

On a 14-dimensional lattice invariant under the simple group $G_2(3)$

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Abstract. We study the 14-dimensional real representation of the finite simple group $G_2(3)$ and relate it to (a) the compact real form of the Lie group G_2 , (b) the smallest Ree group and (c) representations in characteristic 3. In particular, we give a set of generators which leads to a new and easy proof that the group is indeed $G_2(3)$.

1 Introduction

The 14-dimensional real representation of the simple group $G_2(3)$ is a small example of a phenomenon studied in depth by Kostrikin and Tiep [3], whereby the compact real form of a simple Lie algebra gives rise to an interesting lattice, whose automorphism group is larger than one has a right to expect. This study was motivated by the construction of Thompson's sporadic simple group by P. E. Smith, preserving a 248-dimensional lattice, which, when reduced modulo 3, illustrates the inclusion of the Thompson group in $E_8(3)$. Since this construction involved the Dempwolff group $2^5 \cdot \text{SL}_5(2)$ acting irreducibly, it seemed natural to generalize to other finite groups acting irreducibly on Lie algebras, and in particular the normalizers of so-called Jordan subgroups of exceptional Lie groups. These are the following subgroups:

$$\begin{aligned}2^3 \cdot \text{SL}_3(2) &< G_2(\mathbb{C}), \\3^3 \cdot \text{SL}_3(3) &< F_4(\mathbb{C}), \\5^3 \cdot \text{SL}_3(5) &< E_8(\mathbb{C}), \\3^{3+3} \cdot \text{SL}_3(3) &< E_6(\mathbb{C}), \\2^{5+10} \cdot \text{SL}_5(2) &< E_8(\mathbb{C}).\end{aligned}$$

In the first of these cases, the group $G_2(3)$ arises as the group analogous to the Thompson group. In the second and third cases, the groups $\text{PSL}_4(3)$ and $\text{PSL}_4(5)$ arise, while in the fourth case one obtains both $\Omega_7(3)$ and the Fischer group Fi_{22} .

The example studied in this paper was first worked out in detail by Kostrikin, Kostrikin and Ufnarovskii [2], who began with the group $2^3 \cdot \mathrm{SL}_3(2)$, preserving the compact real form of the Lie algebra of type G_2 . After classifying all the lattices preserved by this group, they used a computer to find an automorphism of order 13 of one of the lattices, which they then used to show that the full automorphism group of the lattice is $2 \times G_2(3)$.

In [3, Chapter 8] some variants of this approach are given. One approach uses detailed knowledge of the group $G_2(3)$ (including information from the character table and the maximal subgroups) to observe that $G_2(3)$ must preserve one of the six lattices $\Lambda_0, \dots, \Lambda_5$ given in [3, Chapter 8], and since it does not preserve any of $\Lambda_0, \dots, \Lambda_4$ it must preserve Λ_5 . Another constructs explicitly by hand a lattice automorphism of order 3 outside the group $2^3 \cdot \mathrm{SL}_3(2)$. In either case, intrinsic properties of the lattice are used to show that the automorphism group is no bigger than $2 \times G_2(3)$.

It is the aim of this note to provide some simplifications to these proofs, all of which are quite long and involved. The key observation is that $G_2(3)$ is generated by two subgroups, of shapes $2^3 \cdot \mathrm{SL}_3(2)$ and ${}^2G_2(3) \cong \mathrm{PSL}_2(8):3$, intersecting in $2^3:7:3$. Explicit generators for the former, acting on the compact real form of the Lie algebra of type G_2 , in 14 real dimensions, are given in [2, 3] and also in [7]. Explicit generators for the latter, in 7 dimensions over \mathbb{F}_3 , are given in [5, 6], and in 7 real dimensions in [4, Section 4.5.4]. Now it turns out that, with small modifications, these two constructions can be combined to give a construction of the 14-dimensional real representation of $G_2(3)$.

