Inverse limits of finite rank free groups

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Abstract. We will show that the inverse limit of finite rank free groups with surjective connecting homomorphism is isomorphic either to a finite rank free group or to a fixed universal group. In other words, any inverse system of finite rank free groups which is not equivalent to an eventually constant system has the universal group as its limit. This universal inverse limit is naturally isomorphic to the first shape group of the Hawaiian earring. We also give an example of a homomorphic image of the Hawaiian earring group which lies in the inverse limit of free groups but is neither a free group nor isomorphic to the Hawaiian earring group.

One of the first to consider the inverse limits of finite rank free groups was Higman. In [10], he studies the inverse limit of finite rank free groups which he calls the unrestricted free product of countably many copies of $\mathbb{Z}$. There he proves that this group is not a free group and that each of its free quotients has finite rank. He considers a subgroup P of the unrestricted product which turns out to be isomorphic to the Hawaiian earring group though he does not prove that fact there. In [6, Section 2], de Smit gives a proof that the Hawaiian earring group embeds in an inverse limit of free groups and gives a characterization of the elements of the image. Daverman and Venema in [5, Theorem 1.0 and Corollary 2.1] showed that a one-dimensional Planar Peano continuum either has the shape of a finite bouquet of circles or of the Hawaiian earring. Hence any inverse limit of finite rank free groups that arises from the inverse system of fundamental groups of nerves of a one-dimensional Peano continuum is either a finite rank free group or the standard Hawaiian inverse limit. In Section 2, we will show that the following group-theoretic analogue to the result of Daverman and Venema is true. Every inverse limit of finite rank free groups with surjective connecting homomorphisms is isomorphic to a free group or the standard Hawaiian inverse limit. Hence the system coming from the fundamental groups of nerves of a one-dimensional Peano continuum is not necessary.

Eda in [7, Theorem 1.3] and Conner and Eda in [3, Theorem 5.1] (see also [4]) show that the fundamental group of a one-dimensional Peano continuum which is not semilocally simply connected at any point determines the homotopy type of the space. This was done using uncountable homomorphic images of the Hawai-

ian earring group. It was believed that any uncountable homomorphic image of the Hawaiian earring group which embedded in an inverse limit of free groups was itself isomorphic to the Hawaiian earring group. In Section 3, we will show that there exists an uncountable homomorphic image of the Hawaiian earring group which embeds in an inverse limit of free groups but is not isomorphic to the Hawaiian earring group or a free group. This is done by using two propositions which were originally proved by Higman in [10] to construct a homomorphism with our desired image.

1 Definitions

The Hawaiian earring group, which we will denote by $\mathbb{H}$, is the fundamental group of the Hawaiian earring, $\mathbf{E}$, which is the subspace of the plane formed by the union of circles centered at $(0, \frac{1}{n})$ with radius $\frac{1}{n}$. The Hawaiian earring group is uncountable and locally free. Cannon and Conner in [1, Section 2] and [2, Section 4] showed that the Hawaiian earring group is generated, in the sense of infinite products, by a countable sequence of loops corresponding to the circles, where an infinite product is legal if each loop is transversed only finitely many times. (For more information on infinite products, see [2].) We will refer to this infinite generating set for the fundamental group of $\mathbf{E}$ as $\{a_n\}$, i.e. a_n represents the loop which transverses counterclockwise one time the circle of radius $\frac{1}{n}$ centered at $(0, \frac{1}{n})$. We will frequently denote the base point $(0, 0)$ of $\mathbf{E}$ by just 0.

An *inverse system of groups* is a collection of groups F_α indexed by a partially ordered set J along with a collection of homomorphisms

$$\{\varphi_{\alpha,\beta} : F_\alpha \to F_\beta \mid \alpha \geq \beta\},$$

which are called *connecting homomorphisms*. The connecting homomorphisms must satisfy the following condition. For every triple $\{\alpha, \beta, \gamma\}$ such that $\alpha \geq \beta \geq \gamma$, the connecting homomorphisms satisfy $\varphi_{\alpha,\beta} \circ \varphi_{\beta,\gamma} = \varphi_{\alpha,\gamma}$. An inverse limit of an inverse system is the subgroup of the direct product which consists of functions $f : J \to \bigcup_{\alpha \in J} F_\alpha$ such that $f(\beta) \in F_\beta$ and $f(\beta) = \varphi_{\alpha,\beta}(f(\alpha))$, for every α, β such that $\beta \leq \alpha$. We will denote the inverse limit by $\varprojlim\{F_\alpha, \varphi_{\alpha,\beta}\}$.

