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A finitely presented subgroup of the automorphism group of a right-angled Artin group

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Abstract. Let G_Γ be a right-angled Artin group. We use geometric methods to compute a presentation of the subgroup $\mathrm{Conj}(G_\Gamma)$ of $\mathrm{Aut}(G_\Gamma)$ consisting of the automorphisms that send each generator to a conjugate of itself. This generalizes a result of McCool on basis-conjugating automorphisms of free groups.

1 Introduction

In 1936, Whitehead proved what is now known as 'Whitehead's theorem': there is an algorithm which, given two m-tuples $(u_1, \dots, u_m)$ and $(v_1, \dots, v_m)$ of elements of F_n, decides whether there exists an automorphism $\alpha \in \mathrm{Aut}(F_n)$ such that $\alpha(u_i) = v_i$ for all $i \in \{1, \dots, m\}$ (see [12]). To this end, he introduced a set of transformations of F_n, now known as the 'Whitehead automorphisms'. Whitehead's proof used topological methods. In 1958, Rapaport gave an algebraic proof of Whitehead's theorem (see [10]), which was later simplified by Higgins and Lyndon (see [5]). Using a refinement of the argument of Higgins and Lyndon, McCool obtained a finite presentation for $\mathrm{Aut}(F_n)$, with the Whitehead automorphisms as generating set (see [7]). McCool also proved that the stabilizer of an m-tuple of cyclic words in F_n is finitely presented (see [8]). (A *cyclic word* in F_n can be thought of as the set of all cyclic permutations of a given cyclically reduced word.) Thereafter McCool obtained a finite presentation for the subgroup of $\mathrm{Aut}(F_n)$ consisting of the automorphisms that send each generator to a conjugate of itself (see [9]).

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite simplicial graph. The *right-angled Artin group* associated to Γ is the group G_Γ defined by the presentation:

$$G_\Gamma = \langle \mathcal{V} \mid vw = wv,\ \forall \{v, w\} \in \mathcal{E} \rangle.$$

At the two extremes are the free group F_n of rank n (Γ is discrete), and the free abelian group $\mathbb{Z}^n$ (Γ is complete). Right-angled Artin groups are sometimes called *graph groups*, or *free partially commutative groups*. In recent years, right-angled Artin groups have received considerable attention due to the fact that they contain many interesting subgroups, and also because of their actions on CAT(0) cube complexes. For a general survey of right-angled Artin groups, see [1].

Automorphisms of right-angled Artin groups were first studied by Servatius in [11]. Drawing on Nielsen automorphisms for free groups, Servatius defined four classes of automorphisms—namely, inversions, partial conjugations, transvections, and symmetries (see Section 2)—and conjectured that they generate $\mathrm{Aut}(G_\Gamma)$. Servatius proved his conjecture for some classes of right-angled Artin groups, for example, when Γ is a tree. Thereafter, Laurence proved the conjecture for arbitrary right-angled Artin groups in [6]. In [2], Day extended the concept of Whitehead automorphism to arbitrary right-angled Artin groups, and gave a finite presentation for $\mathrm{Aut}(G_\Gamma)$. This presentation will be described in detail in Section 2.

Our focus here is on the automorphisms of G_Γ that satisfy the following definition:

Definition 1.1. We say that an automorphism φ of G_Γ is vertex-conjugating if $\varphi(v)$ is conjugate to v for all $v \in \mathcal{V}$.

Vertex-conjugating automorphisms were first introduced by Laurence in [6], where they are called conjugating. They also appear in the recent work of Duncan, Kazachkov and Remeslennikov (see [3], see also [4]). As one of the steps in the proof of Servatius' conjecture, Laurence proved that the set of vertex-conjugating automorphisms coincides with the subgroup $\mathrm{Conj}(G_\Gamma)$ of $\mathrm{Aut}(G_\Gamma)$ generated by the partial conjugations (see Section 2 for the definition of a partial conjugation). Let S denote the set of all partial conjugations of G_Γ. In Section 3, we define a finite set R of relations satisfied by the elements of S. Our main result is the following:

Theorem 1.2. *The group* $\mathrm{Conj}(G_\Gamma)$ *of vertex-conjugating automorphisms has the presentation* $\langle S \mid R \rangle$.

