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Presentations for rigid solvable groups

Nikolay S. Romanovskiy

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Abstract. A group G is said to be m-rigid if it has a normal series

$$G = G_1 > G_2 > \cdots > G_m > G_{m+1} = 1$$

in which each factor G_i/G_{i+1} is abelian and torsion-free as a $\mathbb{Z}[G/G_i]$-module. Denote by Σ_m the class of all m-rigid groups and by $\Sigma_m(R)$ the set of groups in Σ_m generated by $x_1, \dots, x_n$ that satisfy a given set of relations R. We say that a group in $\Sigma_m(R)$ is maximal if it has no proper covering in $\Sigma_m(R)$. It is proved that, for every R, the set $\Sigma_m(R)$ contains only finitely many maximal groups. The set of relations R is said to be complete if $\Sigma_m(R)$ contains a unique maximal group. It is shown that every finitely generated group in Σ_m is completely finitely presented. We give a definition of a canonical presentation for a rigid group with the generators $x_1, \dots, x_n$. If such a presentation is given, the group at least has decidable word problem. Given a finite set of relations $R = R(x_1, \dots, x_n)$, we effectively construct a finite set of canonical presentations in the generators $x_1, \dots, x_n$ for groups in $\Sigma_m(R)$ among which all the maximal groups in $\Sigma_m(R)$ are contained.

1 Introduction

In this section, we give some definitions and state the main results. Following [10], we say that a group G is *m-rigid* if it has a normal series

$$G = G_1 > G_2 > \cdots > G_m > G_{m+1} = 1 \tag{1.1}$$

with abelian factors G_i/G_{i+1} each of which, when viewed as a right $\mathbb{Z}[G/G_i]$-module, has no module torsion. A group is *rigid* if it is m-rigid for some m. Important examples of rigid groups are the free solvable groups in which the required series is the derived series. In [10] it is shown that if the series (1.1) with the required properties exists at all, then it is uniquely determined by the group, in which case the derived length of G is precisely m. A crucial fact about rigid groups is the theorem that says that an arbitrary rigid group is equationally Noetherian, i.e., every system of equations in finitely many variables over the base group is equivalent to a finite subsystem [14]. This made it possible to develop algebraic geometry over rigid groups [3, 11, 13–16].

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A subgroup H of a rigid group G is also rigid, with the corresponding series being obtained by intersecting H with (1.1) and omitting repetitions. A normal subgroup H is called an *ideal of* G if the factor group G/H is rigid.

Theorem 1.1. *In an m-rigid group generated by n elements the length of every strictly increasing (decreasing) chain of ideals is bounded by a function in m and n.*

We denote by Σ_m the class of all rigid groups of length $\leq m$. Clearly, every n-generated group in Σ_m is a quotient of the free solvable group $F_{m,n}$ of length m with basis $\{x_1, \dots, x_n\}$ by some ideal. We consider the problem of specifying a group in Σ_m by defining relations. Let $R = R(x_1, \dots, x_n)$ be a set of group words in $x_1, \dots, x_n$. In the classic situation, a group with the presentation $\langle x_1, \dots, x_n \mid R \rangle$ is the quotient of the corresponding free group by the smallest normal subgroup that includes R. In our case, the free solvable group $F_{m,n}$ need not contain the least ideal that includes R. This can be seen from the following example. Let $m = 2$, $n = 3$, and let R contain the single element $[x_1, x_2]^{x_3 - 1}$. If a group $G \in \Sigma_2$ generated by x_1, x_2, x_3 satisfies the relation $[x_1, x_2]^{x_3-1} = 1$, then either $[x_1, x_2] = 1$ or $x_3 \in G_2$. The first possibility is realized for the group defined in the variety $\mathfrak{A}^2$ of solvable groups of length 2 by the generators x_1, x_2, x_3 and the defining relation $[x_1, x_2] = 1$. In this group we have $x_3 \notin G_2$. The second possibility is realized for the group defined by the relations $[x_3, F'_{2,3}] = 1$. In this group, $[x_1, x_2] \neq 1$ and $x_3 \in G_2$. There exists no group in Σ_2 that would cover both of these groups and satisfy the relation $[x_1, x_2]^{x_3-1} = 1$.

Denote by $\Sigma_m(R)$ the set of rigid groups in Σ_m generated by $x_1, \dots, x_n$ that satisfy the relations R. A group in $\Sigma_m(R)$ is said to be *maximal* if it has no proper covering in $\Sigma_m(R)$. Theorem 1.1 implies that every group in $\Sigma_m(R)$ has maximal coverings. Therefore, the set of defining relations R in the generators $x_1, \dots, x_n$ generally defines in Σ_m not a single group but rather a set of groups which is the set of all maximal groups in $\Sigma_m(R)$.

Theorem 1.2. *For every R, the set $\Sigma_m(R)$ has only finitely many maximal groups.*

We say that R is a complete set of defining relations if $\Sigma_m(R)$ has a unique maximal group which is therefore uniquely determined by the relations R.

Theorem 1.3. *Every finitely generated group in Σ_m is completely finitely generated, i.e., can be defined in Σ_m by a complete finite set of defining relations.*

With respect to Theorem 1.3 we recall the fact from [17] that demonstrates the distinction with the classical case that the free solvable group of length $m - 1 \geq 2$ is not finitely presented in the variety $\mathfrak{A}^m$ of solvable groups of length $\leq m$. In the classical case, we also have the well-known example of a solvable finitely presented group with algorithmically undecidable word problem which was constructed by O. G. Kharlampovich [5].

In Section 5.2, we give the definition of a canonical presentation for a given rigid group in its generators $x_1, \dots, x_n$. If such a presentation is given, then the group at least has decidable word problem.

Theorem 1.4. *Given a finite set of relations $R = R(x_1, \dots, x_n)$, one can effectively construct a finite set $\Omega_m(R)$ of canonical presentations in $x_1, \dots, x_n$ for groups in $\Sigma_m(R)$ among which all the maximal groups in $\Sigma_m(R)$ are contained.*

Unfortunately, we do not know a way to distinguish between the maximal groups in $\Omega_m(R)$ and the nonmaximal ones. This problem is equivalent to the following question: given a canonical presentation for a rigid group, can one effectively construct a complete finite set of defining relations?

