding because v is relatively prime to p. Thus $d \mid q'^2 - 1$ and $d \mid q'^p + 1$. Now $\gcd(q' - 1, q'^p + 1)$ is 1 if q' is even and is 2 otherwise. On the other hand, since p is odd, $(q'^p + 1)/(q' + 1)$ is odd if q is odd. The claim follows. Suppose next that v is odd. In this case, $(-q')^v = -q'^v$, and $d = \gcd(q+1, q'^v + 1)$. Again, since v is relatively prime to p and p is odd, the claim follows.

Since $q+1$ is a factor of $(-q')^{pm} - 1$, it follows from the claim that $\gcd(((-q')^{pm} - 1)(q+1), ((-q')^v - 1)(q+1))$ is a factor of $((-q')^{pm} - 1)(q'+1)$, from which we get that

$$\zeta^{(q'+1)} = \lambda^{((-q')^{pm}-1)(q'+1)} = 1.$$

The above shows that $Z(t)$ has order dividing $q' + 1$. □

Notation. For any positive integer m, let m_+ denote the p-part of m and let m_- denote the p'-part of m.

Lemma 5.6. *Suppose that $p \mid q - 1$ and $p \mid n$. Suppose also that $\mathrm{SL}_n(q)$ has a block c with central defect group. Let b be a block of $\mathrm{GL}_n(q)$ covering $\mathrm{SL}_n(q)$, let $[t]$ be the semisimple label of b, and let $f(x)$ be the characteristic polynomial of t. Then one of the following holds*:

(i) *$f(x)$ is irreducible and $n_+ \leqslant (q-1)_+$;*

(ii) *$n_+ \geqslant (q-1)_+$, and $f(x)$ is a product $f(x) = p_1(x)p_2(x)$, where $p_1(x)$ and $p_2(x)$ are irreducible polynomials such that $\deg(f_1(x)) \neq \deg(f_2(x))$ and neither $\deg(f_1(x))$ nor $\deg(f_2(x))$ is divisible by p.*

In particular, $f(x)$ has an irreducible factor whose degree is distinct from any other irreducible factor of $f(x)$.

Proof. Let $f(x) = \prod_{1 \leqslant i \leqslant u} p_i(x)^{m_i}$ be a prime factorization of $f(x)$ in $\mathbb{F}_q[x]$. Let R be a defect group of the block b of $k\mathrm{GL}_n(q)$. Since p divides $q-1$, R is conjugate to a Sylow p-subgroup of $C_G(t)$. Then $R \cap \mathrm{SL}_n(q)$ is a defect group of $k\mathrm{SL}_n(q)c$. But by hypothesis, the Sylow p-subgroup of $Z(\mathrm{SL}_n(q))$ is the (unique) defect group of $k\mathrm{SL}_n(q)c$. Since the Sylow p-subgroup of $Z(\mathrm{SL}_n(q))$ is a cyclic group of order d_+, where $d = \gcd(n, q-1)$ and since $R/(R \cap \mathrm{SL}_n(q))$ is a cyclic group of order dividing $q-1$, it follows that R is metacyclic and has p-rank at most 2. Now by the prime decomposition of $f(x)$ above, if n_i is the degree of $p_i(x)$ for each i then we have

$$C_{\mathrm{GL}_n(q)}(t) \cong \prod \mathrm{GL}_{m_i}(q^{n_i}).$$

Since p divides $q-1$, the fact that R is metacyclic and is a Sylow p-subgroup of $C_{\mathrm{GL}_n(q)}(t)$ forces $u \leqslant 2$, and either $m_i = 1$ for all i, or $u = 1$ and $m_1 = 2$.

Suppose that $u = 1$ and that $f(x)$ is irreducible and $|R| = (q^n - 1)_+ = (q-1)_+ n_+$. Suppose, if possible that $n_+ > (q-1)_+$. Then $|R| > (q-1)d_+$, a contradiction. Hence in this case, case (i) of the proposition holds.

