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On the vertices of indecomposable summands of certain Lefschetz modules

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Abstract. We study the reduced Lefschetz module of the complex of p-centric and p-radical subgroups. We assume that the underlying group G has parabolic characteristic p and the centralizer of a certain noncentral p-element has a component with central quotient H a finite group of Lie type in characteristic p. A nonprojective indecomposable summand of the associated Lefschetz module lies in a nonprincipal block of kG and it is a Green correspondent of an inflated, extended Steinberg module for a Lie subgroup of H. The vertex of this summand is the defect group of the block in which it lies. The application of these results to sporadic finite simple groups yields nine groups when $p = 2$ and eight groups when $p = 3$ for which the reduced Lefschetz module has precisely one nonprojective summand.

1 Introduction

For an action of a finite group on a finite simplicial complex, we define the reduced Lefschetz module as a formal alternating sum of the chain groups. The best known example of a reduced Lefschetz module is the Steinberg module for a finite simple group of Lie type acting on its Tits building, with coefficients having the defining characteristic. This modular representation is both irreducible and projective. Another important example is the reduced Lefschetz module of the Brown complex of nontrivial p-subgroups of an arbitrary finite group, with coefficients in a field of characteristic p. Although not necessarily indecomposable it is always projective both on the level of characters in the Grothendieck ring [36, Corollary 4.3] and as a virtual module in the Green ring [52].

More generally, Webb [52, Theorem A'] proved that if the fixed point sets of elements of order p are contractible, for an admissible action of a finite group on a finite simplicial complex, then the reduced Lefschetz module is a projective virtual module.

Ryba, Smith and Yoshiara [39] applied Webb's theorem to various geometries admitting a flag transitive action by a sporadic or alternating group, or such ge-

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ometries other than the buildings for Lie type groups. Their "Table 1" summarizes information on 18 cases where the reduced Lefschetz module is projective, including descriptions of many indecomposable summands as projective covers of irreducible modules. Later work of Smith and Yoshiara [47] used a different approach, verifying equivariant homotopy equivalences between such sporadic geometries and the Quillen complex of nontrivial elementary abelian p-subgroups.

Many complexes of p-subgroups have been constructed; a methodical treatment of nine such complexes can be found in [20]. All of these, in particular the complex of p-centric and p-radical subgroups $\mathcal{D}_p(G)$, have relevance to the work on cohomology decompositions that was systematically developed by Dwyer [14] and continued by Grodal [19]. This complex is also of interest to finite group theorists and can be regarded as the best candidate for a p-local geometry and a generalization of the notion of building to general finite groups. A comprehensive study of 2-local sporadic geometries (in which all the vertex stabilizers are 2-local subgroups) with applications to decompositions of the mod-2 group cohomology can be found in Benson and Smith [9]. In many cases, notably when the group has parabolic characteristic 2 (see Section 2 for an explanation of this term), the 2-local geometry studied by Benson and Smith is equivariantly homotopy equivalent with the complex of 2-centric and 2-radical subgroups.

The reduced Lefschetz module of a complex of subgroups is not in general projective. One of the first to study nonprojective reduced Lefschetz modules was Thévenaz [49, Theorem 2.1]. More recently, Sawabe [41, Proposition 5] determined an upper bound for the orders of vertices of their indecomposable summands. Further results regarding the Lefschetz characters associated to several 2-local sporadic geometries were obtained by Grizzard [18]. He studied the distribution into blocks of the indecomposable summands for several sporadic groups whose 2-local geometries have nonprojective Lefschetz characters.

In the present paper we analyze in detail the reduced Lefschetz module

$$\widetilde{L} = \widetilde{L}_G(\mathcal{D}_p(G); \mathrm{k})$$

of the complex of p-centric and p-radical subgroups in a finite group G of parabolic characteristic p. Our approach is to use a theorem of Robinson, quoted in Section 5 of this paper as Theorem 5.1, which relates the indecomposable summands with vertex V in $\widetilde{L}_G(\mathcal{D}_p(G); \mathrm{k})$ to the indecomposable summands with vertex V in the reduced Lefschetz module $\widetilde{L}_{N_G(V)}(\mathcal{D}_p(G)^V; \mathrm{k})$ of the fixed point subcomplex under the action of V.

We first find the homotopy type of the fixed point set of the action of a p-subgroup T on the complex $\mathcal{D}_p(G)$ of p-centric and p-radical subgroups, using a string of poset homotopy equivalences. Our approach is to focus on the set of p-central elements in the group, elements of order p that lie in the center of a Sylow p-subgroup of G. In [27] we introduced distinguished collections, consisting of p-subgroups of G that contain p-central elements in their centers. For a group G of parabolic characteristic p, the complex $\mathcal{D}_p(G)$ of p-centric and p-radical subgroups equals the complex $\widehat{\mathcal{B}}_p(G)$ of distinguished p-radical subgroups.