In Section 3, we shall state a more precise version of Theorem 1.2 which yields an explicit finite presentation for $\mathrm{Conj}(G_\Gamma)$ (see Theorem 3.1).

Our proof relies on geometric methods. Following arguments from McCool [8, 9], we construct a finite, connected 2-complex K with fundamental group

$$\mathrm{Conj}(G_\Gamma) = \langle S \mid R \rangle.$$

An important observation is that every partial conjugation is a long-range Whitehead automorphism in the sense of [2].

Note that we cannot hope for a generalization of the presentation given in the theorem of [9] (see Remark 3.2 below).

2 Preliminaries

Let $\Gamma = (\mathcal{V}, \mathcal{E})$ be a finite simplicial graph, and let G_Γ be the right-angled Artin group associated to Γ. Let v be a vertex of Γ. The *link* of v, denoted by $\mathrm{lk}(v)$, is the subset of $\mathcal{V}$ consisting of all vertices that are adjacent to v. The *star* of v, denoted by $\mathrm{st}(v)$, is $\mathrm{lk}(v) \cup \{v\}$. We set $L = \mathcal{V} \cup \mathcal{V}^{-1}$. Let $x \in L$. The *vertex* of x, denoted by $v(x)$, is the unique element of $\mathcal{V} \cap \{x, x^{-1}\}$. We set

$$\mathrm{lk}_L(x) = \mathrm{lk}(v(x)) \cup \mathrm{lk}(v(x))^{-1} \quad \text{and} \quad \mathrm{st}_L(x) = \mathrm{st}(v(x)) \cup \mathrm{st}(v(x))^{-1}.$$

Let w be a word in $\mathcal{V} \cup \mathcal{V}^{-1}$. The *support* of w, denoted by $\mathrm{supp}(w)$, is the subset of $\mathcal{V}$ of all vertices v such that v or v^{-1} is a letter of w. A word w in $\mathcal{V} \cup \mathcal{V}^{-1}$ is said to be *reduced* if it contains no subwords of the form $vw'v^{-1}$ or $v^{-1}w'v$ with $\mathrm{supp}(w') \subset \mathrm{st}(v)$. For a word w in $\mathcal{V} \cup \mathcal{V}^{-1}$, we denote by $|w|$ the length of w. The *length* of an element g of G_Γ is by definition the minimal length of any word representing g. Note that the length of g is equal to the length of any reduced word representing g. We say that an element g of G_Γ is *cyclically reduced* if it cannot be written vhv^{-1} or $v^{-1}hv$ with $v \in \mathcal{V}$, and $|g| = |h| + 2$. By [11, Proposition 2], every element of G_Γ is conjugate to a unique (up to cyclic permutation) cyclically reduced element. The *length* of a conjugacy class is by definition the minimal length of any of its representative elements. Observe that the length of a conjugacy class is equal to the length of a cyclically reduced element representing it. For an n-tuple of conjugacy classes W, we define the *length* of W, denoted by $|W|$, as the sum of the lengths of its elements ($n \geq 1$).

Let v, w be vertices of Γ. We use the notation $v \geq w$ to mean $\mathrm{lk}(w) \subset \mathrm{st}(v)$. We use the notation $v \sim w$ to mean $v \geq w$ and $w \geq v$.

The Laurence–Servatius generators for $\mathrm{Aut}(G_\Gamma)$ are defined as follows:

Inversions: Let $v \in \mathcal{V}$. The automorphism ι_v that sends v to v^{-1} and fixes all other vertices is called an *inversion*.

Partial conjugations: Let $x \in L$, and let Y be a non-empty union of connected components of $\Gamma \setminus \mathrm{st}(v(x))$. The automorphism $c_{x,Y}$ that sends each vertex y in Y to $x^{-1}yx$ and fixes all vertices not in Y is called a *partial conjugation*.

Transvections: Let $v, w \in \mathcal{V}$ be such that $v \geq w$. The automorphism $\tau_{v,w}$ that sends w to vw and fixes all other vertices is called a *transvection*.

Symmetries: Let φ be an automorphism of the graph Γ. The automorphism ϕ given by $\phi(v) = \varphi(v)$ for all $v \in \mathcal{V}$ is called a *symmetry*.

Our aim is to compute a presentation of the subgroup $\mathrm{Conj}(G_\Gamma)$ of $\mathrm{Aut}(G_\Gamma)$ generated by the partial conjugations. Our proof will use the fact that partial conjugations are long-range Whitehead automorphisms.