Next, suppose that $u = 2$, so that $f(x) = p_1(x)p_2(x)$, and $R \cong R_1 \times R_2$ where R_i is a Sylow p subgroup of $\mathrm{GL}_{m_i}(q^{n_i} - 1)$, for $i = 1, 2$. So we get

$$(q-1)_+ d_+ \geqslant |R| = (q^{n_1} - 1)_+ (q^{n_2} - 1)_+ = (q-1)_+^2 n_{1+} n_{2+},$$
$$d_+ = (q-1)_+ n_{1+} n_{2+},$$

which implies that $n_+ \geqslant (q-1)_+$ and that n_1 and n_2 are not divisible by p. Now if $n_1 = n_2$, then n is a p' number, a contradiction. Finally, consider the case that $u = 1$ and $m_1 = 2$. Then $f(x) = p_1^2(x)$, and the same argument as above leads to a contradiction.

The last assertion is immediate from the description of $f(x)$. □

Notation. For each irreducible polynomial $p(x) \in \mathbb{F}_{q^2}[x]$ different from x, we let $p_-(x)$ be the irreducible polynomial in $\mathbb{F}_{q^2}[x]$ whose roots are of the form λ^{-q}, where λ is a root of $p(x)$. We say that $p(x)$ is of unitary type if $p(x) = p_-(x)$ and of non-unitary type otherwise.

Note that if t is a semisimple element of $\mathrm{GU}_n(q)$, and if $f(x)$ is the characteristic polynomial of t over $\mathbb{F}_{q^2}$, then for any irreducible $p(x) \in \mathbb{F}_{q^2}[x]$, the multiplicity of $p(x)$ as a divisor of $f(x)$ is the same as that of $p_-(x)$.

Lemma 5.7. *Suppose that $p \mid q+1$ and $p \mid n$. Suppose also that c is a block of $\mathrm{SU}_n(q)$ with central defect group. Let b be a block of $\mathrm{GU}_n(q)$ covering $\mathrm{SU}_n(q)$, let $[t]$ be the semisimple label of b, and let $f(x)$ be the characteristic polynomial of t over $\mathbb{F}_{q^2}$. Then one of the following holds*:

(i) *$f(x)$ is irreducible, of unitary type and $n_+ \leqslant (q+1)_+$*;

(ii) *$n_+ \geqslant (q+1)_+$, and $f(x)$ is a product $f(x) = p_1(x)p_2(x)$, where $p_1(x)$ and $p_2(x)$ are irreducible polynomials of unitary type such that $\deg(p_1(x)) \neq \deg(p_2(x))$ and neither $\deg(f_1(x))$ nor $\deg(f_2(x))$ is divisible by p*;

(iii) *$f(x) = p(x)p_-(x)$ where $p(x)$ is irreducible of non-unitary type*;

(iv) *$f(x) = p(x)p_-(x)g(x)g_-(x)$, where $p(x)$ and $g(x)$ are irreducible polynomials of non-unitary type such that $\deg(p(x)) \neq \deg(g(x))$ and neither $\deg(p(x))$ nor $\deg(g(x))$ is divisble by p*;

(v) *$f(x) = p(x)g(x)g_-(x)$, where $p(x)$ is of unitary type and $g(x)$ is of non-unitary type.*

In particular, t has an eigenvalue λ satisfying the conditions of Proposition 5.5.

Proof. The proof is entirely similar to that for the linear case. Let

$$f(x) = \prod_{1 \leqslant i \leqslant u} p_i(x)^{m_i} \prod_{1 \leqslant i \leqslant v} [(g_i(x)(g_{i\_}(x))]^{n_i}$$

be a prime factorization of $f(x)$ in $\mathbb{F}_q[x]$, where the $p_i(x)$ are of unitary type and the $g_i(x)$ are of non-unitary type. Then if d_i is the degree of $p_i(x)$ for $i \leqslant u$ and e_i is the degree of $g_i(x)$ for $i \leqslant v$, we have (see for instance [4, Proposition 1A])

$$C_{\mathrm{GU}_n(q)}(t) \cong \prod_i \mathrm{GU}_{m_i}(q^{d_i}) \prod_i \mathrm{GL}_{n_i}(q^{2e_i}).$$