Theorem 1.4 yields, in particular, that, for every word $v(x_1, \dots, x_n)$, it can be effectively determined whether or not this word represents the identity in all n-generated groups in Σ_m that satisfy the relations R or, in other words, whether or not the relation v is a consequence of the relations R in the class Σ_m. We observe that this fact can also be deduced from the property that a finitely generated rigid group is embedded in an iterated wreath product of several free abelian groups of finite ranks (proven in [10]) and from the decidability of the universal theory of such a product (proven in [2]).

2 Auxiliary definitions and facts

2.1. Let G be a group. A *G-group* is a group that includes G as a fixed subgroup. For G-groups one can naturally define G-subgroups, G-homomorphisms, generating sets, etc.

As usual, given group elements x, y, we set

$$x^y = y^{-1}xy, \quad [x, y] = x^{-1}y^{-1}xy.$$

The derived subgroup of G is denoted by either G' or $[G, G]$.

We say that a group G *is discriminated* by the groups $\{H_i \mid i \in I\}$ if, for every finite set of nonidentity elements of G, there exists a homomorphism $G \to H_i$ such that the images of all elements of the set remain nonidentity. This condition is equivalent to the condition that for every finite set of elements of G, there exists a homomorphism $G \to H_i$ that is injective on this set.

2.2. Let G have an abelian normal subgroup A. We set $\overline{G} = G/A$ and $\overline{g} = gA$ for $g \in G$. The group A may be viewed as a right $\mathbb{Z}\overline{G}$-module with the action of an element $u = \alpha_1 \overline{g}_1 + \dots + \alpha_n \overline{g}_n \in \mathbb{Z}[G/A]$ on $a \in A$ defined by the formula $a^u = (a^{g_1})^{\alpha_1} \cdot \ldots \cdot (a^{g_n})^{\alpha_n}$. Suppose that we also have a group that can be decomposed into a semidirect product of its subgroup $\overline{G}$ and an abelian normal subgroup $D(G)$. We represent this group in the matrix form

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}.$$

The latter group is called a *splitting of G over A* provided that it comes with an embedding of G into it given by the rule

$$g = \begin{pmatrix} \overline{g} & 0 \\ d(g) & 1 \end{pmatrix}$$

so that $D(G)$ is generated as an $\mathbb{Z}\overline{G}$-module by the elements $d(g)$, $g \in G$.

By analogy with [7], we say that the splitting

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$$

is *free* if, for every epimorphism $\gamma : G \to H$, where H has a normal abelian subgroup B such that $A\gamma \leqslant B$, and every splitting

$$\begin{pmatrix} \overline{H} & 0 \\ D(H) & 1 \end{pmatrix}$$

of H over B, the map $d(g) \to d(g\gamma)$ defines a module epimorphism $D(G) \to D(H)$ consistent with the ring epimorphism $\mathbb{Z}\overline{G} \to \mathbb{Z}\overline{H}$. Then we clearly have an epimorphism of splittings

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix} \to \begin{pmatrix} \overline{H} & 0 \\ D(H) & 1 \end{pmatrix}$$

whose restriction to G coincides with γ. A general argument shows that a free splitting, if exists, is defined uniquely up to isomorphism. This means that, for two free splittings

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}, \quad \begin{pmatrix} \overline{G} & 0 \\ D_1(G) & 1 \end{pmatrix},$$

the map $d(g) \to d_1(g)$ defines a module isomorphism $D(G) \to D_1(G)$ which, in turn, gives a group isomorphism

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix} \to \begin{pmatrix} \overline{G} & 0 \\ D_1(G) & 1 \end{pmatrix}.$$

We say that *the splitting*

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$$

has differential if the map $d(g) \to \overline{g} - 1$ defines an epimorphism (differential) δ of $D(G)$ to the augmentation ideal $(\overline{G} - 1) \cdot \mathbb{Z}\overline{G}$ of the group ring $\mathbb{Z}\overline{G}$, with A being the kernel of the epimorphism (here A is naturally identified with a submodule of $D(G)$).

We now construct a particular splitting with differential, starting from the Magnus embedding; we call it the *Magnus splitting*. In order to do this we represent G as a factor group of the free group F with basis $\{x_i \mid i \in I\}$. Let $\varphi_1 : F \to G$ and $\varphi_2 : F \to \overline{G}$ be the canonical epimorphisms and let $g_i = x_i \varphi_1$, $i \in I$. Denote by T the right free $\mathbb{Z}\overline{G}$-module with basis $\{t_i \mid i \in I\}$. Consider the module epimorphism $\psi : T \to (\overline{G} - 1) \cdot \mathbb{Z}\overline{G}$ which is given by $(\sum t_i u_i)\psi = \sum(\overline{g}_i - 1)u_i$. Consider also the Magnus group homomorphism

$$\tau : F \to \begin{pmatrix} \overline{G} & 0 \\ T & 1 \end{pmatrix}$$

defined by the map

$$x_i \to \begin{pmatrix} \overline{g}_i & 0 \\ t_i & 1 \end{pmatrix}, \quad i \in I.$$

We have $\ker \tau \leqslant \ker \varphi_1 \leqslant \ker \varphi_2$ and

$$(\ker \varphi_2)\tau = \begin{pmatrix} 1 & 0 \\ L & 1 \end{pmatrix},$$

where $L = \ker \psi$ (see [12]). Then

$$(\ker \varphi_1)\tau = \begin{pmatrix} 1 & 0 \\ U & 1 \end{pmatrix},$$

where U is a submodule of L. By construction we see that if we set $D(G) = T/U$, then G is embedded into

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}.$$

The image of A under this embedding equals

$$\begin{pmatrix} 1 & 0 \\ L/U & 1 \end{pmatrix}$$

and can be identified with the module L/U. Using ψ we can determine the homomorphism $\delta : D(G) \to (\overline{G} - 1) \cdot \mathbb{Z}\overline{G}$ whose kernel is A. Since, by construction, the generators g_i of G satisfy $d(g_i)\delta = \overline{g}_i - 1$, we have $d(g)\delta = \overline{g} - 1$ for all $g \in G$.