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We will now describe the inverse system of finite rank free groups constructed by Higman. Let $A_i = \langle \mathbf{a}_1, \ldots, \mathbf{a}_i \rangle$ be the free group on i generators with the natural inclusions $A_1 \subset \cdots \subset A_i \subset A_{i+1} \subset \cdots$. For $i \geq j$, the connecting homomorphism $P_{i,j} : A_i \to A_j$ sends $\mathbf{a}_k$ to $\mathbf{a}_k$ for $k \leq j$ and $\mathbf{a}_k$ to 1 for $k > j$. We will

denote this inverse limit of this system by $\mathbb{G}$. We will use $P_i : \mathbb{G} \rightarrow A_i$ to denote the standard projection homomorphism.

Eda [8] pointed out to the authors that Proposition 2.1 does not hold in the case that the connecting homomorphisms are not surjective. However, it turns out the proof of Proposition 2.1 is still sufficient to enumerate all possible isomorphism types of inverse limits of countable rank free groups (see Remark 2.4).

Proposition 2.1. *Let G be an inverse limit of finite rank free groups, F_i, with surjective connecting homomorphisms indexed over the natural numbers. Then G is isomorphic to $\mathbb{G}$ or a finite rank free group.*

The following two lemmas are well known.

Lemma 2.2. *If G is an inverse limit of groups $\{F_\alpha\}$ and G' is the inverse limit of some cofinal sequence of $\{F_\alpha\}$, then G is isomorphic to G'.*

Lemma 2.3. *Any morphism of inverse systems which consists of isomorphisms induces an isomorphism of limits.*

Proof of Proposition 2.1. Let $\pi_{i,j}$ be the connecting homomorphisms of G. If the rank of F_n is eventually constant, then the connecting homomorphisms must eventually be isomorphisms (see [12, Proposition 2.12]). Hence G is a finite rank free group. Otherwise by passing to a cofinal subsequence, we may assume that the rank of F_n is a strictly increasing sequence.

Let B_1 be a basis for F_1. By induction, suppose that for all $m < n$, $B_m \cup K_m$ is a basis for F_m such that $\langle K_m \rangle \subset \ker(\pi_{m,m-1})$ and $\pi_{m,m-1}$ maps B_m bijectively onto $B_{m-1} \cup K_{m-1}$.

There exists a free basis $B'_n \cup K_n$ of F_n with the property that

$$\langle K_n \rangle \subset \ker(\pi_{n,n-1})$$

and $\pi_{n,n-1}$ restricted to $\langle B'_n \rangle$ is an isomorphism (again by [12, Proposition 2.12]). Now restricting $\pi_{n,n-1}$ to this set we may define $B_n = (\pi_{n,n-1}|_{\langle B'_n \rangle})^{-1}(B_{n-1})$. Since $\pi_{n,n-1}$ restricted to $\langle B'_n \rangle$ is an isomorphism, B_n is a free basis for $\langle B'_n \rangle$. It is a simple exercise to show that $B_n \cup K_n$ is still a free basis for F_n.