Furthermore, if R is a defect group of the block b of $\mathrm{GU}_n(q)$ then since p divides $q+1$, R is conjugate to a Sylow p-subgroup of $C_G(t)$. On the other hand, R is metacyclic of order at most $(q+1)^2$. Now proceeding as for the general linear groups, we get that $f(x)$ satisfies one of (i)–(v). If $f(x)$ is of type (i), (ii), (iii) or (iv) take for λ a root of $f(x)$. Let λ' be any eigenvalue of $f(x)$ and let $\zeta \in \mathbb{F}_{q^2}$ satisfy $\zeta^{q+1} = 1$. Since $\mathbb{F}_{q^2}[\lambda^{-q'}] = \mathbb{F}_{q^2}[\lambda]$ and $\mathbb{F}_{q^2}[\lambda'] = \mathbb{F}_{q^2}[\zeta\lambda']$, if either λ or $\lambda^{-q'}$ equals $\zeta\lambda'$, then $\mathbb{F}_{q^2}[\lambda'] = \mathbb{F}_{q^2}[\lambda]$, and the degree constraints in (i)–(iv) yield that $\lambda' = \lambda^{(-q)^m}$ for some m. If $f(x)$ is of type (v), take for λ a root of $p(x)$. Let λ' be any eigenvalue of $f(x)$ and let $\zeta \in \mathbb{F}_{q^2}$ be such that $\zeta^{q+1} = 1$ and such that either λ or $\lambda^{-q'}$ equals $\zeta\lambda'$. Since λ is $\mathbb{F}_{q^2}$-conjugate to λ^{-q}, $\lambda^{-q'}$ is $\mathbb{F}_{q^2}$-conjugate to $(\lambda^{-q'})^{-q}$, hence the same is true of $\lambda'\zeta$ and of λ'. But $p(x)$ is the only unitary factor of $f(x)$, and so $\lambda' = \lambda^{(-q)^m}$ for some m. □

Proof of Theorem 1.4. Assume that the conditions of Theorem 1.4 hold. Let b be a block of G covering c and let $[t]$ be the semisimple label of b. Since c is ϕ-stable, $\phi^{-1}(b)$ also covers c. By Lemma 5.1, $\phi^{-1}(b)$ has semisimple label $[\phi(t)]$. Thus, by Proposition 5.3, we have that $[\phi(t)] = [tz]$ for some $z = \mathrm{diag}(\eta, \eta, \ldots, \eta) \in Z(G)$. It follows from Lemmas 5.4–5.7 that $|Z(t)|$ is a factor of $q' - \varepsilon$. On the other hand, by Proposition 5.3, $[G : I(c)]$ is a factor of $|Z(t)|$. This means that if $h \in G$ is such that $\det(h) = \alpha^{q'-\varepsilon}$ for some $\alpha \in \mathbb{F}_q$, then $h \in I(c)$. Now let $g \in G$ and set $h = \phi(g)g^{-1}$. Then $\det(h) = \det(g)^{q'-\varepsilon}$, hence $h \in I$. But this means exactly that $I(c) \rtimes \langle\phi\rangle$ is normal in $G \rtimes \langle\phi\rangle$. □

6 Proof of Theorem 1.2

Notation. We keep the notation of Theorem 1.2. In addition, if the simple quotient of N is $\mathrm{PSL}_n(q)$, let $K_0 := \mathrm{GL}_n(q)$ and let $N_0 := \mathrm{SL}_n(q)$. If the simple quotient of N is $\mathrm{PSU}_n(q)$, let $K_0 := \mathrm{GU}_n(q)$ and let $N_0 := \mathrm{SU}_n(q)$. Set $Z := Z(K_0)$.