The proof that the group we construct is indeed isomorphic to $G_2(3)$ proceeds by constructing an invariant lattice spanned by an orbit of 2×378 vectors $\pm v$, so that the group is finite, and then finding a map onto the standard copy of $G_2(3)$ as automorphisms of the octonion algebra, such that the kernel is trivial.

It is worth noting that this lattice is also described in the Atlas [1, p. 60], with respect to a completely different basis, so that a subgroup $\mathrm{SL}_3(3)$ fixes one coordinate, and acts monomially on the other 13. (This must be the same lattice, since, up to scalar multiplications and automorphisms, $G_2(3)$ only preserves one lattice in 14 dimensions.)

2 The action of $2^3 \cdot \mathrm{SL}_3(2)$ on the Lie algebra

We begin by summarizing the results of [7]. There the compact real form of the Lie algebra of type G_2 was constructed in such a way that an irreducible subgroup $2^3 \cdot \mathrm{SL}_3(2)$ of the automorphism group was visible. Indeed, the algebra constructed

there is really an *integral* form of the Lie algebra, whose automorphism group is exactly $2^3 \cdot \text{SL}_3(2)$.

The underlying real vector space is written as a direct sum of seven 2-spaces, labelled by the elements $t \in \mathbb{F}_7$, each spanned by three vectors u_t, v_t, w_t such that $u_t + v_t + w_t = 0$. First, let us change the notation by replacing u, v, w by the complex numbers $1, \omega, \bar{\omega}$, where ω is a primitive cube root of unity, so that the real 2-spaces are identified with complex 1-spaces, and the 14-dimensional real vector space is written as a 7-dimensional complex space. It is straightforward to check that the following elements $\alpha, \beta, \gamma, \delta$ generate a group $2^3 \cdot \text{SL}_3(2)$:

$$\begin{aligned} \alpha : x_t &\mapsto \pm x_t, \text{ minus sign just if } t \in \{0, 3, 5, 6\}, \\ \beta : x_t &\mapsto x_{t+1}, \\ \gamma : x_t &\mapsto \omega x_{2t}, \\ \delta : x_1 &\mapsto -x_1, x_2 \mapsto -\bar{x}_2, x_4 \mapsto \bar{\omega} \bar{x}_4, \\ &x_0 \leftrightarrow \bar{x}_3, x_5 \leftrightarrow \omega x_6. \end{aligned}$$

The main result of [7] is that there is a unique (up to real scalar multiplication) 14-dimensional Lie algebra invariant under this group. The product is given by $[1_1, \omega_2] = \bar{\omega}_4$, and images under the group. Notice that the argument given in [7] actually works in every characteristic except 3. In characteristic 3 there is exactly one other solution (in the notation of [7], we can have $\lambda = \mu = \frac{1}{2}$), in which $[1_1, \omega_2] = \omega_4$.

3 Representations of the Ree group ${}^2G_2(3)$

In [5] the subgroup $2^3:7:3$ was used to construct the Ree groups ${}^2G_2(3^{2n+1})$ in their natural 7-dimensional representations in characteristic 3. Now in the case $n = 0$, this representation actually lifts to characteristic 0. An explicit proof of this statement is given for example in [4, p. 139], where the well-known isomorphism ${}^2G_2(3) \cong \text{PSL}_2(8):3$ is also proved. In fact there are three 7-dimensional representations of this group, obtained from each other by tensoring with a linear character of the quotient of order 3.

These representations are given by generators $\alpha, \beta, \gamma, \varepsilon$ of the group acting as follows:

$$\begin{aligned} \alpha : x_t &\mapsto -x_t \text{ if } t \in \{0, 3, 5, 6\}, \\ \beta : x_t &\mapsto x_{t+1}, \\ \gamma : x_t &\mapsto \lambda x_{2t}, \\ \varepsilon : x_t &\mapsto -\frac{1}{2}(-x_{-t} + x_{1-t} + x_{2-t} + x_{4-t}) \end{aligned}$$

where $\lambda = 1, \omega, \bar{\omega}$ in the three cases. Reducing these matrices modulo $\theta = \omega - \bar{\omega}$ (so that $-\frac{1}{2} = 1$ and $\lambda = 1$) gives the matrices over \mathbb{F}_3 as in [5].