It is now trivial to find isomorphisms between the groups F_n and $A_{|B_n \cup K_n|}$ which commute with the connecting homomorphisms. The result follows from Lemma 2.3 and Lemma 2.2. $\qquad\square$

Remark 2.4. Given a system of countable rank free groups $(F_n, \pi_{n,n-1})$ with inverse limit G, one can pass to the inverse system $(\pi_n(G), \pi'_{n,n-1})$ without changing the isomorphism type of the inverse limit, where π_n is the canonical projection of G to F_n and $\pi'_{n,n-1}$ is the restriction of $\pi_{n,n-1}$ to $\pi_n(G)$. The new system then

has surjective connecting homomorphisms. However, $\pi_n(G)$ is a free group of possibly infinite rank. Then as in the proof of Theorem 2.1, we can find a basis $B_n \cup K_n$ of $\pi_n(G)$ with the same properties as before. After passing to a cofinal inverse system, we may assume that B_n and K_n are eventually trivial, finite, or countably infinite. The isomorphism type of the inverse limit is determined by the cardinalities of B_n and K_n. This gives five possible isomorphism types. Eda [8] shows that these five types can all be realized and are distinct.

3   Images of the Hawaiian earring group

Note that A_i embeds in $\mathbb{G}$ by sending $\mathbf{a}_i$ to the element which is 1 in the first $i-1$ coordinates and $\mathbf{a}_i$ in all other coordinates. In [6, Section 2], de Smit shows that this defines a canonical embedding of $\mathbb{H}$ into $\mathbb{G}$.

We will give $\mathbb{G}$ a metric such that $\mathbb{G}$ under the induced topology is a topological group. In [10, Section 4], Higman defines a topology which gives the same topology as our metric. We introduce a metric for convenience of notation only. Recall that $\mathbb{G} = \varprojlim\{A_i, P_{i,j}\}$ where A_i is a free group of rank i and $P_{i,j}$ are the connecting homomorphisms defined in Section 2. Let d_i be the $\{0, 1\}$-metric on A_i, i.e. $d_i(x, y) = 0$ if $x = y$ and 1 otherwise. For $(g_n), (h_n) \in \mathbb{G}$, let

$$d((g_n), (h_n)) = \sum_n \frac{1}{2^n} d_n(g_n, h_n).$$

Remark 3.1. Under this topology, $g_i = (g_n^i)$ converges to $h = (h_n)$ if and only if for each n there exists an $M(n)$ such that $g_n^i h_n^{-1} = 1$ for all $i \geq M(n)$. It follows that if g_i converges $g_i g_{i+1}^{-1}$ must converge to 1. Suppose that $g_i g_{i+1}^{-1}$ converges to 1. Then g_n^i considered as a sequence in i is eventually constant for every n. Hence g_i converges.

We will leave it to the reader to verify that this topology makes $\mathbb{G}$ into a topological group. An interesting note is that under this metric $\mathbb{G}$ is not complete [9] and all three of the sets $\mathbb{H}$, $\mathbb{G} - \mathbb{H}$, and the free group generated by $\{\mathbf{a}_i \mid i \in \mathbb{N}\}$, which we will denote by $\langle \mathbf{a}_1, \mathbf{a}_2, \dots \rangle$, are dense.

The following two propositions of Higman demonstrate the elegance of this topology. For completeness and to make the proof readily accessible to the reader, we will include their proofs here.

Proposition 3.2. *Any endomorphism of $\mathbb{G}$ is continuous.*

Proof. Let $\varphi : \mathbb{G} \to \mathbb{G}$ be an endomorphism. Then $d((g_n), (h_n)) \leq \frac{1}{2^i}$ if and only if $g_n = h_n$ for all $n \leq i$. Higman in [10, Theorem 1] showed that $\tilde{P}_i \circ \varphi$ factors

through some $P_{n(i)}$. Cannon and Conner (see [2, Theorem 4.4]) have shown that the same holds true for any $\varphi : \mathbb{H} \to F$ where F is a free group. Hence,

$$P_i \circ \varphi = \varphi \circ P_{n(i)}$$

for some $n(i)$ which depends on i. Thus $d((g_n), (h_n)) \leq \frac{1}{2^{n(i)}}$ implies that

$$d\big(\varphi((g_n)), \varphi((h_n))\big) \leq \frac{1}{2^i}. \qquad \square$$

Proposition 3.3. *Any set function φ from $\{\mathbf{a}_1, \mathbf{a}_2, \dots\}$ to $\mathbb{G}$ such that $d(\varphi(\mathbf{a}_i), 1)$ converges to 0 extends to an endomorphism of $\mathbb{G}$.*

Proof. Let $\varphi : \{\mathbf{a}_1, \mathbf{a}_2, \dots\} \to \mathbb{G}$ be a function such that $d(\varphi(\mathbf{a}_i), 1)$ converges to 0. Then φ will extend to the free group $\langle \mathbf{a}_1, \mathbf{a}_2, \dots \rangle \leq \mathbb{G}$. We want to be able to extend φ to all of $\mathbb{G}$.