A *Whitehead automorphism* is an automorphism $\alpha \in \mathrm{Aut}(G_\Gamma)$ of one of the following two types:

Type 1: α restricted to $\mathcal{V} \cup \mathcal{V}^{-1}$ is a permutation of $\mathcal{V} \cup \mathcal{V}^{-1}$.

Type 2: There is an element $a \in L$, called the *multiplier* of α, such that $\alpha(a) = a$, and for each $x \in \mathcal{V}$, the element $\alpha(x)$ lies in $\{x, xa, a^{-1}x,\ a^{-1}xa\}$.

One can show that the set of type 1 Whitehead automorphisms is the subgroup of $\mathrm{Aut}(G_\Gamma)$ generated by inversions and symmetries.

Following [2], we say that a Whitehead automorphism α is *long-range* if α is of type 1 or if α is of type 2 and α fixes the vertices of $\mathrm{lk}(v(a))$ (where a is the multiplier of α).

We denote by $\mathcal{W}$ the set of Whitehead automorphisms, by $\mathcal{W}_1$ the set of Whitehead automorphisms of type 1, and by $\mathcal{W}_2$ the set of Whitehead automorphisms of type 2. We also denote by $\mathcal{W}_\ell$ the set of long-range Whitehead automorphisms.

We use the following notation for type 2 Whitehead automorphisms. Let A be a subset of L, and let $a \in L$, such that $a \in A$ and $a^{-1} \notin A$. Provided that it exists, (A, a) denotes the automorphism given by

$$(A, a)(a) = a,$$

and, for all $x \in \mathcal{V} \setminus \{v(a)\}$,

$$(A, a)(x) = \begin{cases} x & \text{if } x \notin A \text{ and } x^{-1} \notin A, \\ xa & \text{if } x \in A \text{ and } x^{-1} \notin A, \\ a^{-1}x & \text{if } x \notin A \text{ and } x^{-1} \in A, \\ a^{-1}xa & \text{if } x \in A \text{ and } x^{-1} \in A. \end{cases}$$

If A is a subset of L, we set $A^{-1} = \{a^{-1} \mid a \in A\}$. If A and B are subsets of L, and a is an element of L, we use the notations $A - B$ for $A \setminus B$, $A + B$ for $A \sqcup B$ (if $A \cap B = \emptyset$), $A - a$ for $A \setminus \{a\}$ and $A + a$ for $A \sqcup \{a\}$ (if $a \notin A$).

The following remark will be of particular importance in our proof:

Remark 2.1. Let $x \in L$, and let Y be a non-empty union of connected components of $\Gamma \setminus \mathrm{st}(v(x))$. Set $A = Y \cup Y^{-1} \cup \{x\}$, and $a = x$. Then the Whitehead automorphism (A, a) is nothing but the partial conjugation $c_{x,Y}$. In particular, the Whitehead automorphism $(L - \mathrm{lk}_L(a) - a^{-1}, a)$ is the inner automorphism ω_a induced by a. Note that there is not a unique way to write a partial conjugation as a type 2 Whitehead automorphism. More specifically, if $B \subset \mathrm{lk}(v(a))$, then the Whitehead automorphisms (A, a) and $(A + B + B^{-1}, a)$ represent the same element of S.

In [2], Day proved that $\mathrm{Aut}(G_\Gamma)$ is generated by the Whitehead automorphisms, subject to the relations:

(R1) $(A,a)^{-1} = (A - a + a^{-1}, a^{-1})$, for $(A,a) \in \mathcal{W}_2$.

(R2) $(A,a)(B,a) = (A \cup B, a)$, for $(A,a),(B,a) \in \mathcal{W}_2$ with $A \cap B = \{a\}$.

(R3) $(B,b)(A,a)(B,b)^{-1} = (A,a)$, for $(A,a),(B,b) \in \mathcal{W}_2$ such that $a \notin B$, $a^{-1} \notin B$, $b \notin A$, $b^{-1} \notin A$, and at least one of the following holds:

  (a) $A \cap B = \emptyset$,

  (b) $b \in \mathrm{lk}_L(a)$.