In the sequel, we shall not use the facts quoted without proof in this section. We merely use them as motivation for the definition of ε .

4 An orbit of 2×378 vectors

If we now put $\lambda = \omega$ so that the two definitions of γ agree, then all the maps $\alpha, \beta, \gamma, \delta, \varepsilon$ act semilinearly on the 7-dimensional complex space. Let G be the group generated by $\alpha, \beta, \gamma, \delta, \varepsilon$. We shall show that $G \cong G_2(3)$. This is perhaps not completely obvious from what we have done already, but can be proved easily once we have constructed the lattice.

We show first that the vector $(2\bar{\omega}, 2, 0, 0, 0, 0, 0)$ lies in an orbit of just 2×378 vectors under the action of the group G . To see this we first calculate the image $(\omega\theta, \omega, -\bar{\omega}, -1, -\bar{\omega}, 0, 1)$ of this vector under ε , and then use elements of the monomial group $\langle \alpha, \beta, \gamma, \delta \rangle \cong 2^3 \cdot \text{SL}_3(2)$ to map this to $(0, -\theta, \omega, \omega, \bar{\omega}, 1, \bar{\omega})$. Now we observe that the orbit contains at least

- the 2×42 images under the monomial group of $(2\bar{\omega}, 2, 0, 0, 0, 0, 0)$, and
- the 2×336 images under the monomial group of $\pm(0, -\theta, \omega, \omega, \bar{\omega}, 1, \bar{\omega})$.

Conversely, we must show that this set of 2×378 vectors is invariant under ε . It is easy to check that ε has order 2, and normalizes the subgroup $\langle \beta, \gamma \rangle \cong 7:3$, extending it to a group $7:6$. Therefore we only need to calculate the images under ε of representatives of each of the 18 orbits of $\langle \beta, \gamma \rangle$ on the vectors (up to sign). Indeed, since ε has order 2, the calculation reduces to the following 12 cases:

$$\begin{aligned}
 (2\bar{\omega}, 2, 0, 0, 0, 0, 0) &\leftrightarrow (\omega\theta, \omega, -\bar{\omega}, -1, -\bar{\omega}, 0, 1), \\
 (2\bar{\omega}, -2, 0, 0, 0, 0, 0) &\leftrightarrow (-\omega, -\omega\theta, -\bar{\omega}, 1, -\bar{\omega}, 0, -1), \\
 (0, 1, -\omega\theta, \omega, \bar{\omega}, 1, \bar{\omega}) &\leftrightarrow (1, -\bar{\omega}, 0, 1, -\omega, \omega\theta, -\bar{\omega}), \\
 (0, -1, \omega\theta, -\omega, \bar{\omega}, 1, -\bar{\omega}) &\leftrightarrow (\omega\theta, \bar{\omega}, -1, -\bar{\omega}, 0, 1, -\omega), \\
 (0, -1, -\omega\theta, \omega, -\bar{\omega}, 1, -\bar{\omega}) &\leftrightarrow (-\bar{\omega}, -1, -\bar{\omega}, 0, 1, \omega, -\omega\theta), \\
 (0, 1, -\omega\theta, -\omega, \bar{\omega}, -1, -\bar{\omega}) &\leftrightarrow (1, -\omega, -\omega\theta, -\bar{\omega}, -1, \bar{\omega}, 0), \\
 (0, -1, -\omega\theta, -\omega, -\bar{\omega}, -1, \bar{\omega}) &\mapsto (-\bar{\omega}, 0, 1, \omega\theta, \omega, \bar{\omega}, 1), \\
 (0, 1, \omega\theta, -\omega, -\bar{\omega}, 1, \bar{\omega}) &\mapsto (0, 1, \omega\theta, -\omega, -\bar{\omega}, 1, \bar{\omega}), \\
 (0, -1, \omega\theta, \omega, \bar{\omega}, -1, \bar{\omega}) &\mapsto (\omega\theta, \omega, \bar{\omega}, -1, \bar{\omega}, 0, -1), \\
 (0, 1, \omega, \omega\theta, -\bar{\omega}, 1, \bar{\omega}) &\mapsto (-\bar{\omega}, 0, -1, -\omega, -\omega\theta, \bar{\omega}, -1), \\
 (0, 1, -\omega, -\omega\theta, \bar{\omega}, 1, \bar{\omega}) &\mapsto (-\omega, -\omega\theta, \bar{\omega}, 1, \bar{\omega}, 0, 1), \\
 (0, 1, \omega, -\omega\theta, \bar{\omega}, 1, -\bar{\omega}) &\mapsto (0, 1, \omega, -\omega\theta, \bar{\omega}, 1, -\bar{\omega}).
 \end{aligned}$$