Lemma 2.1 (see [7] and [10, Lemma 4.1]). *For a splitting*

$$\begin{pmatrix} \overline{G} & 0 \\ D(G) & 1 \end{pmatrix}$$

of a group G over its abelian normal subgroup A, the following conditions are equivalent:

(1) *the splitting is free,*

(2) *the splitting has differential,*

(3) *the splitting is isomorphic to the Magnus splitting,*

(4) *the module $D(G)$ is isomorphic to $(G - 1) \cdot \mathbb{Z}G/(G - 1)(A - 1) \cdot \mathbb{Z}G$ and $d(g) = g - 1 + (G - 1)(A - 1) \cdot \mathbb{Z}G$.*

3 Modules over Ore domains

3.1. Let K be an associative ring without zero divisors. Then K is called a right Ore domain if, for arbitrary elements $a, b \in K$, there exists a nonzero pair of elements $x, y \in K$ such that $ax = by$. A left Ore domain is defined similarly. The right Ore domain K is embedded in the right division ring of fractions $Q(K)$ (see [4]). In this ring, every element can be represented in the form ab^{-1}, where $a, b \in K$, $b \neq 0$. If K is embedded in another division ring, then the division subring generated by K is isomorphic to $Q(K)$. This implies that if K is both a left and right Ore domain, then the left and right division rings of fractions coincide.

In this paper, we will use the embeddedness into division rings of fractions of integral group rings. Suppose that the group ring $\mathbb{Z}G$ is a right Ore domain. Then $\mathbb{Z}G$ is also a left Ore domain. It has right division ring (which is also its left division ring) of fractions henceforth denoted by $Q(G)$. From [8] and [6], it follows that the integral group ring of a torsion-free solvable group G is an Ore domain, which is hence embedded into a division ring of fractions $Q(G)$.

We say that a right module T over a ring K is torsion-free if the condition $0 \neq t \in T$, $0 \neq u \in K$ implies $tu \neq 0$. If K is a right Ore domain, the torsion elements of T form a submodule, which is called *the torsion submodule*, with the corresponding factor module having no torsion.

Let T be a right torsion-free module over a right Ore domain K. Then T can be embedded in the right vector space $T \cdot Q(K) = T \otimes_K Q(K)$. The *rank of T*

is the cardinality of any maximal system of elements that are linearly independent over K. The rank of T coincides with the dimension of the vector space $T \cdot Q(K)$. A submodule U of T is said to be *isolated* if the factor module T/U is torsion-free. There exists a smallest isolated submodule that contains the given submodule U, which is called its *isolator*. Clearly, the isolator of U coincides with the intersection $T \cap U \cdot Q(K)$, where $U \cdot Q(K)$ is the subspace of $T \cdot Q(K)$, generated by U.

3.2. Let $\tau : K \to K\tau$ be an epimorphism of right Ore domains. We say that this epimorphism satisfies the lifting condition for linear independence if, for every n, the fact that some tuple of elements $\{(u_{11}\tau, \dots, u_{1n}\tau), \dots, (u_{m1}\tau, \dots, u_{mn}\tau)\}$ is linearly independent over $K\tau$ in the right module of rows $(K\tau)^n$ implies that its preimage $\{(u_{11}, \dots, u_{1n}), \dots, (u_{m1}, \dots, u_{mn})\}$ in K^n is linearly independent over K.

Lemma 3.1. *Let $\tau : K \to K\tau$ be an epimorphism of right Ore domains that satisfies the lifting condition for linear independence and let $T = K^n$ be a right free K-module. Let τ also denote the module epimorphism $T \to T\tau = (K\tau)^n$ which is consistent with the given ring epimorphism. Let U be a submodule of T, $\overline{U}$ the isolator of U, and $\overline{U\tau}$ the isolator of the submodule $U\tau$ of $T\tau$. Suppose that* $\operatorname{rank} U = \operatorname{rank} U\tau$. *Then $\overline{U}\tau \leqslant \overline{U\tau}$ and, therefore, τ defines an epimorphism of factor modules $T/\overline{U} \to T\tau/\overline{U\tau}$.*

Proof. The lifting condition implies that $\operatorname{rank} U \geq \operatorname{rank} U\tau$. By hypothesis, we have $\operatorname{rank} U = \operatorname{rank} U\tau$. Assume to the contrary that there is an element $v \in \overline{U}$ such that $v\tau \notin \overline{U\tau}$. Let $\{u_1\tau, \dots, u_r\tau\}$ be a maximal linearly independent subsystem in $U\tau$. Then $\operatorname{rank} U\tau = r$. Clearly, the system $\{v\tau, u_1\tau, \dots, u_r\tau\}$ is linearly independent over $K\tau$. Consequently, the system $\{v, u_1, \dots, u_r\}$ must be linearly independent over K. We have a contradiction with the fact that $\operatorname{rank} U\tau = \operatorname{rank} U = \operatorname{rank} \overline{U}$. The proof is complete. □

Lemma 3.2. *Let $T = t_1 \cdot K + \dots + t_n \cdot K$ denote a right free module with basis $\{t_1, \dots, t_n\}$ over an effectively defined (constructible) two-sided Ore domain K and let $\overline{U}$ be the isolator of the submodule U that is generated by a given finite tuple $u_1, \dots, u_m \in T$. The factor module $T/\overline{U}$ can be effectively embedded into a free K-module of rank $n - r$, where $r =$* $\operatorname{rank} U$.