Suppose that $g_i \in \langle \mathbf{a}_1, \mathbf{a}_2, \dots \rangle$ converges in $\mathbb{G}$. We must show that $\varphi(g_i)$ also converges. By Remark 3.1, it is enough to show that $\varphi(g_i)\varphi(g_{i+1})^{-1}$ converges. Since φ is a homomorphism on $\langle \mathbf{a}_1, \mathbf{a}_2, \dots \rangle$, it follows that

$$\varphi(g_i)\varphi(g_{i+1})^{-1} = \varphi(g_i g_{i+1}^{-1}).$$

Hence it is sufficient to show that if $g_i \to 1$, then $\varphi(g_i) \to 1$.

Let $\varphi(\mathbf{a}_i) = (\varphi(\mathbf{a}_i)_n)$. Suppose that $g_i = (g_n^i)$ converges to 1. Fix n. We will show that $\varphi(g_i)_n$ when considered as a sequence in i is eventually trivial. Fix M such that $\varphi(\mathbf{a}_i)_n = 1$ for all $i \geq M$. Fix M' such that $g_n^i = 1$ for all $i \geq M'$. By construction, g_i is a word $w_i(\{\mathbf{a}_j\})$ in $\langle \mathbf{a}_1, \mathbf{a}_2, \dots \rangle$. Then $\varphi(g_i) = w_i(\{\varphi(\mathbf{a}_j)\})$; hence, $\varphi(g_i)_n = w_i(\{\varphi(\mathbf{a}_j)_n\})$. Then for all $i \geq \max\{M, M'\}$,

$$\varphi(g_i)_n = w_i(\{\varphi(\mathbf{a}_j)_n\}) = 1.$$

Suppose that $g_i, g_i' \in \langle \mathbf{a}_1, \mathbf{a}_2, \dots \rangle$ converge to the same element of $\mathbb{G}$. Then $g_i' g_i^{-1}$ converges to 1. Then $\varphi(g_i')\varphi(g_i^{-1}) = \varphi(g_i' g_i^{-1}) \to 1$. Thus φ extends to a well defined continuous function $\overline{\varphi} : \mathbb{G} \to \mathbb{G}$ which is independent of the chosen sequence.

Suppose that $g, h \in \mathbb{G}$. Then there exist $g_i, h_i \in \langle \mathbf{a}_1, \mathbf{a}_2, \dots \rangle$ such that $g_i \to g$ and $h_i \to h$. Since $\overline{\varphi}$ is independent of the chosen sequence,

$$\varphi(g_i)\varphi(h_i) \to \overline{\varphi}(g)\overline{\varphi}(h) \quad \text{and} \quad \varphi(g_i)\varphi(h_i) = \varphi(g_i h_i) \to \overline{\varphi}(gh).$$

Hence φ extends to a homomorphism $\overline{\varphi}$. $\qquad \square$

Cannon and Conner in [2] showed that in one-dimensional spaces there exists a unique (up to reparametrization) reduced representative for each path class. We will use $[\,\cdot\,]_{\natural}$ (or $\varphi(\,\cdot\,)_{\natural}$) to represent the unique reduced representative for the

path class $[\cdot]$ (or $\varphi(\cdot)$). For a path $\alpha : [0, t] \to X$, let $\overline{\alpha}$ be the path defined by $\overline{\alpha}(s) = \alpha(t - s)$. As well, let $\widehat{\alpha}$ be the change of base point isomorphism induced by the path α, i.e. $\widehat{\alpha}([g]) = [\overline{\alpha} * g * \alpha]$. We will also use the following theorem.