5 Reduction modulo θ

Let Λ be the \mathbb{Z} -lattice spanned by the 2×378 vectors listed above (but *not* their multiples by ω or $\bar{\omega}$). First note that $\theta\Lambda \subseteq \Lambda$, since by transitivity we need only check one vector, and we have

$$\begin{aligned} \theta(2\bar{\omega}, 2, 0, 0, 0, 0, 0) &= (2\bar{\omega}, 2, 0, 0, 0, 0, 0) \\ &\quad + (2, 0, 0, 0, 0, 0, 2\bar{\omega}) + (2, 0, 0, 0, 0, 0, -2\bar{\omega}) \\ &\quad + (0, 2\omega, 0, 0, 0, 2\bar{\omega}, 0) + (0, 2\omega, 0, 0, 0, -2\bar{\omega}, 0) \in \Lambda. \end{aligned}$$

The semilinear maps $\alpha, \beta, \gamma, \delta, \varepsilon$ all preserve Λ and $\theta\Lambda$, so induce \mathbb{F}_3 -linear maps on $\Lambda/\theta\Lambda$. (The only case in which this is not completely obvious is the element δ : in this case, multiplying the coordinates by θ has the effect of changing the sign in the definition of δ for all those coordinates which involve complex conjugation, namely coordinates 0, 2, 3, 4; since this sign-change is effected by α^{β^4} , the result follows.)

Now $(4, 0, 0, 0, 0, 0, 0) \in \Lambda$, so $(4\theta, 0, 0, 0, 0, 0, 0) \in \theta\Lambda$, and similarly for all 7 coordinates. But multiplication by 4 induces the identity map on $\Lambda/\theta\Lambda$, so we may obtain coordinates for $\Lambda/\theta\Lambda$ by reducing the coordinates for Λ modulo θ .

We obtain in this way 2×378 vectors in $\Lambda/\theta\Lambda \cong \mathbb{F}_3^7$. These fall into two orbits under the monomial group $2^3 \cdot \text{SL}_3(2)$, as follows:

- 2×42 vectors of shape $(\pm 1, \pm 1, 0, 0, 0, 0, 0)$, and
- 2×336 vectors of shape $(0, 0, \pm 1, \pm 1, \pm 1, \pm 1, \pm 1)$.

Moreover, the natural Hermitian form on Λ , suitably scaled, induces the natural quadratic form on $\Lambda/\theta\Lambda$, that is the form given by the sum of the squares of the coordinates. Our 2×378 vectors are then exactly the vectors of norm 2 with respect to this quadratic form.