(R4) $(B,b)(A,a)(B,b)^{-1} = (A,a)(B-b+a,a)$, for $(A,a),(B,b) \in \mathcal{W}_2$ such that $a \notin B$, $a^{-1} \notin B$, $b \notin A$, $b^{-1} \in A$, and at least one of the following holds:

  (a) $A \cap B = \emptyset$,

  (b) $b \in \mathrm{lk}_L(a)$.

(R5) $(A - a + a^{-1}, b)(A,a) = (A - b + b^{-1}, a)\sigma_{a,b}$, for $(A,a) \in \mathcal{W}_2$, $b \in L$ such that $b \in A$, $b^{-1} \notin A$, $b \neq a$, and $v(b) \sim v(a)$. Here $\sigma_{a,b}$ denotes the type 1 Whitehead automorphism that sends a to b^{-1} and b to a, and fixes the other generators.

(R6) $\sigma(A,a)\sigma^{-1} = (\sigma(A), \sigma(a))$, for $(A,a) \in \mathcal{W}_2$, and $\sigma \in \mathcal{W}_1$.

(R7) The entire multiplication table of $\mathcal{W}_1$, which forms a finite subgroup of $\mathrm{Aut}(G_\Gamma)$.

(R8) $(A,a) = (L - a^{-1}, a)(L - A, a^{-1})$, for $(A,a) \in \mathcal{W}_2$.

(R9) $(A,a)(L-b^{-1},b)(A,a)^{-1} = (L-b^{-1},b)$, for $(A,a) \in \mathcal{W}_2$, $b \in L$ such that $b \notin A$, $b^{-1} \notin A$.

(R10) $(A,a)(L-b^{-1},b)(A,a)^{-1} = (L-a^{-1},a)(L-b^{-1},b)$, for $(A,a) \in \mathcal{W}_2$, $b \in L$ such that $b \in A$, $b^{-1} \notin A$, and $b \neq a$.

Note that relation (R8) is a direct consequence of relations (R1) and (R2).

In order to prove Theorem 1.2, we need to introduce the following technical definitions.

Let $\alpha, \beta \in \mathcal{W}$, and let W be an n-tuple of conjugacy classes ($n \geq 1$). Following [2], we say that α is a *peak* of $\beta\alpha$ with respect to W if:

$$|W| \leq |\alpha.W|, \quad |\beta\alpha.W| \leq |\alpha.W|,$$

and at least one of these inequalities is strict. Let $\alpha_1, \dots, \alpha_k \in \mathcal{W}$ ($k \geq 1$). We say that α_i is a *peak* of the product $\alpha_k \cdots \alpha_1$ with respect to W if $1 \leq i < k$ and α_i is a peak of $\alpha_{i+1}\alpha_i$ with respect to $\alpha_{i-1} \cdots \alpha_1.W$. We say that the product $\alpha_k \cdots \alpha_1$

is *peak-reduced* with respect to W if it has no peaks with respect to W. The *height* of a peak α_i is $|\alpha_i \cdots \alpha_1.W|$.

3 Proof of the main theorem

In this section, we prove the following:

Theorem 3.1. *The group* $\mathrm{Conj}(G_\Gamma)$ *has a presentation with generators* $c_{x,Y}$, *for* $x \in L$ *and* Y *a non-empty union of connected components of* $\Gamma \setminus \mathrm{st}(v(x))$, *and relations:*

(1) $(c_{x,Y})^{-1} = c_{x^{-1},Y}$,

(2) $c_{x,Y}c_{x,Z} = c_{x,Y\cup Z}$ *if* $Y \cap Z = \emptyset$,

(3) $c_{x,Y}c_{y,Z} = c_{y,Z}c_{x,Y}$ *if* $v(x) \notin Z$, $v(y) \notin Y$, $x \neq y, y^{-1}$, *and at least one of* $Y \cap Z = \emptyset$ *or* $y \in \mathrm{lk}_L(x)$ *holds,*

(4) $\omega_y c_{x,Y} \omega_y^{-1} = c_{x,Y}$ *if* $v(y) \notin Y$, $x \neq y, y^{-1}$.

Proof. Our proof is based on arguments developed by McCool in [8] and [9] (similar arguments were used in [2]). Recall that S denotes the set of partial conjugations. Let R denote the set of relations given in the statement of Theorem 3.1. We shall construct a finite, connected 2-complex K with fundamental group

$$\mathrm{Conj}(G_\Gamma) = \langle S \mid R \rangle.$$

We identify a partial conjugation with any of its representatives in $\mathcal{W}_2$ (see Remark 2.1 above). Note that for every $(A,a) \in \mathcal{W}_2$, we have $(A,a) \in S$ if and only if $(A-a)^{-1} = A - a$.