If the largest normal p-subgroup $O_p(TC_G(T))$ of T times its centralizer contains p-central elements then the fixed point set $\mathcal{D}_p(G)^T$ is contractible; this is Theorem 3.3 and was initially proven in [29] for T cyclic. If $O_p(TC_G(T))$ is purely noncentral, we assume that $TC_G(T)$ has a component with central quotient H which also has parabolic characteristic p. This assumption seems natural if we keep in mind the p-local structure of the sporadic simple groups of parabolic characteristic p. Our main result Theorem 3.6 asserts that the fixed point set $\mathcal{D}_p(G)^T$ is equivariantly homotopy equivalent to the complex of distinguished p-radical subgroups in H.

Next we study the vertices of the indecomposable summands of $\widetilde{L}$. We focus on the case when the centralizer of noncentral p-elements has a component with central quotient H which is a finite group of Lie type in characteristic p. We find in Theorem 5.4 that a nonprojective indecomposable summand of $\widetilde{L}$ is the Green correspondent of an inflated, extended Steinberg module which corresponds to a Lie subgroup of H. This summand lies in a nonprincipal block of the group ring kG, and has vertex equal to the defect group of the block in which it lies.

We discuss applications to the sporadic simple groups of parabolic characteristic p for $p = 2$ and $p = 3$. Our results verify for these groups a conjecture of Grizzard [18] and of Benson and Smith [9, p. 164] who noticed: *"…for all 15 sporadic groups G …for which the 2-local geometry Δ is not homotopy equivalent to"* the complex of nontrivial 2-subgroups *"$\mathcal{S}_2(G)$, it seems that the reduced Lefschetz module involves an indecomposable in a suitable nonprincipal 2-block of G of small positive defect."* We give an explanation for twelve of the sporadic simple groups that have parabolic characteristic 2.

In Proposition 6.2 we determine that for nine of these groups, there is a unique nonprojective summand of the reduced Lefschetz module, and we give partial results for the remaining three groups. Proposition 6.10 deals with the sporadic simple groups of parabolic characteristic 3. Tables 1 and 2 from Section 6 contain information about centralizers $C_G(t)$ of p-elements in these sporadic groups, the homotopy type of fixed point sets, and possible vertices V for an indecomposable summand of the reduced Lefschetz module. The headings of these tables are explained in Notations 6.1 and 6.9; also see Notation 3.4 for the formula $C_G(t) = O_C.H.K$ when $T = \langle t \rangle$.

We use the standard notation for finite groups and conjugacy classes of elements of order p as in the Atlas [13]. If p is a prime, then p^n denotes an elementary abelian p-group of rank n and p a cyclic group of order p. Also $H.K$ denotes an extension of the group H by a group K and $H : K$ denotes a split extension. The simplified notation of the form $5A^2$ stands for an elementary abelian group 5^2 whose 24 nontrivial elements are all of type $5A$. The notation of the form $2A_2B_1$ stands for a group of order 4 which contains two elements from class $2A$ and one element from class $2B$.

The structure of our paper is as follows. In Section 2 we review properties of groups of local characteristic p and parabolic characteristic p, as well as standard facts on collections of p-subgroups. We also recall a few useful results on distinguished collections of p-subgroups. In Section 3 we determine the nature of the fixed point sets under the action of p-subgroups. Section 4 is dedicated to standard facts from modular representation theory, in particular to properties of group algebras for groups of parabolic characteristic p. In Section 5 we prove a theorem that characterizes the Lefschetz module of the collection of p-centric and p-radical subgroups. Applications to the sporadic simple groups are discussed in Section 6.

2 Collections of p-subgroups

In this section we introduce the necessary notation, review the basic terminology regarding subgroup complexes and list a few known facts which will be used later in the proofs.

Groups of characteristic p and local characteristic p

The finite groups we are working with in the present paper have properties known as characteristic p, local characteristic p and parabolic characteristic p. For completeness of our presentation, we define these notions and list some of their features; the results mentioned here are well known and can be found elsewhere in the literature.

If G is a finite group, we denote by $O_p(G)$ the largest normal p-subgroup in G. A *p-local subgroup* is the normalizer of a nontrivial p-subgroup in G.

Definition 2.1. The group G has *characteristic p* if $C_G(O_p(G)) \leq O_p(G)$. If all p-local subgroups of G have characteristic p, then G has *local characteristic p*.

We remark here that the notion of "local characteristic p" is what group theorists usually call "characteristic p-type".

Proposition 2.2. *Let G be a finite group and let p be a prime dividing its order.*

(i) *Assume G has characteristic p. If P is a p-subgroup of G and H a subgroup of G with*
$$PC_G(P) \leq H \leq N_G(P),$$

then H has characteristic p, see [48, Lemma 1]. *In particular, G has local characteristic p, see* [15, 12.6].

(ii) *Let P be a p-subgroup of G. The subgroup $C_G(P)$ has characteristic p if and only if $N_G(P)$ has characteristic p, see* [15, 5.12].

Definition 2.3. A *parabolic subgroup* of G is an overgroup of a Sylow p-subgroup of G. The group G has *parabolic characteristic p* if all p-local, parabolic subgroups of G have characteristic p.