If $q = q'^p$ and $K_0 = \mathrm{GL}_n(q)$, we define $\phi : K_0 \to K_0$ to be the automorphism $(a_{ij}) \mapsto (a_{ij}^{q'})$. If $q = q'^p$ and $K_0 = \mathrm{GU}_n(q)$, we define $\phi : K_0 \to K_0$ be the automorphism $(a_{ij}) \mapsto (a_{ij}^{q'})^{t^{-1}}$. Note that ϕ is an automorphism of K_0 of order p, N_0 is ϕ-stable, and since K_0/N_0 is cyclic, any subgroup of K_0 containing N_0 is also ϕ-stable.

If $K_0 = \mathrm{GL}_n(q)$ we set $\varepsilon = 1$ and if $K_0 = \mathrm{GU}_n(q)$, we set $\varepsilon = -1$.

For an abelian group H, we will let H_+ denote the Sylow p-subgroup of H and let H_- denote the Hall p'-subgroup of H.

We first prove the following result detailing the structure of G.

Proposition 6.1. *With the notation and assumptions of Theorem* 1.2, *suppose that W is not self-dual. Then there exists a subgroup Z_0 of $Z(N_0)$ containing the Sylow p-subgroup of $Z(N_0)$ such that N is isomorphic to N_0Z_+/Z_+Z_0. Further, $q = q'^p$ for some prime power q', the index of N_0Z_+/Z_+Z_0 in K_0/Z_+Z_0 is divisible by p and letting M be the unique subgroup of K_0/Z_+Z_0 containing N_0Z_+/Z_+Z_0 as a subgroup of index p, there exists an isomorphism, $G \cong M \rtimes \langle\phi\rangle$ sending N to N_0Z_+/Z_+Z_0.*

Proof. We make a series of reductions.

6.2. $O_p(G) = 1$.

Proof. If not, then W is isomorphic to $\mathrm{Inf}^P_{P/O_p(G)}\, W'$ for some endo-permutation module for the cyclic group $P/O_p(G)$, and clearly W is self-dual.

6.3. *p is a divisor of* $\gcd(q - \epsilon, n)$ *and $q = q'^p$ for some prime power q'.*

Proof. By Lemma 4.3, G/N is isomorphic to a subgroup of $\mathrm{Out}(N)$. Since N is quasi-simple, $\mathrm{Out}(N)$ is in turn isomorphic to a subgroup of $\mathrm{Out}(N/Z(N))$. In particular, $\mathrm{Out}(N/Z(N))$ has an elementary abelian subgroup of order p^2. Since p is odd, the result is immediate from the nature of the outer automorphism groups of $\mathrm{PSL}_n(q)$ and of $\mathrm{PSU}_n(q)$ (see [5, Theorem 2.5.1]).

6.4. *N is isomorphic to N_0/Z_0, where Z_0 is a central subgroup of N_0 containing the Sylow p-subgroup of $Z(N_0)$.*

Proof. By 6.3, and since p is odd, $N/Z(N)$ is not one of the groups $\mathrm{PSL}_2(4)$, $\mathrm{PSL}_3(2)$, $\mathrm{PSL}_3(4)$, $\mathrm{PSL}_4(2)$, or $\mathrm{PSL}_2(9)$, $\mathrm{PSU}_4(2)$, $\mathrm{PSU}_6(2)$ or $\mathrm{PSU}_4(3)$. Hence, the exceptional part of the Schur multiplier of $N/Z(N)$ is trivial (see [5, Table 6.1.3]). By [5, Table 6.1.2], N_0 is a universal covering group of $N/Z(N)$. By [5, Corollary 5.1.5]), N is a quotient of N_0 by a central subgroup, say Z_0. Finally, since $Z(N)$ is assumed to be a p'-group, it follows that Z_0 contains the Sylow p-subgroup of $Z(N_0)$.

It follows from 6.3 that $|K_0/Z_+Z_0 : N_0Z_+/Z_+Z_0|$ is divisible by p and that ϕ is defined. Let M be the unique subgroup of K_0/Z_+Z_0 containing N_0Z_+/Z_+Z_0 as a subgroup of index p.

