J. Group Theory **12** (2009), 401–405
DOI 10.1515/JGT.2008.085

Journal of Group Theory

On groups admitting a fixed-point-free four-group of automorphisms

Pavel Shumyatsky and Carmela Sica

(Communicated by E. I. Khukhro)

Abstract. Let G be a locally finite group admitting a four-group of automorphisms without non-trivial fixed points. It is shown that the derived group G' is a product of normal nilpotent subgroups.

Introduction

In the present paper we consider locally finite groups G admitting a fixed-point-free four-group of automorphisms. It is easy to see that G has no involutions and therefore, by the Feit–Thompson theorem [2], G is locally soluble. A theorem of Bauman tells us that if G is finite, then G' is nilpotent (see [1] or [3, Theorem 10.5.3]). A simple inverse limit argument along the lines of Kegel and Wehrfritz [4, p. 54] shows that in general G' is locally nilpotent. The first author showed in [7] that in fact the normal closure of any element of G' is hypercentral. Thus, G' is a product of normal hypercentral subgroups. The proof can also be found in [8].

Lie algebras admitting a fixed-point-free four-group of automorphisms have been studied in [8]. One of the results obtained there is that if a locally soluble Lie algebra L admits a fixed-point-free four-group of automorphisms, then the derived subalgebra $[L, L]$ is a sum of nilpotent ideals.

The aim of the present paper is to prove a similar result for locally finite groups.

Theorem A. *Let G be a locally finite group admitting a fixed-point-free four-group of automorphisms. Then the derived group G' is a product of nilpotent normal subgroups of G.*

It was proved in [6] that if G is soluble with derived length d, then G' is nilpotent of class at most $2^d - d$. However, there are examples showing that under the hypothesis of Theorem A the group G need not be soluble and therefore G' need not be nilpotent. In particular, this can be shown using an example from [5].

Let p be an odd prime and let t denote the largest odd divisor of $p-1$. Let G_k be the group formed by the matrices

$$M = \begin{pmatrix} u+pa & pb \\ pc & v+pd \end{pmatrix}$$

of determinant 1, where a, b, c, d, u, v lie in the ring of residue classes modulo p^{k+1} and $uv = u^t = 1$ modulo p. Then G_k is of derived length m or $m+1$ where m is the least integer such that $2^m \geqslant k+1$. Let α_k and β_k be the automorphisms of G_k such that for any $M = \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix} \in G_k$ we have

$$M^{\alpha_k} = (M^{-1})^T \quad \text{and} \quad \begin{pmatrix} a_1 & a_2 \\ a_3 & a_4 \end{pmatrix}^{\beta_k} = \begin{pmatrix} a_1 & -a_2 \\ -a_3 & a_4 \end{pmatrix}.$$

It is easy to check that $V_k = \langle \alpha_k, \beta_k \rangle$ is a four-group acting fixed-point-freely on G_k. Now let G be the direct product of the groups G_k. It is clear that G is not soluble and admits a fixed-point-free four-group of automorphisms.

1 Preliminaries

In what follows G will be a locally finite group and $V = \{1, v_1, v_2, v_3\}$ a non-cyclic group of order 4 acting fixed-point-freely on G. Put $G_i = C_G(v_i)$, for $i \in \{1, 2, 3\}$. Recall that G has no non-trivial elements of even order. Otherwise, some of them would necessarily lie in $C_G(V)$.

The first lemma is very well known. It is an immediate corollary of [3, Theorem 6.2.2 (iv)].

Lemma 1.1. *If N is a normal V-invariant subgroup of G then for any subgroup W of V we have $C_{G/N}(W) = C_G(W)N/N$.*

The proofs of the next few lemmas can be found in [6].

Lemma 1.2. *Each G_i is abelian and if $i \neq j$ then v_j acts on G_i by the rule $x^{v_j} = x^{-1}$ for each $x \in G_i$.*

Lemma 1.3. *Let x be an element of G such that $x^{v_1} = x^{-1}$. Suppose that $x \in S$, where S is some V-invariant subgroup of G. Then there exists a unique pair of elements $y \in G_2 \cap S$ and $t \in G_3 \cap S$ such that $x = yty$.*

Lemma 1.4. $G = G_1 G_2 G_3$.

Lemma 1.5. *$G' = \bigcap_{1 \leqslant i < j \leqslant 3} \langle G_i, G_j \rangle$. In particular, the subgroups $\langle G_i, G_j \rangle$ are normal and contain G'.*

Remark 1. Of course, Lemmas 1.3 and 1.4 also hold with the indices 1, 2, 3 permuted in any way.