Our usage of the term "parabolic" differs from the classical notion of parabolics, in the special case of a Lie type group over a field of characteristic p, for in the latter, the parabolics are actually the overgroups not of just a Sylow p-subgroup S, but of a full Borel subgroup, i.e. of the normalizer $N_G(S)$. Our notion of "parabolic characteristic p" appears elsewhere in group theory: notably for $p = 2$ it appears as "even characteristic" in Aschbacher and Smith [6, p. 3]; this provides further evidence for the naturality of this concept.

Collections of p-subgroups. Notation and known results

The fundamentals of the homotopy theory for complexes of p-subgroups of a finite group were established by Quillen [36]. Significant developments were also made by Bouc, Brown, Thévenaz and Webb, among others. The interested reader can find historic details and specific references in [8, Secttion 6.6] and [46, Chapter 4].

A *collection* $\mathcal{C}$ of p-subgroups of G is a set of p-subgroups which is closed under conjugation; a collection is a G-poset under the inclusion relation with G acting by conjugation. The *order complex* or the *nerve* $\Delta(\mathcal{C})$ is the simplicial complex which has as simplices proper inclusion chains from $\mathcal{C}$; for a more detailed discussion see [9, 2.7]. If we let $|\Delta(\mathcal{C})|$ to denote the geometric realization of the nerve $\Delta(\mathcal{C})$, then the correspondence $\mathcal{C} \to |\Delta(\mathcal{C})|$ allows assignment of topological concepts to posets [36, Section 1]. A collection $\mathcal{C}$ is contractible if $|\Delta(\mathcal{C})|$ is contractible.

Notation 2.4. Given a subgroup P of G, let $\mathcal{C}^P = \{Q \in \mathcal{C} \mid P \leq N_G(Q)\}$ denote the subcollection of $\mathcal{C}$ fixed under the action of P. Next

$$\mathcal{C}_{>P} = \{Q \in \mathcal{C} \mid P < Q\}.$$

Similarly define $\mathcal{C}_{\geq P}$ and also $\mathcal{C}_{<P}$ and $\mathcal{C}_{\leq P}$. We will use the notation $\mathcal{C}_{>P}^{\leq H}$ for the set

$$\mathcal{C}_{>P}^{\leq H} = \{Q \in \mathcal{C} \mid P < Q \leq H\}.$$

The following statement was initially proved in a more general context by James and Segal [23], also see [51, 8.2.4]:

2.5. A poset map $f : \mathcal{C} \to \mathcal{C}'$ is a G-homotopy equivalence if and only if the induced map $f^H : \mathcal{C}^H \to \mathcal{C}'^H$ on H-fixed points is a homotopy equivalence for all subgroups H of G.

The results (ii) and (iii) from the next theorem are well known and appear in several places in the literature. We chose to use a more recent paper by Grodal and Smith [20] as a convenient reference.

Theorem 2.6. *Let $\mathcal{C}$ and $\mathcal{C}'$ be two collections of subgroups of G.*

(i) [50, Theorem 1] *Let $\mathcal{C} \subseteq \mathcal{C}'$. Assume either that $\mathcal{C}_{\geq P}$ is $N_G(P)$-contractible for all $P \in \mathcal{C}'$, or that $\mathcal{C}_{\leq P}$ is $N_G(P)$-contractible for all $P \in \mathcal{C}'$. Then the inclusion $\mathcal{C} \hookrightarrow \mathcal{C}'$ is a G-homotopy equivalence.*

(ii) [20, 2.2(3)] *Suppose that F is a G-equivariant poset endomorphism of $\mathcal{C}$ satisfying either $F(P) \geq P$ or $F(P) \leq P$, for all $P \in \mathcal{C}$. Then, for any collection $\mathcal{C}'$ containing the image of F, the inclusions $F(\mathcal{C}) \subseteq \mathcal{C}' \subseteq \mathcal{C}$ are G-homotopy equivalences.*

(iii) [20, in proof of Lemma 2.7 (1)] *Let $\mathcal{C}$ be a collection of p-subgroups that is closed under passage to p-overgroups. Let Q be an arbitrary p-subgroup in G. Then the inclusion $\mathcal{C}_{\geq Q} \hookrightarrow \mathcal{C}^Q$ is an $N_G(Q)$-homotopy equivalence.*

Theorem 2.7 ([50, Theorem 1]). *Let G be a group and let $\varphi : \mathcal{C} \to \mathcal{C}'$ be a poset map between two collections of subgroups of G. Suppose that for all $P \in \mathcal{C}'$, the subcollection $\varphi^{-1}(\mathcal{C}'_{\geq P})$ is $N_G(P)$-contractible. Then φ is a G-homotopy equivalence.*

We end this succinct review with the following observation [11]; for a proof see [19, pp. 420–421] or [29, Proposition 4.3]. We need one definition first: a p-subgroup Q of G is called *p-radical* if $Q = O_p(N_G(Q))$.