6.5. *There exists an isomorphism $G \cong M \rtimes \langle\phi\rangle$ sending N to N_0Z_+/Z_+Z_0.*

Proof. Since Z_0 contains $Z_+ \cap N_0$, the inclusion of N_0 in K_0 induces an isomorphism from N_0/Z_0 to N_0Z_+/Z_+Z_0. Henceforth, we will identify N with the subgroup N_0Z_+/Z_+Z_0 of K_0/Z_+Z_0.

Let $G' = M \rtimes \langle \phi \rangle$. Then clearly $O_p(G') = 1$ and G'/N is elementary abelian of order p^2. Now since p is odd it follows from the structure of the outer automorphism groups of $\mathrm{PSL}_n(q)$ and $\mathrm{PSU}_n(q)$ that $\mathrm{Out}(N)$ has metacyclic Sylow p-subgroups. On the other hand, for odd p, metacyclic p-groups have at most one elementary abelian subgroup of order p^2 (see for instance [7, Lemma 2.1]). Thus, in the notation of Lemma 4.4, $\pi \circ \gamma_G(G)$ and $\pi \circ \gamma_{G'}(G')$ are conjugate subgroups of $\mathrm{Out}(N)$. The claim follows from Lemma 4.4. $\square$

Notation. If $N/Z(N) \cong \mathrm{PSL}_n(q)$, let $\tau : \mathrm{GL}_n(q) \to \mathrm{GL}_n(q)$ be the transpose automorphism. If $N/Z(N) \cong \mathrm{PSU}_n(q)$, let $\tau : \mathrm{GU}_n(q) \to \mathrm{GU}_n(q)$ be the automorphism which raises every entry to its qth power.

Proof of Theorem 1.2. Suppose if possible that W is not self-dual. By Proposition 6.1, we may assume that $G = M \rtimes \langle \phi \rangle$, where Z_0 and M are as in the statement of Proposition 6.1. We identify N with $N_0 Z_+/Z_+ Z_0$ as before.

Let c be the block (necessarily of defect 0) of kN containing the simple kN-module U and let c_0 be the unique block of kN_0Z_+ whose image under the canonical surjection of kN_0Z_+ onto kN is c. Let I be the inertial subgroup of c in K_0/Z_0Z_+ and let I_0 be the the inertial subgroup of c_0 in K_0.

6.6. *For any* $g \in K_0$, $g^\phi g^{-1} \in I_0$. *Further*, $K_{0+} \leqslant I_0$.

Proof. Since Z_+ is a central p-group, c_0 is also a block of kN_0. Furthermore, since c is a block of defect 0 of kN_0Z_+/Z_+Z_0, as block of kN_0, c_0 has central defect group. Thus, I_0 is the inertial subgroup in K_0 of a central defect block of kN_0. By hypothesis, U is G-stable, from which it follows that c_0 is ϕ-stable. The first assertion follows from Theorem 1.4 applied to K_0, N_0 and c_0 and the second follows from Proposition 5.3.

Set $J = N_0 I_{0-} Z_+/Z_+Z_0$ and set $L = M_0 I_{0-}/Z_+Z_0 \rtimes \langle \phi \rangle$, where M_0 is the inverse image of M in K_0. Then we have the following diagram of group inclusions:

$$(I_0/Z_+Z_0) \rtimes \langle \phi \rangle$$

$$L = M_0 I_{0-}/Z_+Z_0 \rtimes \langle \phi \rangle$$

$$G = M \rtimes \langle \phi \rangle \qquad\qquad J = N_0 I_{0-} Z_+/Z_+Z_0$$

$$N = N_0 Z_+/Z_+Z_0$$

Note that J and N are normal in $(I_0/Z_+Z_0) \rtimes \langle \phi \rangle$, that $(I_0/Z_+Z_0) \rtimes \langle \phi \rangle / J$ is a p-group and that J/N is isomorphic to a quotient of $I_{0-}/(I_{0-} \cap N_0 Z_+)$, hence is a cyclic p'-group. Thus by Lemma 4.1 (applied with $H = (I_0/Z_+Z_0) \rtimes \langle\phi\rangle$) there is a block f of kJ such that $f = bf$ and such that f is $(I_0/Z_+Z_0) \rtimes \langle\phi\rangle$-stable.