Set $\mathcal{V} = \{v_1,\dots,v_n\}$ $(n \geq 1)$. Let W denote the n-tuple $(v_1,\dots,v_n)$.

The set of vertices $K^{(0)}$ of K is the set of n-tuples $\alpha.W$, where α ranges over the set $\mathcal{W}_1$ of type 1 Whitehead automorphisms. For any $\alpha, \beta \in \mathcal{W}_1$, the vertices $\alpha.W$ and $\beta\alpha.W$ are joined by a directed edge $(\alpha.W, \beta\alpha.W; \beta)$ labelled β. Note that, at this stage, K is just the Cayley graph of $\mathcal{W}_1$. Next, for any $\alpha \in \mathcal{W}_1$, and $(A,a) \in S$, we add a loop $(\alpha.W, \alpha.W; (A,a))$ labelled (A,a) at $\alpha.W$. This defines the 1-skeleton $K^{(1)}$ of K.

We shall define the 2-cells of K. These 2-cells will derive from the relations (R1)–(R10) of [2]. First, let K_1 be the 2-complex obtained by attaching 2-cells corresponding to relation (R7) to $K^{(1)}$. Note that, if C is the 2-complex obtained from K_1 by deleting the loops $(\alpha.W, \alpha.W; (A,a))$ $(\alpha \in \mathcal{W}_1$, $(A,a) \in S)$, then C is just the Cayley complex of $\mathcal{W}_1$, and therefore is simply connected.

We now explore the relations (R1)–(R5) and (R8)–(R10) of [2] to determine which of these will give rise to relations on the elements of S.

Relation (R1) will give rise to the following:

$$(A,a)^{-1} = (A - a + a^{-1}, a^{-1}), \tag{3.1}$$

for $(A,a) \in S$.

Relation (R2) will give rise to

$$(A,a)(B,a) = (A \cup B, a), \tag{3.2}$$

for $(A,a), (B,a) \in S$, with $A \cap B = \{a\}$.

Relation (R3) will give rise to

$$(A,a)(B,b) = (B,b)(A,a), \tag{3.3}$$

for $(A,a), (B,b) \in S$, such that $a \notin B$, $a^{-1} \notin B$, $b \notin A$, and $b^{-1} \notin A$, and at least one of (a) $A \cap B = \emptyset$ or (b) $b \in \mathrm{lk}_L(a)$ holds.

From (R4), no relations arise. Indeed, suppose that (A,a), (B,b) are in S with $a^{-1} \notin B$, $b \notin A$, and $b^{-1} \in A$. Then $b^{-1} = a$ (because $(A-a)^{-1} = A-a$). But then $a^{-1} = b \in B$, leading to a contradiction with our assumption on a.

From (R5), no relations arise (by the same argument as above).

From (R8), we obtain a relation which is a direct consequence of (3.1) and (3.2).

Relation (R9) will give rise to the following:

$$(A,a)(L - \mathrm{lk}_L(b) - b^{-1}, b)(A,a)^{-1} = (L - \mathrm{lk}_L(b) - b^{-1}, b), \tag{3.4}$$

for $(A,a) \in S$, and $b \in L$ such that $b \notin A$, and $b^{-1} \notin A$.

From (R10), no relations arise (by the same argument as above).

We rewrite the relations (3.1)–(3.4) in the form

$$\sigma_k^{\varepsilon_k} \cdots \sigma_1^{\varepsilon_1} = 1,$$

where $\sigma_1, \dots, \sigma_k \in S$, and $\varepsilon_1, \dots, \varepsilon_k \in \{-1, 1\}$. Let K_2 be the 2-complex obtained from K_1 by attaching 2-cells corresponding to the relations (3.1)–(3.4). Note that the boundary of each of these 2-cells has the form

$$(\alpha.W, \alpha.W; \sigma_1)^{\varepsilon_1} (\alpha.W, \alpha.W; \sigma_2)^{\varepsilon_2} \cdots (\alpha.W, \alpha.W; \sigma_k)^{\varepsilon_k},$$

for $\alpha \in \mathcal{W}_1$.