Proposition 2.8. *Let $\mathcal{C}$ be a collection of nontrivial p-subgroups of G, which is closed under passage to p-overgroups. Let Q be a p-subgroup of G. If $\mathcal{C}' \subseteq \mathcal{C}$ is a subcollection which contains every p-radical subgroup in $\mathcal{C}$, then $\mathcal{C}_{>Q}$ is $N_G(Q)$-homotopy equivalent to $\mathcal{C}'_{>Q}$.*

Standard collections of p-subgroups

Every p-subgroup Q of G is contained in a p-radical subgroup of G uniquely determined by Q and G. This is called the *radical closure* of Q in G and is the last term R_Q of the chain $P_{i+1} = O_p(N_G(P_i))$ starting with $Q = P_0$. It is easy to see that $N_G(Q) \leq N_G(R_Q)$. A p-subgroup R is called *p-centric* if $Z(R)$ is a Sylow p-subgroup of $C_G(R)$, in which case $C_G(R) = Z(R) \times H$, with H a subgroup of order relatively prime to p. If R is p-centric and Q is a p-subgroup of G which contains R, then Q is also p-centric and $Z(Q) \leq Z(R)$.

In what follows $\mathcal{S}(G)$ will denote the Brown collection of nontrivial p-subgroups and $\mathcal{B}(G)$ will denote the Bouc collection of nontrivial p-radical subgroups[1]. The inclusion $\mathcal{B}(G) \subseteq \mathcal{S}(G)$ is a G-homotopy equivalence, see [11, Corollary, p. 50] and [50, Theorem 2]. Let $\mathcal{D}(G)$ denote the subcollection of $\mathcal{S}(G)$ consisting of the nontrivial p-centric and p-radical subgroups of G. This collection is not always homotopy equivalent to $\mathcal{S}(G)$.

Distinguished collections of p-subgroups

In this section we review the definitions and some properties of certain subfamilies of $\mathcal{S}(G)$, called distinguished collections of p-subgroups. These collections were introduced in [26] and further analyzed in [27] and in [29].

2.9. An element x of order p in G is *p-central* if x is in the center of a Sylow p-subgroup of G. Let $\widehat{\Gamma}_p(G)$ denote the family of p-central elements of G.

Definition 2.10. For $\mathcal{C}(G)$ a collection of p-subgroups of G denote

$$\widehat{\mathcal{C}}(G) = \{P \in \mathcal{C}(G) \mid Z(P) \cap \widehat{\Gamma}_p(G) \neq \emptyset\}$$

the collection of subgroups in $\mathcal{C}(G)$ which contain p-central elements in their centers. We call $\widehat{\mathcal{C}}(G)$ the *distinguished* $\mathcal{C}(G)$ *collection*. We shall refer to the subgroups in $\widehat{\mathcal{C}}(G)$ as *distinguished subgroups*. Also, denote

$$\widetilde{\mathcal{C}}(G) = \{P \in \mathcal{C}(G) \mid P \cap \widehat{\Gamma}_p(G) \neq \emptyset\}$$

the collection of subgroups in $\mathcal{C}(G)$ which contain p-central elements. Obviously, the following inclusions hold:

$$\widehat{\mathcal{C}}(G) \subseteq \widetilde{\mathcal{C}}(G) \subseteq \mathcal{C}(G).$$

2.11. For later reference, we collect a few useful facts. The proofs are short and elementary; details can be found in [29, Lemma 3.3, Lemma 3.5 and Proposition 3.7].

[1] It is customary to denote these collections by $\mathcal{S}_p(G)$ and $\mathcal{B}_p(G)$, but since the prime p is fixed throughout the first sections of the paper, in order to simplify the notation we will drop the subscript p.

(i) The collection $\widetilde{\mathcal{S}}(G)$ is closed under passage to p-overgroups.

(ii) Let $P \in \mathcal{S}(G)$. If $Q \in \widehat{\mathcal{S}}(G)_{>P}$, then $N_Q(P) \in \widehat{\mathcal{S}}(G)$.

(iii) If $N_G(P)$ has characteristic p, then $O_p(N_G(P))$ is p-centric and distinguished.

(iv) If G has parabolic characteristic p and if $P \in \widetilde{\mathcal{S}}(G)$, then $N_G(P)$ has characteristic p.

Some of the properties of the distinguished collections of p-subgroups are gathered below.

Theorem 2.12. *Let G be a finite group.*

(i) [29, Propostion 3.4] *All p-centric subgroups of G are distinguished, hence*

$$\mathcal{D}(G) \subseteq \widehat{\mathcal{B}}(G).$$

(ii) [29, Propostion 3.6] *If G has local characteristic p, then*

$$\mathcal{D}(G) = \widehat{\mathcal{B}}(G) = \mathcal{B}(G).$$

(iii) [29, Propostion 3.7] *If G has parabolic characteristic p, then*

$$\mathcal{D}(G) = \widehat{\mathcal{B}}(G) = \widetilde{\mathcal{B}}(G).$$

(iv) [29, Propostions 3.9, 3.10] *If G has parabolic characteristic p, then $\widehat{\mathcal{B}}(G)$, $\widehat{\mathcal{S}}(G)$ and $\widetilde{\mathcal{S}}(G)$ are G-homotopy equivalent.*

2.13. In the case that G has parabolic characteristic p, the better known collection of p-centric p-radical subgroups is equal to the collection of distinguished p-radical subgroups. The reason for using the "distinguished" point of view for this collection resides in the fact that our approach is far more useful and easier to work with in proofs.