Proof. For an ordered system $\{u_1, \dots, u_m\} \subseteq T$, an *elementary transformation over K* means, as usual, either (1) swapping of elements, (2) the replacement $u_i \to u_i\alpha + u_j\beta$, where $i \neq j$, $\alpha, \beta \in K$, $\alpha \neq 0$, or (3) omitting the zero elements. Clearly none of these transformations change the isolator of the submodule generated by the system's elements. Using elementary transformations the system can be made generalized scalar, i.e., when the matrix composed of the coefficients of $t_1, \dots, t_n$ has, to within a column permutation, the form

$$\begin{pmatrix} \alpha & 0 & \dots & 0 & * & \dots & * \\ \dots & & & & & & \\ 0 & 0 & \dots & \alpha & * & \dots & * \end{pmatrix}, \quad \alpha \neq 0.$$

Observe that, while performing the transformations, we will have to effectively solve Ore equations $ax = by$, which can be done, for instance, by exhaustive search. Thus, we may assume that $m = r$ and $u_1 = t_1\alpha + u_1', \dots, u_r = t_r\alpha + u_r'$, where $u_i' = t_{r+1}\alpha_{i,r+1} + \dots + t_n\alpha_{i,n}$. Clearly, $\{u_1, \dots, u_r, t_{r+1}, \dots, t_n\}$ is a basis of the vector space

$$T \cdot Q(K) = t_1 \cdot Q(K) + \dots + t_n \cdot Q(K).$$

We have the congruences

$$t_1 \equiv -u_1'\alpha^{-1}, \dots, t_r \equiv -u_r'\alpha^{-1}$$

modulo the subspace

$$U \cdot Q(K) = u_1 \cdot Q(K) + \dots + u_r \cdot Q(K).$$

Let γ be the left common denominator of all coefficients $\alpha_{ij}\alpha^{-1}$, which can be effectively calculated. Then T/U is included in a right free K-submodule with the basis $\{t_{r+1}\gamma^{-1}, \dots, t_n\gamma^{-1}\}$ of the space $T \cdot Q(K)/U \cdot Q(K)$. The decomposition of each given element of $T/\overline{U}$ in this basis can be found effectively. The proof is complete. □

4 Lifting the linear independence

In this section, we will show that the epimorphism of group rings that is induced by an epimorphism of m-rigid groups satisfies the lifting condition for linear independence.

Consider the following condition $(*)$ for a group:

$(*)$ if S_1 and S_2 are two nonempty finite subsets of a given group, then there exist elements $x \in S_1$, $y \in S_2$ such that the product xy is distinct from the product of any other pair of elements of S_1 and S_2.

This condition is satisfied, for instance, by the right-ordered groups. The condition implies that the group ring of the corresponding group over an integral domain (not necessarily commutative) is itself an integral domain.

Let A be a torsion-free abelian group and let $\Delta = (A-1)\cdot\mathbb{Z}A$ be the augmentation ideal of $\mathbb{Z}A$. We define a function ω on $\mathbb{Z}A$ by setting $\omega(u) = n$ if $u \in \Delta^n \setminus \Delta^{n+1}$ and $\omega(0) = \infty$. Then ω is a valuation, i.e., the conditions

$$\omega(uv) = \omega(u) + \omega(v) \quad \text{and} \quad \omega(u+v) \geq \min\{\omega(u), \omega(v)\}$$

hold. Let A be included as a normal subgroup in a group G and suppose that the factor group G/A is a solvable torsion-free group. Let B be a system of coset representatives of G by A. Every nonzero element of $\mathbb{Z}G$ can be uniquely represented in the form $b_1u_1 + \cdots + b_nu_n$, where b_i are distinct elements of B and $0 \neq u_i \in \mathbb{Z}A$. We extend ω to $\mathbb{Z}G$ by setting $\omega(b_1u_1 + \cdots + b_nu_n) = \min \omega(u_i)$. Clearly, ω does not depend on the choice of coset representatives B and $\omega(uv) \geq \omega(u) + \omega(v)$ for all $u, v \in \mathbb{Z}G$. It is easy to see that if the condition $(*)$ holds for G/A, then ω satisfies the condition $\omega(uv) = \omega(u) + \omega(v)$ and is thus a valuation of $\mathbb{Z}G$. However, we do not know whether $(*)$ holds for an arbitrary solvable torsion-free group. Nevertheless, the following fact holds:

Lemma 4.1. *Let $u, v \in \mathbb{Z}G$ and $\omega(u) = 0$. Then $\omega(uv) = \omega(v)$.*

Proof. We may assume that G is decomposed into a semidirect product of groups A and B, since otherwise we may use the splitting of G over A (see Lemma 2.1). Let $u \equiv b_1\gamma_1 + \cdots + b_n\gamma_n \mod \mathbb{Z}B \cdot \Delta$, where $b_i \in B$, $0 \neq \gamma_i \in \mathbb{Z}$, and let $v = b'_1v_1 + \cdots + b'_mv_m$, where $b'_j \in B$, $v_j \in \mathbb{Z}A$. Suppose that $\omega(v) = k$. Choose a finitely generated subgroup A' of A such that all $v'_j \in \mathbb{Z}A'$. Let $\{a_1, \dots, a_l\}$ be a basis of A'. Every element of $\mathbb{Z}A'$ that lies in Δ^k can be decomposed uniquely modulo Δ^{k+1} into an integral linear combination of monomials of weight k in $a_1 - 1, \dots, a_l - 1$, i.e., elements of the form $(a_1-1)^{\alpha_1} \cdot \ldots \cdot (a_l - 1)^{\alpha_l}$, where $\alpha_1 + \cdots + \alpha_l = k$. Since we may neglect the summands from $\mathbb{Z}B \cdot \Delta$ in the representation of u and the summands from $\mathbb{Z}B \cdot \Delta^{k+1}$ in the representation of v, we assume that $u = b_1\gamma_1 + \cdots + b_n\gamma_n$ and all v_j are nontrivial linear combinations of monomials of weight k. Let us forget for now that A and B are subgroups of G and consider the group ring R of B with coefficients in $\mathbb{Z}A$. Then u and v represent nontrivial elements of R. From [6], it follows that the group ring of a solvable torsion-free group over an Ore domain is itself an Ore domain. Thus, the product of u and v, which equals $\sum_{i,j} b_ib'_j\gamma_iv_j$, is nonzero. But then the product of u and v is nonzero in $\mathbb{Z}G$ too, and if we decompose this product in B with coefficients in $\mathbb{Z}A$, then these coefficients will be integral linear combinations of monomials of weight k. The fact that one of the coefficients is nonzero also means that it does not lie in Δ^{k+1}. Therefore, $\omega(uv) = k$. The proof is complete. □

Denote by $\overline{u}$ the image of $u \in \mathbb{Z}G$ in $\mathbb{Z}\overline{G}$, where $\overline{G} = G/A$.