Theorem 3.4 (Eda [7]). *Let $\psi : \mathbb{H} \to \pi_1(X, x_0)$ be a homomorphism into the fundamental group of a one-dimensional Peano continuum X. Then there exists a continuous function $f : (E, 0) \to (X, x)$ and a path $\alpha : (I, 0, 1) \to (X, x_0, x)$, with the property that $f_* = \widehat{\alpha} \circ \varphi$. Additionally, if the image of ψ is uncountable, the α is unique up to homotopy rel endpoints.*

Another proof, as well as a proof for a planar version, of this theorem can be found in the Masters Thesis of the second author (see [11]).

We are now ready to give our counter-example. We will begin by defining a set function $\varphi : \{\mathbf{a}_1, \mathbf{a}_2, \dots\} \to \mathbb{G}$ by

$$\varphi(\mathbf{a}_i) = \begin{cases} \mathbf{a}_i & \text{if } i \text{ is odd,} \\ \mathbf{a}_1 \mathbf{a}_i \mathbf{a}_1^{-1} & \text{if } i = 2 \bmod 4, \\ \mathbf{a}_{i-2} & \text{if } i = 0 \bmod 4. \end{cases}$$

Then φ extends to an endomorphism of $\mathbb{G}$ which we may then restrict to the naturally embedded set $\mathbb{H}$. Thus after extending and restricting, $\varphi : \mathbb{H} \to \mathbb{G}$ is a homomorphism with uncountable image. Suppose there existed an isomorphism $\psi : \varphi(\mathbb{H}) \to \pi_1(\mathbf{E}, 0)$. Then $\psi \circ \varphi : \mathbb{H} \to \pi_1(\mathbf{E}, 0)$ is a homomorphism from $\mathbb{H}$ to a one-dimensional Peano continuum which by Theorem 3.4 must be conjugate to a homomorphism induced by a continuous function. Let T be the path such that $\widehat{T} \circ \varphi$ is induced by a continuous function. Then by construction $\psi \circ \varphi(\mathbf{a}_{4i-2}) = \psi(\mathbf{a}_1)^{-1} \psi(\mathbf{a}_{4i-2}) \psi(\mathbf{a}_1)$ and $\psi \circ \varphi(\mathbf{a}_{4i}) = \psi(\mathbf{a}_{4i-2})$ which imply that

$$[\overline{T} * \psi \circ \varphi(\mathbf{a}_{4i-2})_\natural * T]$$
$$= [\overline{T} * (\psi(\mathbf{a}_1)^{-1} \psi(\mathbf{a}_{4i-2}) \psi(\mathbf{a}_1))_\natural * T]$$
$$= [\overline{T} * (\psi(\mathbf{a}_1)^{-1} \psi \circ \varphi(\mathbf{a}_{4i}) \psi(\mathbf{a}_1))_\natural * T]$$
$$= [\overline{T} * \overline{\psi(\mathbf{a}_1)}_\natural * T * \overline{T} * \psi \circ \varphi(\mathbf{a}_{4i})_\natural * T * \overline{T} * \psi(\mathbf{a}_1)_\natural * T]$$
$$= [(\overline{\overline{T} * \psi(\mathbf{a}_1)_\natural * T}) * \overline{T} * \psi \circ \varphi(\mathbf{a}_{4i})_\natural * T * (\overline{T} * \psi(\mathbf{a}_1)_\natural * T)]$$
$$= \overline{[\overline{T} * \psi(\mathbf{a}_1)_\natural * T]} \cdot [\overline{T} * \psi \circ \varphi(\mathbf{a}_{4i})_\natural * T] \cdot [\overline{T} * \psi(\mathbf{a}_1)_\natural * T].$$

By our choice of T, $\{[\overline{T} * \psi \circ \varphi(\mathbf{a}_{4i-2})_\natural * T]_\natural\}$ and $\{[\overline{T} * \psi \circ \varphi(\mathbf{a}_{4i})_\natural * T]_\natural\}$ are null sequences of loops in $\mathbf{E}$. However, the second sequence of loops is conjugate to the first by the non-trivial loop $[\overline{T} * \psi(\mathbf{a}_1)_\natural * T]_\natural$ which is a contradiction.

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