3 The structure of the fixed point sets

We investigate the fixed point sets of p-elements of G acting on the order complex of the collection $\mathcal{D}(G)$ of p-centric and p-radical subgroups. We are interested in groups of parabolic characteristic p, in which case, as seen in Theorem 2.12, the p-centric and p-radical subgroups are the distinguished p-radical subgroups of G and the collections $\widehat{\mathcal{B}}(G)$, $\widehat{\mathcal{S}}(G)$ and $\widetilde{\mathcal{S}}(G)$ are G-homotopy equivalent.

Notation 3.1. Throughout this section, G is a finite group of parabolic characteristic p, where p is a prime dividing its order. If T is a p-subgroup of G, we use the shorthand notation $C = TC_G(T)$ and $O_C = O_p(TC_G(T))$. Recall Notation 2.4.

Lemma 3.2. *Let G be a finite group of parabolic characteristic p and let T be a p-subgroup of G. The inclusion*

$$\mathcal{X} := \widehat{\mathcal{S}}(G)^{\leq C}_{>M} \hookrightarrow \mathcal{Y} := \widetilde{\mathcal{S}}(G)^{\leq C}_{>M}$$

is an $N_G(T)$-homotopy equivalence, for every p-subgroup M that satisfies

$$T \leq M \leq C \quad \text{and} \quad N_G(T) \leq N_G(M).$$

Proof. Let $P \in \mathcal{Y}$; we will show that $\mathcal{X}_{\geq P}$ is equivariantly contractible and apply Theorem 2.6 (i). Let R_P be the radical closure of P and let $Q \in \mathcal{X}_{\geq P}$. Consider the string of poset maps given by

$$Q \geq N_Q(P) \leq N_Q(P)Z(R_P) \geq PZ(R_P).$$

Note that $T \leq M < P \leq R_P$, so that $Z(R_P) \leq C$. By 2.11 (iii) and (iv), R_P is p-centric and distinguished. This implies that $Z(R_P)$ is distinguished, and also $PZ(R_P)$ is distinguished, since $Z(R_P) \leq Z(PZ(R_P))$. Next 2.11 (ii) implies that $N_Q(P)$ is distinguished. Observe that $N_Q(P) \leq N_G(P) \leq N_G(R_P)$, implying that $N_Q(P)Z(R_P)$ is a group. Choose $S_R \in \mathrm{Syl}_p(N_G(R_P))$ satisfying $N_Q(P) \leq S_R$, and extend it to $S \in \mathrm{Syl}_p(G)$. Note $R_P \leq S_R \leq S$. Then we have $Z(S) \leq C_G(N_Q(P))$ and $Z(S) \leq C_G(R_P) = Z(R_P)$, since R_P is p-centric and p-radical. This implies that $Z(S) \leq Z(N_Q(P)Z(R_P))$, so that $N_Q(P)Z(R_P)$ is distinguished. Thus the above is a string of equivariant poset maps $\mathcal{X}_{\geq P} \to \mathcal{X}_{\geq P}$, which proves the equivariant contractibility of the subcollection $\mathcal{X}_{\geq P}$. Note the second condition on M is needed so that $N_G(T)$ acts on $\mathcal{X}$ and $\mathcal{Y}$. □

The next result was first obtained for T a cyclic group of order p in [29], and the arguments from Propositions 4.4, 4.5 and 4.7 of that paper also apply to any p-group T. For completeness we include these proofs below. Notation 3.1 is maintained.

Theorem 3.3. *Let G be a group of parabolic characteristic p and let T be a p-subgroup of G. If $O_C \in \widetilde{\mathcal{S}}(G)$, then the fixed point set $\mathcal{D}(G)^T$ is $N_G(T)$-contractible.*

Proof. The result is obtained via a string of homotopy equivalences.

Step 1: The inclusions $\mathcal{D}(G)^T = \widehat{\mathcal{B}}(G)^T \hookrightarrow \widehat{\mathcal{S}}(G)^T \hookrightarrow \widetilde{\mathcal{S}}(G)^T$ are $N_G(T)$-homotopy equivalences, since $\mathcal{D}(G) = \widehat{\mathcal{B}}(G) \hookrightarrow \widehat{\mathcal{S}}(G) \hookrightarrow \widetilde{\mathcal{S}}(G)$ are G-homotopy equivalences by Theorem 2.12 (iii) and (iv).

Step 2: The inclusion $\widetilde{\mathcal{S}}(G)_{\geq T} \hookrightarrow \widetilde{\mathcal{S}}(G)^T$ is an $N_G(T)$-homotopy equivalence by Theorem 2.6 (iii) and 2.11 (i).

Step 3: If $T \in \widetilde{\mathcal{S}}(G)$, then $\widetilde{\mathcal{S}}(G)_{\geq T}$ is contractible, a cone on T, and we are done. Thus we assume $T \notin \widetilde{\mathcal{S}}(G)$, so that $\widetilde{\mathcal{S}}(G)_{\geq T} = \widetilde{\mathcal{S}}(G)_{>T}$.