Remark 2. Since v_j acts on G_i by the rule $x^{v_j} = x^{-1}$, it follows that every subgroup generated by a subset of $G_1 \cup G_2 \cup G_3$ is V-invariant.

For any $x \in G_i$ and $y \in G_j$ with $i \neq j$ it is clear that v_i sends y^x to its inverse. Therefore Lemma 1.3 guarantees that there exists a unique pair $(s,t) \in G_k \times G_j$ such that $y^x = sts$ and $\{i,j,k\} = \{1,2,3\}$. Thus we can define $x * y = s$. According to the same lemma, if y^x is an element of a V-invariant subgroup S, then $x * y \in S$. In particular, it follows that $x * y \in \langle x, y\rangle$. In fact, it is not difficult to prove that $x * y \in \langle x, y\rangle'$ (see Corollary 1.7 below). If N is a normal V-invariant subgroup of G and $\bar{G} = G/N$, then we have $\bar{x} * \bar{y} = \overline{x * y}$. This is immediate from Lemma 1.1 and the fact that the pair s, t in Lemma 1.3 is unique. If X and Y are subsets of G_i and G_j respectively, we write $X * Y$ for the set $\{x * y \mid x \in X, y \in Y\}$.

It is easy to see that if $[x, y] = 1$, then $x * y = 1$. The next lemma shows that the converse is also true.

Lemma 1.6. *Let $x \in G_i$ and $y \in G_j$, where $i \neq j$. If $x * y = 1$, then $[x, y] = 1$.*

Proof. The condition $x * y = 1$ means that $y^x = y_1$ for some $y_1 \in G_j$. This implies that $(y^x)^{v_j} = (y_1)^{v_j}$. Hence $y^{x^{-1}} = y_1 = y^x$. It follows that x^2 commutes with y. Since the order of x is odd, the lemma follows. □

Corollary 1.7. *Whenever $x \in G_i$ and $y \in G_j$, we have $x * y \in \langle x, y\rangle'$.*

2 Proof of the theorem

The following proposition plays a crucial role in the proof of the theorem.

Proposition 2.1. *Let $x \in G_i$ and $y \in G_j$, where $i \neq j$. If $[x, y] = 1$, then for any $x' \in G_i$ we have $x' * y \in C_G(\langle x^G\rangle)$.*

Proof. Without loss of generality, we may assume that $i = 1$ and $j = 2$. Denote by D the centralizer in G of x. It is clear that D is V-invariant, so by Lemma 1.4 we have $D = D_1D_2D_3$, where $D_i = C_D(v_i)$. Put $H = C_{G_1}(D_2)$ and $L = C_{G_3}(H)$. Choose an arbitrary element $g \in G$. We can write $g = g_1g_2g_3$, where $g_i \in G_i$ for $i \in \{1,2,3\}$. Since G_1 is abelian, x and g_1 commute and so we have $x^g = x^{g_1g_2g_3} = x^{g_2g_3}$. Using that $x \in C_G(D_2)$ and G_2 is abelian, it follows that $x^{g_2} \in C_G(D_2)$. By Lemma 1.3 applied with $C_G(D_2)$ in place of S, it follows that $x^{g_2} = tx_1t$ where $t \in C_G(D_2) \cap G_3$ and $x_1 \in C_G(D_2) \cap G_1 = H$. Therefore x^{g_2} centralizes L. Moreover, since g_3 centralizes L too, it follows that x^g also centralizes L. Recall that g was chosen arbitrarily. We now conclude that $L \leqslant C_G(\langle x^G\rangle)$. Note that $y \in C_G(H)$ because $y \in D_2$. Since $x' \in G_1$ and G_1 is abelian, it follows that $x' \in C_G(H)$. Therefore $y^{x'} \in C_G(H)$ and consequently $x' * y \in C_G(H) \cap G_3 = L$. We already know that $L \leqslant C_G(\langle x^G \rangle)$. Hence $x' * y \in C_G(\langle x^G \rangle)$, as required. □

We are now ready to complete the proof of Theorem A.

Proof. We wish to prove that G' is a product of normal nilpotent subgroups. For every pair $x \in G_i$, $y \in G_j$, where $i \neq j$, we define $T_{x,y}$ to be the minimal normal subgroup of G containing $x * y$. Our first aim is to show that each of the subgroups $T_{x,y}$ is nilpotent. Choose elements $b \in G_2$ and $c \in G_3$ and let $a = b * c$. Thus, $T_{b,c} = \langle a^G \rangle$. We will denote this subgroup by T.