Lemma 4.2. *Under the assumptions on G above, the epimorphism of group rings $\mathbb{Z}G \to \mathbb{Z}\overline{G}$ satisfies the lifting condition for linear independence.*

Proof. Let $\{u_1 = (u_{11}, \ldots, u_{1n}), \ldots, u_m = (u_{m1}, \ldots, u_{mn})\}$ be a tuple of rows of length n over $\mathbb{Z}G$ and let $\{\overline{u}_1 = (\overline{u}_{11}, \ldots, \overline{u}_{1n}), \ldots, \overline{u}_m = (\overline{u}_{m1}, \ldots, \overline{u}_{mn})\}$ be the corresponding tuple of images over $\mathbb{Z}\overline{G}$. Suppose that the latter tuple is linearly independent over $\mathbb{Z}\overline{G}$. As in the proof of Lemma 3.1, we will consider elementary transformations over suitable rings. Clearly, the elementary transformations do not change the linear dependence or independence of a tuple. Since the latter tuple is linearly independent and the ring $\mathbb{Z}\overline{G}$ satisfies the (right) Ore condition, we can use elementary transformations over $\mathbb{Z}\overline{G}$ to bring this tuple to a generalized scalar form. Because elementary transformations of the latter tuple over $\mathbb{Z}\overline{G}$ can be lifted to elementary transformations of the former tuple over $\mathbb{Z}G$, we may initially assume that $n = m$ and the matrix $(\overline{u}_{ij})$ is scalar with a nonzero element along the diagonal. This means that $\omega(u_{ii}) = 0$ and $\omega(u_{ij}) > 0$ for $\neq j$. Suppose that the system $\{u_1, \ldots, u_n\}$ is linearly dependent and $u_1\alpha_1 + \cdots + u_n\alpha_n = 0$, where not all $\alpha_i \in \mathbb{Z}G$ are zero. Let $k = \min \omega(\alpha_i)$ and to be certain let $\omega(\alpha_1) = k$. Then, by Lemma 3.2, we have $\omega(u_{11}\alpha_1) = k$ and, in addition, $\omega(u_{i1}\alpha_1) \geq k + 1$ for $i > 1$. This contradicts the equality $u_{11}\alpha_1 + \cdots + u_{n1}\alpha_n = 0$. The proof is complete. □

Lemma 4.3. *Let $\varphi : G \to H$ be an epimorphism of rigid groups. Then the corresponding epimorphism of group rings $\mathbb{Z}G \to \mathbb{Z}H$ satisfies the lifting condition for linear independence.*

Proof. Let G_i be the suitable member of the rigid series for G. We set $C = \ker \varphi$, $C_i = C \cap G_i$. Since the quotients G/G_i and G/C are torsion-free, so are the quotients G/C_i. We obtain a sequence

$$G \to G/C_m \to G/C_{m-1} \to \cdots \to G/C \cong H$$

of epimorphisms of solvable torsion-free groups whose kernels C_i/C_{i+1} are abelian torsion-free groups. The claim follows from the previous lemma. The proof is complete. □

5 Rigid groups and their algebraic geometry

5.1. Consider a rigid group G with rigid series (1.1). Denote by $\chi_i(G)$ the rank of the $\mathbb{Z}[G/G_i]$-module G_i/G_{i+1}. The ordered tuple $\chi(G) = (\chi_1(G), \ldots, \chi_m(G))$ is called *the rank of G*.

Lemma 5.1 (see [10, Lemma 5.8]). *Let G be an m-rigid group generated by n elements. Then $\chi_1(G) \leq n$, $\chi_i(G) \leq n-1$ ($2 \leq i \leq m$), and G can be embedded into an iterated wreath product, denoted by $W(m,n)$, of m free abelian groups of rank n.*

Lemma 5.2 (see [10, Lemma 5.11]). *Let $\varphi : G \to H$ be a proper epimorphism of finitely generated m-rigid groups. Then $G_i\varphi \leqslant H_i$ and $\chi(G) > \chi(H)$ with respect to the lexicographic order.*

Lemma 5.3. *If a solvable group G is discriminated by rigid groups, then G is itself rigid.*

Proof. This is similar to the proof of [10, Lemma 6.1]. We therefore give only a sketch of the proof. Suppose that G is solvable of length m and $1 \neq c \in G^{(m-1)}$. We may consider only the discriminating homomorphisms φ such that $c\varphi \neq 1$. Then the corresponding images $G\varphi$ are m-rigid. It is known [10] that the mth term of the rigid series of an m-rigid group coincides with the centralizer of each of its nonidentity elements. Let C be the centralizer of c in G. Using the discrimination property it can be easily shown that C is a normal abelian subgroup which has no torsion as an $\mathbb{Z}[G/C]$-module and that the factor group G/C is discriminated by $(m-1)$-rigid groups. Then by induction G/C is rigid, which implies the rigidness of G. The proof is complete. □

5.2. In this subsection, using induction on m we will give a definition of a canonical presentation for an m-rigid group G in the generators $x_1, \ldots, x_n$. For $m = 1$ (in the abelian case), the definition amounts to specifying a finite set of defining relations for the abelian group G.