Further, we see that the conditions of Lemma 4.2 hold. Hence for some W', (P, W') is a vertex–source pair for the unique simple module, say V, of kLf and (P, W') is also a vertex–source pair of the kG-module U. We may assume without loss that $W' = W$.

Note that K_0 acts by conjugation on J.

6.7. *The group K_{0-} acts transitively on the K_0-orbit of f.*

Proof. By choice of f, I_0/Z_+Z_0, and hence I_0, stabilizes f. On the other hand, by 6.6 I_0 contains K_{0+}. The claim follows.

6.8. *K_{0-} normalizes L.*

Proof. Let $g \in K_{0-}$. By 6.6, $g^\phi g^{-1} \in I_0$, but since $g \in K_{0-}$, in fact $g^\phi g^{-1} \in I_{0-}$, proving the claim.

6.9. *The simple kL-module V is automorphically dual.*

Proof. Let e be a block of K_0/Z_+Z_0 such that $ef^\vee \neq 0$. Then ${}^\tau e {}^\tau f^\vee \neq 0$. But by Lemma 3.1 and its proof, ${}^\tau e = e^\vee$ and $e^\vee f \neq 0$, from which it follows that f and ${}^\tau f^\vee$ are covered by the same block of K_0, namely $e^\vee$. By 6.7 we get that ${}^\tau f^\vee = {}^g f$ for some $g \in K_{0-}$. Now, by definition, τ and ϕ commute as automorphisms of K_0, hence the action of τ on K_0 extends to an automorphism of $K_0 \rtimes \langle\phi\rangle$, and the group L is clearly invariant under this automorphism.

Let $\omega : L \to L$ be the map $x \mapsto g^{-1}\tau(x)g$. By 6.8, ω is well defined and is an automorphism of L. The claim follows by setting $\psi = \omega^{-1}$.

Now, as observed above, (P, W) is a vertex–source pair of the kL-module V. So, by 6.9 and by Proposition 1.3 (applied to the simple kL-module V), it follows that W is self-dual. □

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References

[1] C. Bonnafé. Sur les caractères des groupes réductifs finis à centre non connexe: applications aux groupes spéciaux linaires et unitaires. *Astérisque* **306** (2006).

[2] E. C. Dade. Endo-permutation modules over p-groups. II. *Ann. of Math* (2) **108** (1978), 317–346.

[3] F. Digne and J. Michel. *Representations of finite groups of Lie type*. London Math. Soc. Student Texts 21 (Cambridge University Press, 1991).

[4] P. Fong and B. Srinivasan, The blocks of finite general linear and unitary groups. *Invent. Math.* **69** (1982), 109–153.

[5] D. Gorenstein, R. Lyons and R. Solomon. *The classification of finite simple groups*, no. 3. Math. Surveys and Monographs 40.3 (American Mathematical Society, 1998).
[6] R. Gow. On the Schur indices of characters of finite classical groups. *J. London Math. Soc* (2) **24** (1981), 135–147.
[7] N. Mazza. The Dade group of a metacyclic p-group. *J. Algebra* **266** (2003), 102–111.
[8] N. Mazza. Endo-permutation modules as sources of simple modules. *J. Group Theory* **6** (2003), 477–497.
[9] G. Navarro, P. H. Tiep and A. Turull. Brauer characters with cyclotomic field of values. *J. Pure Appl. Algebra* **212** (2008), 628–635.
[10] A. Salminen. On the sources of simple modules in nilpotent blocks. *J. Algebra* **319** (2008), 4559–4574.
[11] A. Salminen. Endo-permutation modules arising from the action of a p-group on a defect zero block. *J. Group Theory* **12** (2009), 201–207.
[12] J. Thevenaz. Endo-permutation modules, a guided tour. In *Group representation theory* (EPFL press, 2007), pp. 115–147.
[13] M. Wonenburger. Transformations which are products of two involutions. *J. Math. Mech.* **16** (1966), 327–338.

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