By Bauman's theorem, $\langle b, c\rangle'$ is nilpotent. Let Z be the center of $\langle b, c\rangle'$. Obviously Z is V-invariant, so by Lemma 1.4, we have $Z = Z_1 Z_2 Z_3$, where $Z_i = Z \cap G_i$.

Further, $a \in \langle b, c\rangle'$. Hence, by Proposition 2.1, we have $G_1 * Z_2 \leqslant C_G(T)$ and $G_1 * Z_3 \leqslant C_G(T)$. Set $\bar{G} = G/C_G(T)$. If X is a subset of G, we denote by $\bar{X}$ the image of X in $\bar{G}$. By Lemma 1.6, $\bar{Z}_2$ and $\bar{Z}_3$ commute with $\bar{G}_1$. Since $\bar{Z}_2$ also commutes with $\bar{G}_2$, it commutes with $\langle \bar{G}_2, \bar{G}_1 \rangle$. By Lemma 1.5, $\bar{T} \leqslant \langle \bar{G}_2, \bar{G}_1 \rangle$. Therefore $\bar{Z}_2$ commutes with $\bar{T}$. In the same way it can be shown that $\bar{Z}_3$ commutes with $\bar{T}$. Hence Z_2 and Z_3 are contained in $\zeta_2(T)$, the second term of the upper central series of T.

Now let g be an arbitrary element of Z_1. Since Z is normal in $\langle b, c\rangle$, it is clear that $g^b \in Z$ and therefore $b * g \in Z_3 \leqslant \zeta_2(T)$. Set $\tilde{G} = G/\zeta_2(T)$. Then $[\tilde{g}, \tilde{b}] = 1$ and we can apply once again Proposition 2.1. We derive that $\tilde{G}_2 * \tilde{g}$ centralizes $\langle \tilde{b}^{\tilde{G}} \rangle$. Since $\tilde{T}$ is contained in $\langle \tilde{b}^{\tilde{G}} \rangle$, it follows that $G_2 * g$ is contained in $\zeta_3(T)$. Again using that the subgroup $\langle G_2, G_1 \rangle$ is normal and contains T, we deduce now that $g \in \zeta_4(T)$. Recall that g was chosen in Z_1 arbitrarily, so we now conclude that $Z \leqslant \zeta_4(T)$.

Repeating the same argument with the quotient $G/\zeta_4(T)$ in place of G, we obtain that $\zeta_2(\langle b, c\rangle') \leqslant \zeta_8(T)$ etc. If the nilpotency class of $\langle b, c\rangle'$ is m, then it follows that T is nilpotent of class at most $4m$.

Now let S be the product of all these nilpotent subgroups $T_{x,y}$, where x and y range independently through $G_1 \cup G_2 \cup G_3$. When x, y are in the same G_i we just put $T_{x,y} = 1$. Since $x * y \in S$ for all $x, y \in G_1 \cup G_2 \cup G_3$, Lemma 1.6 shows that G/S is abelian and $G' \leqslant S$. The proof is complete. □

References

[1] S. F. Bauman. The Klein group as an automorphism group without fixed points. *Pacific J. Math.* **18** (1966), 9–13.

[2] W. Feit and J. G. Thompson. Solvability of groups of odd order. *Pacific J. Math.* **13** (1963), 773–1029.

[3] D. Gorenstein. *Finite groups* (Harper and Row, 1968).

[4] O. H. Kegel and B. A. F. Wehrfritz. *Locally finite groups* (North Holland, 1973).

[5] L. G. Kovács and G. E. Wall. Involutory automorphisms of groups of odd order and their fixed point groups. *Nagoya Math. J.* **27** (1966), 113–120.

[6] P. Shumyatsky. Groups with regular elementary 2-groups of automorphisms. *Algebra and Logic* **27** (1988), 447–457.

[7] P. Shumyatsky. Locally finite groups with Carter subgroup which is with a fixed-point-free four-group. *Ural. Gos. Univ. Mat. Zap.* **14**, no. 4 (1989), 131–133.

[8] P. Shumyatsky. Groups and Lie algebras with a fixed-point-free four-group of automorphisms. *Comm. Algebra* **24** (1996), 3771–3785.

Received 5 March, 2008

Pavel Shumyatsky, Department of Mathematics, Universidade de Brasilia, 70.919 Brasilia DF, Brazil
E-mail: pavel@unb.br

Carmela Sica, Dipartimento di Matematica e Informatica, Università di Salerno, Via Ponte don Melillo, 84084 Fisciano (SA), Italy
E-mail: csica@unisa.it