Finally, relation (R6) will give rise to the following:

$$\alpha(A,a)\alpha^{-1} = (\alpha(A), \alpha(a)), \tag{3.5}$$

for $(A,a) \in S$, and $\alpha \in \mathcal{W}_1$. Then K is obtained from K_2 by attaching 2-cells corresponding to the relations (3.5). Observe that the boundary of each of these

2-cells has the form

$$(\beta.W, \beta.W; (\alpha(A), \alpha(a)))^{-1} (\beta.W, \alpha^{-1}\beta.W; \alpha)^{-1}$$
$$(\alpha^{-1}\beta.W, \alpha^{-1}\beta.W; (A, a))(\alpha^{-1}\beta.W, \beta.W; \alpha),$$

for $\beta \in \mathcal{W}_1$.

It remains to show that $\pi_1(K, W) = \mathrm{Conj}(G_\Gamma) = \langle S \mid R \rangle$.

Let T be a maximal tree in the 1-skeleton $K^{(1)}$ of K. Note that T is in fact a maximal tree in the 1-skeleton $C^{(1)}$ of C (i.e., the Cayley graph of $\mathcal{W}_1$). We compute a presentation of $\pi_1(K, W)$ using T. For every vertex V of K, there exists a unique reduced path p_V from W to V in T. To each edge $(V_1, V_2; \alpha)$ of K, we associate the element of $\pi_1(K, W)$ represented by the loop $p_{V_1}(V_1, V_2; \alpha)p_{V_2}^{-1}$. We again denote this by $(V_1, V_2; \alpha)$. Evidently these elements generate $\pi_1(K, W)$. Now, since C is simply connected, we have

$$(\alpha.W, \beta\alpha.W; \beta) = 1 \quad (\text{in } \pi_1(K, W)), \tag{3.6}$$

for all $\alpha, \beta \in \mathcal{W}_1$.

Let $\mathcal{P}$ be the set of combinatorial paths in the 1-skeleton $K^{(1)}$ of K. We define a map $\widehat{\varphi} : \mathcal{P} \to \mathrm{Aut}(G_\Gamma)$ as follows. For an edge $e = (V_1, V_2; \alpha)$, we set $\widehat{\varphi}(e) = \alpha$, and for a path $p = e_k^{\varepsilon_k} \cdots e_1^{\varepsilon_1}$, we set $\widehat{\varphi(p)} = \widehat{\varphi}(e_k)^{\varepsilon_k} \cdots \widehat{\varphi}(e_1)^{\varepsilon_1}$. Clearly, if p_1 and p_2 are loops at W such that $p_1 \sim p_2$, then $\widehat{\varphi}(p_1) = \widehat{\varphi}(p_2)$. Hence, $\widehat{\varphi}$ induces a map $\varphi : \pi_1(K, W) \to \mathrm{Aut}(G_\Gamma)$. It is easily seen that φ is a homomorphism. Then we see from (3.6) that φ maps $\pi_1(K, W)$ to $\mathrm{Conj}(G_\Gamma)$. It follows immediately from the construction of K that $\varphi : \pi_1(K, W) \to \mathrm{Conj}(G_\Gamma)$ is surjective. Thus, it suffices to show that φ is injective. Let p be a loop at W such that $\varphi(p) = 1$. We have to show that $p \sim 1$. Write $p = e_k^{\varepsilon_k} \cdots e_1^{\varepsilon_1}$, where $k \geq 1$ and $\varepsilon_i \in \{-1, 1\}$ for all $i \in \{1, \ldots, k\}$. Using the 2-cells arising from (3.1) and the fact that $\mathcal{W}_1^{-1} = \mathcal{W}_1$, we can restrict our attention to the case where $p = e_k \cdots e_1$. Set $\alpha_i = \varphi(e_i)$ for all $i \in \{1, \ldots, k\}$. Note that $\alpha_i \in S \cup \mathcal{W}_1 \subset \mathcal{W}_\ell$ for all $i \in \{1, \ldots, k\}$.