Induction step $m-1 \to m$. Let G_i denote the suitable term of the rigid series for G and suppose that, for each $i = 2, \ldots, m$, we have constructed by induction a canonical presentation for G/G_i in $x_1, \ldots, x_n$. In particular, suppose that the word problem is decidable and the values of $\chi(G/G_i)$ can be effectively computed. For brevity, we assume that $A = G/G_m$ and consider the free splitting

$$\begin{pmatrix} A & 0 \\ D & 1 \end{pmatrix}$$

of G over G_m. Recall that if $T = t_1 \cdot \mathbb{Z}A + \cdots + t_n \cdot \mathbb{Z}A$ is a free right $\mathbb{Z}A$-module, then D is the quotient T/U of T by an isolated submodule U and the embedding of G into the splitting can be induced by the Manus homomorphism

$$F \to \begin{pmatrix} A & 0 \\ T & 1 \end{pmatrix}, \quad x_i \to \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix},$$

where $F = \langle x_1, \ldots, x_n \rangle$ is a free group and a_i is the image of x_i in A. We assume that U is defined by finitely many elements that generate U as an isolated submodule. Then, by Lemma 3.2, the embedding of D into a free $\mathbb{Z}A$-module can be found effectively, which implies that the word problem in G is decidable. Moreover, both the rank of the module D and the value $\chi(G) = (\chi(A), \operatorname{rank} D - 1)$ can be determined effectively.

5.3. We recall some facts from algebraic geometry over groups. The reader may learn this theory from the papers [1, 9].

Given a group G, we denote by $G[X]$ the free product of G and a free group with basis $X = \{x_1, \ldots, x_n\}$. Consider a G-group H. A set $S \subseteq H^n$ is said to be *algebraic over G in H^n* if it is the solution set of a system of equations $\{v_i(x) = 1 \mid i \in I\}$ in $x = (x_1, \ldots, x_n)$ whose left-hand sides are elements of $G[X]$. Denote by $I(S) = \{v(x) \in G[X] \mid v(s) = 1,\ s \in S\}$ the annihilator of a nonempty algebraic set S and call the factor group $\Gamma(S) = G[X]/I(S)$ *the coordinate group of S*. Clearly, G is embedded into this factor group and $\Gamma(S)$ is generated as a G-group by the images of $x_1, \ldots, x_n$.

Observe that the intersection of any family of algebraic sets in H^n is again algebraic, whereas the union of two algebraic sets need not be algebraic.

The affine space H^n can be endowed with the Zariski topology: one takes the algebraic sets for a subbase of the family of closed sets. Recall that a topology is Noetherian if there exist no infinite decreasing chains of closed sets. In this case, every closed set can be uniquely represented as a noncancellable union of finitely many irreducible closed sets. The study of algebraic geometry over a group in the case of a non-Noetherian topology seems to be difficult. The Noetherian-ness of the Zariski topology amounts to the G-equational Noetherian-ness of H. The latter means that, for every n, every system of equations in $x_1, \ldots, x_n$ over G is equivalent to a finite subsystem. Observe that if $P \leqslant G$ and a G-group H is G-equationally Noetherian, then it is also P-equationally Noetherian.

We have already mentioned in the introduction the important result from [14] that every rigid group is equationally Noetherian (over itself).

Lemma 5.4 (see [1]). *Let a G-group H be G-equationally Noetherian. Then an arbitrary n-generated G-group is the coordinate group of an irreducible algebraic over G set in H^n if and only if it is G-discriminated by H.*

6 Proofs of Theorems 1.1–1.3

6.1. We now prove Theorem 1.1. Let G be an m-rigid n-generated group and let $H_{(1)} < H_{(2)} < \cdots$ be its strictly increasing chain of ideals. We then have the chain of proper epimorphisms of rigid groups $G/H_{(1)} \to G/H_{(2)} \to \cdots$. Let us prove that the length of this chain (i.e., the number of $H_{(i)}$) is bounded by a function $f(m,n)$. For $m = 1$, i.e., in the abelian case, the length of the chain clearly does not exceed $n + 1$. If, at some step, the group $G/H_{(k)}$ has solvable length m and $G/H_{(k+1)}$ has solvable length less than m, then by induction, beginning from this step, the length of the chain is bounded by the function $f(m-1,n)$. For $i \leq k$, we have by Lemma 5.1

$$1 \leq \chi_1(G/H_{(i)}) \leq n,\ 1 \leq \chi_2(G/H_{(i)}) \leq n-1,\ \ldots,\ 1 \leq \chi_m(G/H_{(i)}) \leq n-1$$

and by Lemma 5.2

$$\chi(G/H_{(1)}) > \chi(G/H_{(2)}) > \cdots > \chi(G/H_{(k)})$$

in the lexicographic order. Therefore, $k \leq n(n-1)^{m-1}$ and

$$f(m,n) \leq n(n-1)^{m-1} + f(m-1,n).$$

The proof is complete.

6.2. In order to prove Theorem 1.2, we consider the group $W = W(m,n)$ which we recall to be the iterated wreath product of m free abelian groups of rank n. If $v(x_1,\ldots,x_n)$ is a group word in $x_1,\ldots,x_n$, then the expression $v = 1$ will be understood as an equation over the identity group with the values of the indeterminates taken from W. We may assert that the n-generated subgroups in W are precisely the coordinate groups of the irreducible algebraic sets of the affine space W^n. Indeed, the coordinate group of an algebraic set is n-generated, is discriminated by W in view of Lemma 5.4, is rigid of solvable length at most m by Lemma 5.3, and is embedded into W by Lemma 5.1. Conversely, an n-generated subgroup of W is clearly discriminated by W and thus is the coordinate group of an irreducible algebraic set by Lemma 5.1.

Given a set of relations $R = R(x_1,\ldots,x_n)$, consider the algebraic set $S\,(\subseteq W^n)$ which is defined by the system of equations

$$\{r(x_1,\ldots,x_n) = 1 \mid r \in R\}.$$

We represent S as the union $S_1 \cup \cdots \cup S_p$ of the irreducible components. Consider the coordinate group $\Gamma(S_i)$ of the set S_i. Let G be an arbitrary group in $\Sigma_m(R)$. G is embedded in W and its generating elements $g_1,\ldots,g_n$ are the images of $x_1,\ldots,x_n$. Since the tuple $(g_1,\ldots,g_n)$ lies in S, it belongs to one of the sets S_i, and thus G is covered by $\Gamma(S_i)$. Clearly, $\Gamma(S_i)$ are precisely the maximal groups in $\Sigma_m(R)$. The proof is complete.