Step 4: The inclusion $\widehat{\mathcal{S}}(G)_{>T} \hookrightarrow \widetilde{\mathcal{S}}(G)_{>T}$ is an $N_G(T)$-homotopy equivalence by Proposition 2.8. We are also using 2.11 (i) and Theorem 2.12 (iii). The latter implies the inclusion $\widetilde{\mathcal{S}}(G) \cap \mathcal{B}(G) \subseteq \widehat{\mathcal{S}}(G)$.

Step 5: The inclusion $\widehat{\mathcal{S}}(G)_{>T}^{\leq C} \hookrightarrow \widehat{\mathcal{S}}(G)_{>T}$ is an $N_G(T)$-homotopy equivalence by Theorem 2.6 (ii), where the poset map used is $F(P) = P \cap C$, for $P \in \widehat{\mathcal{S}}(G)_{>T}$. Since $T < P$, we have $Z(P) \leq C_G(T) \leq C$, so that $Z(P) \leq P \cap C$ and $F(P)$ is distinguished.

Step 6: The inclusion $\widehat{\mathcal{S}}(G)_{>T}^{\leq C} \hookrightarrow \widetilde{\mathcal{S}}(G)_{>T}^{\leq C}$ is an $N_G(T)$-homotopy equivalence by Lemma 3.2, applied with $M = T$.

Step 7: The inclusion $\widetilde{\mathcal{S}}(G)_{\geq O_C}^{\leq C} \hookrightarrow \widetilde{\mathcal{S}}(G)_{>T}^{\leq C}$ is an $N_G(T)$-homotopy equivalence by Theorem 2.6 (ii), with poset map $F(P) = P \cdot O_C$ for $P \in \widetilde{\mathcal{S}}(G)_{>T}^{\leq C}$. Since we have assumed that $O_C \in \widetilde{\mathcal{S}}(G)$, we have $\widetilde{\mathcal{S}}(G)_{\geq O_C}^{\leq C}$ is contractible, a cone on O_C. Thus we have shown that $\mathcal{D}(G)^T$ is $N_G(T)$-contractible. □

We arrive at the main result of this section. It asserts that under certain assumptions on the shape of the centralizer of a purely noncentral p-subgroup T of G, the structure of the fixed point set $\mathcal{D}(G)^T$ is determined by the collection of p-centric and p-radical subgroups in a subquotient (or section) of this centralizer. Our Theorem 3.6 generalizes the result of [29, Theorem 4.14]. Before stating and proving the theorem, we need to introduce some more notation.

Notation 3.4. Let T be any p-subgroup of G and set $C = TC_G(T)$ and $O_C = O_p(TC_G(T))$, maintaining Notation 3.1. Let $L \trianglelefteq C$ be a normal subgroup of C with $O_C \leq L$. Denote the quotient groups by $K = C/L$ and $H = L/O_C$, so we can write $C = O_C.H.K = L.K$.

Let $S_C \in \mathrm{Syl}_p(C)$ and extend it to $S \in \mathrm{Syl}_p(G)$. Since $T \leq S_C \leq S$, it follows that $Z(S) \leq Z(S_C)$. Next, define $S_L = S_C \cap L$, a Sylow p-subgroup of L. As $S_L \trianglelefteq S_C$, we have $Z(S_C) \cap S_L \neq 1$. However, this does not guarantee that $Z(S) \cap S_L \neq 1$. The sequence of subgroups $S_L \trianglelefteq S_C \leq S$ chosen as above will be denoted by (S_L, S_C, S) and will be referred to as a *triple of Sylow p-subgroups*.

3.5. We warn the reader that the "hat" symbols in $\widehat{\mathcal{S}}(G)$ and $\widehat{\mathcal{S}}(H)$ have different meanings; in the former it refers to those groups that are distinguished in G, while in the latter it refers to subgroups that are distinguished in H.

Theorem 3.6. *Let G be a finite group of parabolic characteristic p and let T be a p-subgroup of G. Assume the following hold:*

(N1) *The group O_C is purely noncentral in G.*

(N2) *The group $TC_G(T)$ has the form $C = O_C.H.K$, the group H has parabolic characteristic p, and $L = O_C.H$ is a normal subgroup of $N_G(T)$.*

(N3) *There exists a triple of Sylow p-subgroups (S_L, S_C, S) with the property that $Z(S) \cap S_L \neq 1$.*

Then there is an $N_G(T)$-equivariant homotopy equivalence between $\mathcal{D}(G)^T$ and $\mathcal{D}(H)$.

3.7. Condition (N3) implies that if $S'_C \in \mathrm{Syl}_p(C)$, then there exists $S' \in \mathrm{Syl}_p(G)$ such that $Z(S') \cap (S'_C \cap L) \neq 1$. There is an element $g \in C$ with $S'_C = gS_C g^{-1}$. Define $S' = gSg^{-1}$. Since $L \trianglelefteq C$, we have $S'_L = S'_C \cap L = g(S_C \cap L)g^{-1}$. Thus $Z(S') \cap S'_L = g(Z(S) \cap S_L)g^{-1}$.