Let Z be a tuple containing each conjugacy class of length 2 of G_Γ, each appearing once. We prove the following:

Claim. *We have $p \sim e'_l \cdots e'_1$, such that, if we set $\alpha'_i = \varphi(e'_i)$ for all $i \in \{1, \ldots, l\}$, then $\alpha'_i \in \mathcal{W}_1$ or $\alpha'_i \in \mathcal{W}_2 \cap \mathrm{Inn}(G_\Gamma)$ for each $i \in \{1, \ldots, l\}$.*

First, we examine the case where $\alpha_k \cdots \alpha_1$ is peak-reduced with respect to Z. We claim that the sequence

$$|Z|, |\alpha_1.Z|, |\alpha_2\alpha_1.Z|, \ldots, |\alpha_{k-1} \cdots \alpha_1.Z|, |\alpha_k \cdots \alpha_1.Z| = |Z|$$

is a constant sequence. Suppose the contrary. By [2, Lemma 5.2], $|Z|$ is the least element of the set $\{|\alpha.Z| \mid \alpha \in \langle \mathcal{W}_\ell \rangle\}$. Hence we can find $i \in \{1, \ldots, k-1\}$ such that we have

$$|\alpha_{i-1}\cdots\alpha_1.Z| \leq |\alpha_i\cdots\alpha_1.Z|,$$
$$|\alpha_{i+1}\cdots\alpha_1.Z| \leq |\alpha_i\cdots\alpha_1.Z|,$$

and at least one of these inequalities is strict, which contradicts the fact that the product $\alpha_k\cdots\alpha_1$ is peak-reduced. Therefore we have

$$|\alpha_i\cdots\alpha_1.Z| = |Z|,$$

for all indices $i \in \{1,\ldots,k\}$. We argue by induction on $i \in \{1,\ldots,k\}$ to prove that $\alpha_i\cdots\alpha_1.Z$ is a tuple containing each conjugacy class of length 2 of G_Γ, each appearing once. The result holds for $i = 0$ by assumption. Suppose that $i \geq 1$, and that the result holds for $i-1$. Observe that a type 1 Whitehead automorphism does not change the length of a conjugacy class. Thus, we can assume that α_i is a type 2 Whitehead automorphism. Since $|\alpha_i\alpha_{i-1}\cdots\alpha_1.Z| = |\alpha_{i-1}\cdots\alpha_1.Z|$, α_i is trivial, or an inner automorphism by [2, Lemma 5.2]. Thus, the result holds for i. In this case, p has already the desired form.

We now turn to prove the claim. We define

$$h_p = \max\{|\alpha_i\cdots\alpha_1.Z| \mid i \in \{0,\ldots,k\}\},$$

and

$$N_p = |\{i \mid i \in \{0,\ldots,k\} \text{ and } |\alpha_i\cdots\alpha_1.Z| = h_p\}|.$$

We argue by induction on h_p. The base of induction is $|Z|$: the smallest possible value for h_p by [2, Lemma 5.2]. If $h_p = |Z|$, then the product $\alpha_k\cdots\alpha_1$ is peak-reduced and we are done. Thus, we can assume that $h_p > |Z|$ and that the result has been proved for all loops p' with $h_{p'} < h_p$. Let $i \in \{1,\ldots,k\}$ be such that α_i is a peak of height h_p. An examination of the proof of [2, Lemma 3.18] shows that $e_{i+1}e_i \sim f_j\cdots f_1$ such that, if we set $\beta_\kappa = \varphi(f_\kappa)$ for all $\kappa \in \{1,\ldots,j\}$, then

$$|\beta_\kappa\cdots\beta_1\alpha_{i-1}\cdots\alpha_1.Z| < |\alpha_i\alpha_{i-1}\cdots\alpha_1.Z|, \tag{3.7}$$

for all $\kappa \in \{1,\ldots,j-1\}$. Therefore, we get

$$p \sim e_k\cdots e_{i+2}f_j\cdots f_1e_{i-1}\cdots e_1 = p',$$

and a new product $\alpha_k\cdots\alpha_{i+2}\beta_j\cdots\beta_1\alpha_{i-1}\cdots\alpha_1$. We argue by induction on N_p. If $N_p = 1$, then (3.7) implies that $h_{p'} < h_p$ and we can apply the induction hypothesis on h_p. If $N_p \geq 2$, then (3.7) implies that $h_{p'} = h_p$ and $N_{p'} < N_p$, and we can apply the induction hypothesis on N_p. This proves the claim.