6.3. We now proceed to proving Theorem 1.3. Let G be an n-generated group in the class Σ_m. We represent G as the factor group of a free group with basis $\{x_1, \ldots, x_n\}$ by a normal subgroup H. Consider the system of equations

$$\{h(x_1, \ldots, x_n) = 1 \mid h \in H\},$$

where the values of the indeterminates are taken from W as above. Since W is equationally Noetherian, it follows that the system is equivalent to a finite subsystem $\{r(x_1, \ldots, x_n) = 1 \mid r \in R\}$. We claim that the set R is a complete set of relations for G. Assume the contrary. Then the set $\Sigma_m(R)$ contains a maximal group C that is distinct from G. Since G does not cover C, there exists an element $h \in H$ that is not a relation on C. We know that C is embedded into W and, if $c_1, \ldots, c_n$ are the images of $x_1, \ldots, x_n$ in C, then we conclude that $(c_1, \ldots, c_n)$ is a solution of the system $\{r(x_1, \ldots, x_n) = 1 \mid r \in R\}$ but is not a solution of $\{h(x_1, \ldots, x_n) = 1 \mid h \in H\}$. This leads to a contradiction with the assumption on R. The proof is complete.

7 Proof of Theorem 1.4

7.1. Given a set of relations $R = R(x_1, \ldots, x_n)$, for every group $A \in \Sigma_{m-1}(R)$, we define a group $\widehat{A} \in \Sigma_m(R)$ and an element $\gamma_A \in \mathbb{Z}A$ which is not uniquely determined.

Consider the Magnus homomorphism

$$\varphi_A : F \to \begin{pmatrix} A & 0 \\ T_A & 1 \end{pmatrix}, \quad x_i \to \begin{pmatrix} a_i & 0 \\ t_i & 1 \end{pmatrix},$$

where $T_A = t_1 \cdot \mathbb{Z}A + \cdots + t_n \cdot \mathbb{Z}A$ is a free $\mathbb{Z}A$-module, $F = \langle x_1, \ldots, x_n \rangle$ is a free group, and a_i is the image of x_i in A. Clearly, the images of the elements of R under this homomorphism are represented as unitriangular matrices. Let

$$r\varphi_A = \begin{pmatrix} 1 & 0 \\ u(r) & 1 \end{pmatrix}, \quad r \in R.$$

Denote by $\overline{U}$ the isolator in T_A of the submodule U generated by all $u(r)$, $r \in R$. Using φ_A we define a homomorphism

$$\psi_A : F \to \begin{pmatrix} A & 0 \\ T_A/\overline{U} & 1 \end{pmatrix}$$

and set $\widehat{A} = F\psi_A$. Recall that $\overline{U}$ is included in the submodule

$$L = \{t_1\alpha_1 + \cdots + t_n\alpha_n \mid (a_1 - 1)\alpha_1 + \cdots + (a_n - 1)\alpha_n = 0\}.$$

If $\overline{U} = L$, then $\widehat{A} = A$. Otherwise, $\widehat{A}$ is a proper covering for A. The module U has a maximal linearly independent system of elements $\{u_1, \dots, u_k\}$ which has a generalized scalar form. To be specific, let

$$u_1 = t_1\gamma + u_1', \ \dots, \ u_k = t_k\gamma + u_k',$$

with

$$0 \neq \gamma \in \mathbb{Z}A, \quad u_i' \in t_{k+1} \cdot \mathbb{Z}A + \dots + t_n \cdot \mathbb{Z}A.$$

We set $\gamma_A = \gamma$.

Given a canonical presentation for A in the generators $x_1, \dots, x_n$ and a finite set R, we may, using Lemma 3.2, effectively construct a canonical presentation for $\widehat{A}$ and find an element γ_A.

Lemma 7.1. *Let $A \in \Sigma_{m-1}(R)$, let $G \in \Sigma_m(R)$ be an m-rigid group, and let $B = G/G_m$. Suppose that we have a covering $A \to B$ such that the value of γ_A in $\mathbb{Z}B$ is nonzero. Then the group $\widehat{A}$ covers G.*

Proof. Denote by τ the given group epimorphism $A \to B$. We extend τ to a ring epimorphism $\mathbb{Z}A \to \mathbb{Z}B$ and then consider the epimorphism of free modules $(\mathbb{Z}A)^n \to (\mathbb{Z}B)^n$ that is consistent with the ring epimorphism. All these maps will be denoted by a single symbol τ. The same symbol will denote the induced homomorphism of matrix groups

$$\begin{pmatrix} A & 0 \\ T_A & 1 \end{pmatrix} \to \begin{pmatrix} B & 0 \\ T_B & 1 \end{pmatrix},$$

$T_A = (\mathbb{Z}A)^n$, $T_B = (\mathbb{Z}B)^n$. Consider the Magnus homomorphisms

$$\varphi_A : F \to \begin{pmatrix} A & 0 \\ T_A & 1 \end{pmatrix}, \quad \varphi_B : F \to \begin{pmatrix} B & 0 \\ T_B & 1 \end{pmatrix}$$

that correspond to A and B. We have $\varphi_B = \varphi_A \cdot \tau$. We will use the notation introduced when constructing the group $\widehat{A}$. Consider the free splitting

$$\begin{pmatrix} B & 0 \\ D_B & 1 \end{pmatrix}$$

of G over G_m. Recall that the module D_B can be represented as the factor module T_B/V, where V is an isolated submodule of T_B. Since R lies in the kernel of the group epimorphism $F \to G$, it follows that U lies in the kernel of the composition epimorphism $T_A \to T_B \to D$, i.e., $U\tau \leqslant V$. Thus $\overline{U\tau} \leqslant V$. We know that the rank of U equals k. Since the value of γ_A in $\mathbb{Z}B$ differs from zero, the rank of $U\tau$ cannot be less than k because the image of a maximal linearly independent system