There are several examples involving sporadic finite simple groups in which a condition stronger than (N3) is satisfied: all p-central elements of G which lie in C actually lie in L. This condition was used to prove a similar theorem; see [28, Theorem 2.1 and Lemma 2.3].

3.8. In some of the examples we will consider in Section 6, we have $L = C_G(O_C)$. Then whenever we have a triple (S_L, S_C, S) of Sylow p-subgroups, we will have $Z(S) \leq L$ since $O_C \leq S$. Thus (N3) is satisfied. Also, (N3) will be satisfied if we know that $Z(S_C) \leq L$, which will be true in most of our examples.

There is no direct relationship between the elements which are p-central in G and those which are p-central in C, or in L. In order to deal with this difficulty, we introduce the following subcollection of $\widehat{\mathcal{S}}(G)^{\leq C}_{>O_C}$.

3.9. Use Notations 2.4, 3.1 and 3.4 and set

$$\mathcal{T} = \{P \in \widehat{\mathcal{S}}(G)^{\leq C}_{>O_C} \mid Z(P) \cap Z(S) \cap Z(S_L) \neq 1, \\ \text{for some triple } (S_L, S_C, S) \text{ with } P \leq S_C \leq S\}.$$

This collection is contained in $\widehat{\mathcal{S}}(C)$, and it contains all the p-centric subgroups P in C which properly contain O_C, since if P is p-centric and $P \leq S_C \leq S$, then $Z(S) \leq C_G(P)$, implying $Z(S) \leq Z(P)$. Another collection of interest will be $\mathcal{T}^{\leq L}$, whose p-subgroups are distinguished as subgroups of G and also are distinguished as subgroups of L. The condition $N_G(T) \leq N_G(L)$ in (N2) is needed so that $N_G(T)$ acts on $\mathcal{T}^{\leq L}$. This condition will be satisfied if L is the generalized Fitting subgroup of C (there exists one component E of C with $E/Z(E) = H$).

3.10. Let $P \in \mathcal{T}$ and assume that $O_C < Q \leq P$. Then $N_P(Q) \in \mathcal{T}$. It is easy to see that if $Q \leq N_P(Q) \leq P$, then $Z(P) \leq Z(N_P(Q))$ and this suffices to show $N_P(Q) \in \mathcal{T}$.

Proof of Theorem 3.6. The proof will consist of five steps and a number of homotopy equivalences between various intermediary collections. For the rest of the proof we assume that G and T satisfy all the conditions from the hypotheses of Theorem 3.6.

Step 1: The collections $\mathcal{D}(G)^T$ and $\widehat{\mathcal{S}}(G)^{\leq C}_{>O_C}$ are $N_G(T)$-homotopy equivalent.

The proof of Theorem 3.3 shows that there is an $N_G(T)$-homotopy equivalence between $\mathcal{D}(G)^T$ and $\widetilde{\mathcal{S}}(G)^{\leq C}_{\geq O_C} = \widetilde{\mathcal{S}}(G)^{\leq C}_{>O_C}$, using that $O_C \notin \widetilde{\mathcal{S}}(G)$. Then an application of Lemma 3.2 with $M = O_C$ gives an equivalence with $\widehat{\mathcal{S}}(G)^{\leq C}_{>O_C}$.

Step 2: The inclusion $\mathcal{T} \hookrightarrow \widehat{\mathcal{S}}(G)^{\leq C}_{>O_C}$ is an $N_G(T)$-homotopy equivalence.

We will apply Theorem 2.6 (i) once again. We need to show that $\mathcal{T}_{\geq Q}$ is equivariantly contractible for every $Q \in \widehat{\mathcal{S}}(G)^{\leq C}_{>O_C}$. The equivariance here is with respect to the action of $N_G(T) \cap N_G(Q)$. Set $O_{CQ} := O_p(N_C(Q))$. Assume now that $P \in \mathcal{T}_{\geq Q}$ and consider the contracting homotopy given by the following string of equivariant poset maps:

$$P \geq N_P(Q) \leq N_P(Q)O_{CQ} \geq O_{CQ}.$$

In order to complete the proof we need to prove that each of the subgroups involved in the above string lies in $\mathcal{T}_{\geq Q}$. Since Q is a distinguished p-subgroup of G, $N_G(Q)$ has characteristic p, by 2.11 (iv). Next, because $T < Q \leq C$, it follows that $C_G(Q) \leq C_G(T) \leq C$ and thus $QC_G(Q) \leq N_C(Q) \leq N_G(Q)$, and by Proposition 2.2 (i), $N_C(Q)$ has characteristic p. Hence $C_{N_C(Q)}(O_{CQ}) \leq O_{CQ}$. But $Q \leq O_{CQ}$ so $C_G(O_{CQ}) \leq C_G(Q) \leq N_C(Q)$ and $C_G(O_{CQ}) = Z(O_{CQ})$ which implies $Z(S_C) \leq Z(O_{CQ})$ for any $S_C \in \mathrm{Syl}_p(C)$ which contains O_{CQ}. Extend S_C to a Sylow p-subgroup S of G satisfying $Z(S) \cap S_L \neq 1$, using (N3) and 3.7. Recall that $Z(S) \leq Z(S_C)$. Since $Z(S) \cap Z(S_L) \leq Z(S_C) \leq O_{CQ}$, it follows that $O_{CQ} \in \mathcal{T}$. The fact that $N_P(Q) \in \mathcal{T}_{\geq Q}$ follows from 3.10. Next consider $N_P(Q)O_{CQ}$ which is a p-subgroup of $N_C(Q)$ and which contains $Z(S_C)$ for any $S_C \in \mathrm{Syl}_p(C)$ which contains it. Thus all the groups in the string are in $\mathcal{T}_{\geq Q}$.