Hence, using the 2-cells arising from the relations (3.5), we obtain

$$p \sim h_s\cdots h_1g_r\cdots g_1,$$

where, if we set

$$\gamma_i = \varphi(g_i) \text{ for all } i \in \{1, \dots, r\} \quad \text{and} \quad \delta_j = \varphi(h_j) \text{ for all } j \in \{1, \dots, s\},$$

then $\delta_i \in \mathcal{W}_1$ for all $i \in \{1, \dots, s\}$ and $\gamma_j \in \mathcal{W}_2 \cap \mathrm{Inn}(G_\Gamma)$ for all $j \in \{1, \dots, r\}$. Using relation (3.6), we obtain $p \sim g_r \cdots g_1$. Set $\mathcal{Z} = \bigcap_{v \in \mathcal{V}} \mathrm{st}(v)$. It follows from Servatius' Centralizer Theorem (see [11]) that the center $Z(G_\Gamma)$ of G_Γ is the special subgroup of G_Γ generated by $\mathcal{Z}$. Let Γ' be the full subgraph of Γ spanned by $\mathcal{V} \setminus \mathcal{Z}$. We have

$$G_{\Gamma'} \simeq \mathrm{Inn}(G_\Gamma),$$

where the isomorphism is given by $v \mapsto \omega_v$ (see, for example, [2, Lemma 5.3]). Write

$$\gamma_i = (L - \mathrm{lk}_L(c_i) - c_i^{-1}, c_i),$$

where $c_i \in \mathcal{V} \setminus \mathcal{Z} \cup (\mathcal{V} \setminus \mathcal{Z})^{-1}$ ($i \in \{1, \dots, r\}$). Since $\gamma_r \cdots \gamma_1 = 1$ (in $\mathrm{Inn}(G_\Gamma)$), we have $c_r \cdots c_1 = 1$ (in $G_{\Gamma'}$). Therefore $c_r \cdots c_1$ is a product of conjugates of defining relators of G_Γ. Using the 2-cells corresponding to the relations (3.1) and (3.3) (b), we deduce that $p \sim 1$. We conclude that φ is injective, and thus

$$\mathrm{Conj}(G_\Gamma) = \pi_1(K, W).$$

Now, using the 2-cells arising from the relations (3.5) (with $\alpha = \beta$), we obtain

$$(\alpha.W, \alpha.W; (\alpha(A), \alpha(a))) = (\alpha.W, W; \alpha^{-1})(W, W; (A, a))(W, \alpha.W; \alpha),$$

and then, using (3.6),

$$(\alpha.W, \alpha.W; (\alpha(A), \alpha(a))) = (W, W; (A, a)), \tag{3.8}$$

for all $\alpha \in \mathcal{W}_1$, and $(A, a) \in S$. It then follows that $\mathrm{Conj}(G_\Gamma)$ is generated by the $(W, W; (A, a))$, for $(A, a) \in S$. We identify $(W, W; (A, a))$ with (A, a) for all $(A, a) \in S$. Any relation in $\mathrm{Conj}(G_\Gamma) = \pi_1(K, W)$ will be a product of conjugates of boundary labels of 2-cells of K. Then, using relation (3.8) and identifying $(W, W; (A, a))$ with (A, a), we see that these relations will actually result from the relations (3.1)–(3.4) above. It is easily seen that the relations (3.1)–(3.4) above are equivalent to those of R. We have shown that $\mathrm{Conj}(G_\Gamma)$ has the presentation $\langle S \mid R \rangle$. □

Remark 3.2. We cannot hope for a generalization of the presentation given in the theorem of [9], since, in a general right-angled Artin group, the existence of one-term partial conjugations depends on the existence of domination relations between the vertices of Γ. (A *one-term partial conjugation* is a partial conjugation of the form $c_{x,\{y\}}$ with $x \geq y$.)

Remark 3.3. There is a slightly different definition of partial conjugation which requires that a partial conjugation $\gamma_{v,C}$ conjugates a single connected component C of $\Gamma - \mathrm{st}(v)$ by v, where v is a vertex of Γ. With this definition, the first relation vanishes. The second relation has to be replaced by

$$\gamma_{v,C}\gamma_{v,D} = \gamma_{v,D}\gamma_{v,C} \quad \text{if } C \cap D = \emptyset.$$

But then the fourth relation becomes less simple since, with this definition, an inner automorphism is not a partial conjugation but a product of partial conjugations (or their inverses).

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