In this way, we obtain an action of G on a 7-dimensional orthogonal space over \mathbb{F}_3 . Since the 2×378 given vectors in Λ span Λ , which is an irreducible lattice, the kernel of the permutation action of G on these 2×378 vectors is trivial. Since these vectors remain distinct on reduction modulo θ , the kernel of the action of G on $\Lambda/\theta\Lambda$ is also trivial.

6 Preserving the octonion algebra

Finally, we observe that the induced actions of $\alpha, \beta, \gamma, \delta, \varepsilon$ are exactly the generators for $G_2(3)$ given in [4]: indeed, $\alpha, \beta, \gamma, \delta^{\alpha^3}$ are given on page 120, while ε is in (4.56) on page 139. For the sake of completeness we include a proof here.

First we define the octonion algebra over \mathbb{F}_3 by identifying the 7 coordinates of $\Lambda/\theta\Lambda$ with square roots of -1 , labelled i_t for $t \in \mathbb{F}_7$, and defining the other products by $i_0i_1 = -i_1i_0 = i_3$, and images under the action of β and γ . Then the defining products are all of the form $i_r i_s = \pm i_t$, where $\{r, s, t\}$ is one of the 7 triples $\{x, x + 1, x + 3\}$. Since these are the lines which define the Fano plane, and α negates the coordinates corresponding to the complement of a line, we see that α negates exactly two of the three terms i_r, i_s, i_t , so preserves the octonion product.

Checking that δ preserves the product is a little more tedious, but since the permutation part is an automorphism of the Fano plane, it is only the signs which need checking. Finally, ε normalizes the group generated by β and γ , so it suffices to check that ε maps the equation $i_0i_1 = i_3$ to a true statement. In other words we need only check that

$$(-i_0 + i_1 + i_2 + i_4) \cdot (-i_6 + i_0 + i_1 + i_3) = -i_4 + i_5 + i_6 + i_1.$$

This shows that our five generators all preserve the octonion algebra, so lie in $G_2(3)$. Since $2^3 \cdot \text{SL}_3(2)$ is maximal in $G_2(3)$, they generate $G_2(3)$. (This can also be shown by explicit computations.) This completes the proof that the given matrices in real 14-space do indeed generate a group isomorphic to $G_2(3)$.

7 The minimal vectors of the lattice

We have not however proved that the full automorphism group of the lattice Λ is $2 \times G_2(3)$, since it is not obvious that the full automorphism group preserves $\theta\Lambda$, let alone the octonion product on $\Lambda/\theta\Lambda$. The method used in [2, 3] is to show that the minimal vectors of Λ are precisely the 2×378 vectors listed above, although the proof of this is omitted in [3].

Our ‘complex’ notation permits a relatively straightforward proof of the salient facts.

Lemma 7.1. *If $v, w \in \Lambda$, then $\Re(v.w) \in 2\mathbb{Z}$, where $v.w$ denotes the usual Hermitian inner product, and \Re denotes the real part of a complex number.*

Proof. By transitivity it is sufficient to check the inner products of the vector $(2\bar{\omega}, 2, 0, 0, 0, 0, 0)$ with the spanning vectors. The result is clear for all vectors in the 84-orbit of the monomial group, so it is sufficient to check the inner products of the two vectors $(2\bar{\omega}, \pm 2, 0, 0, 0, 0, 0)$ with all rotations of $(0, -\theta, \omega, \omega, \bar{\omega}, 1, \bar{\omega})$, $(0, 1, -\omega\theta, \omega, \bar{\omega}, 1, \bar{\omega})$ and $(0, 1, \omega, \omega, -\bar{\omega}\theta, 1, \bar{\omega})$. But the difference of any two of these last three vectors is in the 84-orbit, so we only need to check rotations of one of them. This is an easy exercise. \square

As an immediate corollary, we have that the norm of every vector in Λ is a multiple of 4.