$\{u_1, \ldots, u_k\}$ chosen in U, which has a generalized scalar form, remains generalized scalar and therefore linearly independent. The rank of $U\tau$ equals precisely k, since Lemma 4.3 implies that the ring epimorphism $\mathbb{Z}A \to \mathbb{Z}B$ satisfies the lifting condition for linear independence. Then, by Lemma 3.1, we have $\overline{U}\tau \leqslant \overline{U\tau}$ and thus τ defines an epimorphism of factor modules $T_A/\overline{U} \to T_B/V = D_B$. This yields the group covering

$$\begin{pmatrix} A & 0 \\ T_A/\overline{U} & 1 \end{pmatrix} \to \begin{pmatrix} B & 0 \\ D_B & 1 \end{pmatrix}$$

and thus also the group covering $\widehat{A} \to G$. The proof is complete. □

7.2. We now proceed directly to constructing the set $\Omega_m(R)$. Suppose by induction that the required construction for an arbitrary finite set of relations can be realized in Σ_{m-1}.

Starting from the set $\Delta_1 = \Omega_{m-1}(R)$, which can be constructed by induction, we will construct a sequence of finite sets $\Delta_1, \Delta_2, \Delta_3, \ldots$ of presentations for groups in $\Sigma_{m-1}(R)$ and prove that this sequence cannot be infinite. We then set $\Omega_m(R) = \{\widehat{A} \mid A \in \Delta = \bigcup \Delta_i\}$.

The algorithm for constructing the set Δ_i is as follows. Suppose that the sets $\Delta_1, \Delta_2, \ldots, \Delta_j$ have been constructed. We will assume that, for every group $A \in \Delta_j$, a finite set of relations $R_A \supseteq R$ such that $A \in \Omega_{m-1}(R_A)$ is specified. At the first stage, if $A \in \Delta_1$, then we define $R_A = R$. The set Δ_{j+1} is the union of the sets Δ_A which are indexed by the groups $A \in \Delta_j$. If all Δ_A turn out to be empty, the construction process stops.

Given a group $A \in \Delta_j$, we will show how to construct Δ_A. Consider the element $\gamma_A \in \mathbb{Z}A$ defined in the previous section, and the ring equation $\gamma_A = 0$ (the elements of A are represented as words in $x_1, \ldots, x_n$, and the values of the indeterminates are to be found in $W(m-1, n)$). If γ_A does not lie in the augmentation ideal $(A-1) \cdot \mathbb{Z}A$, then the equation $\gamma_A = 0$ is not consistent, and we then set $\Delta_A = \emptyset$. Suppose that $\gamma_A \in (A-1) \cdot \mathbb{Z}A$. In this case, the equation $\gamma_A = 0$ is equivalent to a disjunction of finitely many finite systems of group equations. We take one of these systems and add its left-hand sides to R_A. This will give a set of relations $\widetilde{R} \supset R_A$. The set Δ_A is defined to be the union of $\Omega_{m-1}(\widetilde{R})$ for all possible $\widetilde{R}$ that arise from the equation $\gamma_A = 0$. If $B \in \Omega_{m-1}(\widetilde{R})$, then we set $R_B = \widetilde{R}$. The actual elements of Δ_{j+1} are not merely the groups B defined by their canonical presentations, but rather the tuples consisting of the group B, the group A that gives rise to B, the set of relations R_B, and the index $j+1$. Observe that by construction we have $A \in \Sigma_{m-1}(R_A) \setminus \Sigma_{m-1}(R_B)$. There is a natural map $\Delta_{j+1} \to \Delta_j$ given by $(B, A, R_B, j+1) \to A$.

Suppose that the sequence $\Delta_1, \Delta_2, \ldots$ is infinite. Since the inverse limit of the spectrum $\Delta_1 \leftarrow \Delta_2 \leftarrow \cdots$ of finite sets is nonempty, it follows that every element of the inverse limit gives rise to an increasing chain of finite sets of relations $R_1 \subset R_2 \subset \cdots$ such that, for every j, there is a group that belongs to $\Sigma_{m-1}(R_j) \setminus \Sigma_{m-1}(R_{j+1})$. This contradicts the equational Noetherian-ness of the group $W(m-1, n)$ in which all the considered groups are embedded.

7.3. It remains to prove that the above-constructed set

$$\Omega_m(R) = \left\{\widehat{A} \mid A \in \Delta = \bigcup \Delta_i\right\}$$

contains all the maximal groups in $\Sigma_m(R)$. Let us take such a maximal group G. If its solvable length is less than m then, by induction, $G \in \Omega_{m-1}(R)$. Moreover, $\widehat{G} = G$ (otherwise, we have a proper covering $\widehat{G} \to G$ in $\Sigma_m(R)$) and then, by construction, $G = \widehat{G} \in \Omega_m(R)$.

Suppose that the solvable length of G is exactly m, i.e., G is an m-rigid group. We have $B = G/G_m \in \Sigma_{m-1}(R)$. Clearly, B is covered by some maximal group in $\Sigma_{m-1}(R)$ which lies in $\Omega_{m-1}(R) = \Delta_1$ by induction. Choose a maximal j such that B is covered by $A \in \Delta_j$. We claim that $B = A$ and $G = \widehat{A} \in \Omega_m(R)$. In order to prove this, we consider, for the chosen A, the element γ_A. If the value of γ_A in B is zero, then B satisfies one of the sets of relations $\widetilde{R}$ that arise from the equation $\gamma_A = 0$. Hence, B is covered by one of the maximal groups of the set $\Sigma_{m-1}(\widetilde{R})$ which, by construction, is contained in Δ_{j+1}. We have a contradiction with the choice of j. Therefore, the value of γ_A in B is nonzero. In this situation, $\widehat{A}$ covers G by Lemma 7.1. The maximality of G implies that $G = \widehat{A}$. The proof is complete.

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Author information

Nikolay S. Romanovskiy, Institute of Mathematics, Novosibirsk, Russia.
E-mail: `rmnvski@math.nsc.ru`