Lemma 7.2. *There is no vector of norm 4 in Λ .*

Proof. Suppose that $v \in \Lambda$ has norm 4. If v has a single non-zero coordinate, then there is a rotation of $(0, -\theta, \omega, \omega, \bar{\omega}, 1, \bar{\omega}) \in \Lambda$, whose inner product with v is 2ω , which is a contradiction. Otherwise, v has at least three zero coordinates, and we may assume two of these are consecutive, so that $v = (x, 0, 0, \dots)$ with $x \notin 2\mathbb{Z}[\omega]$. Then taking inner products with $(2\bar{\omega}, 2, 0, 0, 0, 0, 0)$ and $(2, 0, 2\omega, 0, 0, 0, 0)$ yields that both $\Re(x)$ and $\Re(\bar{\omega}x)$ are integers. This is a contradiction. □

Lemma 7.3. *There are exactly 756 vectors of norm 8 in Λ .*

Proof. The same argument shows that if v is any vector of norm 8 which has at least two zero coordinates, then the coordinates immediately before and after a string of at least two zeroes are both divisible by 2. Therefore the only possible distributions of the norm among the coordinates are $(4^2, 0^5)$ or $(3, 1^5, 0)$.

In the first case, we may assume $v = 2(a, b, 0, 0, 0, 0, 0)$, where a, b are units in the ring $\mathbb{Z}[\omega]$. Taking inner products with the vectors $(0, 1, -\omega\theta, \omega, \bar{\omega}, 1, \bar{\omega})$ and $(\bar{\omega}, 0, 1, -\omega\theta, \omega, \bar{\omega}, 1)$, shows that $b = \pm 1$ and $a = \pm \bar{\omega}$.

In the second case, a similar argument taking inner products with vectors of shape $(2^2, 0^5)$ shows that the powers of ω in the coordinates of v are the same as in the known spanning vectors of Λ . Now if we could change sign in one coordinate, we would have a vector of norm 4 in Λ , so this does not happen. Moreover, changing sign in two coordinates can only happen if the powers of ω agree with those in one of the vectors of the 84-orbit. This restriction means that the group of allowable sign-changes to v has order at most 16. But we already have 16 sign combinations for the listed spanning vectors of this type. Thus there can be no more. □

8 The automorphism group of the lattice

Now to complete the proof that the automorphism group of the lattice is no bigger than $2 \times G_2(3)$, we first compute the real parts of the inner products of a selection of minimal vectors, and find some values $\pm 8, \pm 4, \pm 2$, and 0 for this quantity. Hence the rank of the action of the automorphism group on the 378 pairs of opposite vectors is at least 4. (In fact the distribution of these values for one minimal vector with all the minimal vectors is $-8^1 - 4^{52} - 2^{208}0^{234}2^{208}4^{52}8^1$, but we shall not need this.)

We have already shown that $\text{Aut } \Lambda$ has a subgroup $G_2(3)$, acting on an orthogonal 7-space over \mathbb{F}_3 in such a way that the minimal vectors of the lattice correspond

to vectors of norm 2 in this orthogonal space. Moreover, it is known, or can be readily calculated, that the suborbits of $G_2(3)$ on the 378 pairs of vectors of norm 2 are $1 + 52 + 117 + 208$. Therefore $\text{Aut } \Lambda$ also has suborbits of these sizes. Since the suborbit of size 117 consists exactly of those vectors which are perpendicular to the fixed vector, the projective geometry for the orthogonal group $\text{PO}_7(3)$ can be re-constructed from the suborbits. This puts $\text{Aut } \Lambda$ inside the automorphism group of the orthogonal group, which has suborbits $1 + 117 + 260$. Indeed, $\text{P}\Omega_7(3)$ also has suborbits $1 + 117 + 260$, so $\text{Aut } \Lambda$ intersects $\text{P}\Omega_7(3)$ in a proper subgroup, which must be $G_2(3)$ since this is a maximal subgroup of $\text{P}\Omega_7(3)$. Finally, the normalizer of $G_2(3)$ in the conformal orthogonal group is just $2 \times G_2(3)$, which finishes the argument.

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