Step 3: The inclusion $\mathcal{T}^{\leq L} \hookrightarrow \mathcal{T}$ is an $N_G(T)$-homotopy equivalence.

Apply Theorem 2.6 (ii) with poset map $F(P) = P \cap L$ for $P \in \mathcal{T}$. Note that $Z(P) \cap Z(S) \cap Z(S_L) \leq Z(P \cap L)$.

Step 4: Recall that $L \simeq O_C.H$ and the quotient group $H \simeq L/O_C$ has parabolic characteristic p. The quotient map will be denoted $q : L \to H$. For $M \leq L$, let $\overline{M} = q(M)$. For a subgroup Q of L, set $O_{LQ} = O_p(N_L(Q))$ and for $\overline{Q} \leq H$, denote $O_{HQ} = O_p(N_H(\overline{Q}))$. If (S_L, S_C, S) is a triple of Sylow p-subgroups defined as in Notation 3.4, we let $\overline{S}_L = q(S_L) \simeq S_H \in \mathrm{Syl}_p(H)$. We prove the following:

> *The poset map $q_* : \mathcal{T}^{\leq L} \to \widehat{\mathcal{S}}(H)$ induced by the quotient map $q : L \to H$ is an equivariant homotopy equivalence.*

We first show that $q_*(\mathcal{T}^{\leq L}) \subseteq \widehat{\mathcal{S}}(H)$. Since O_C is purely noncentral in G, the map $q : L \to H$ is injective on the elements of $Z(S)$ as they are p-central in G. Therefore, if $P \in \mathcal{T}^{\leq L}$ then $Z(P) \cap Z(S) \cap Z(S_L) \neq 1$ for some triple of Sylow p-subgroups (S_L, S_C, S) with $P \leq S_L$, and this implies $Z(\overline{P}) \cap Z(\overline{S}_L) \neq 1$, and we have $q_*(P) = \overline{P} \in \widehat{\mathcal{S}}(H)$.

According to Theorem 2.7, the poset map $q_* : \mathcal{T}^{\leq L} \to \widehat{\mathcal{S}}(H)$ is an equivariant homotopy equivalence if $q_*^{-1}(\widehat{\mathcal{S}}(H)_{\geq \overline{Q}})$ is equivariantly contractible for any $\overline{Q} \in \widehat{\mathcal{S}}(H)$.

Define $Q = q^{-1}(\overline{Q})$. Recall that $O_C \leq P$, thus we have

$$q_*^{-1}(\widehat{\mathcal{S}}(H)_{\geq \overline{Q}}) = \{P \in \mathcal{T}^{\leq L} \mid \overline{Q} \leq \overline{P}\} = \{P \in \mathcal{T}^{\leq L} \mid Q \leq P\} = \mathcal{T}^{\leq L}_{\geq Q}.$$

For $P \in \mathcal{T}^{\leq L}_{\geq Q}$, consider the string of equivariant poset maps given by

$$P \geq N_P(Q) \leq N_P(Q) O_{LQ} \geq O_{LQ}.$$

We need to show that all of these terms lie in $\mathcal{T}^{\leq L}_{\geq Q}$. The fact that $N_P(Q) \in \mathcal{T}^{\leq L}_{\geq Q}$ follows from 3.10. Next, we have $N_L(Q) = q^{-1}(N_H(\overline{Q}))$, using $O_C \leq Q$. Thus O_{LQ} is equal to $q^{-1}(O_{HQ})$ by the correspondence theorem for normal subgroups applied to $N_L(Q) \to N_H(\overline{Q})$. Since $\overline{Q} \in \widehat{\mathcal{S}}(H)$, $N_H(\overline{Q})$ has characteristic p, by 2.11 (iv) and using our assumption that H has parabolic characteristic p. Thus $C_H(O_{HQ}) \leq O_{HQ}$ and $N_H(O_{HQ})$ also has characteristic p, which follows by an application of Proposition 2.2 (ii). Next, we have $C_L(O_{LQ}) \leq q^{-1}(C_H(O_{HQ})) \leq q^{-1}(O_{HQ}) = O_{LQ}$, which shows that the group O_{LQ} is p-centric in L. It follows that $Z(S_L) \leq Z(O_{LQ})$ for every Sylow p-subgroup of L which contains O_{LQ}. But by our assumption (N3), and using 3.7, the subgroup $Z(S_L)$ contains p-central elements of G; therefore it follows that O_{LQ} is distinguished in G. Consequently O_{LQ} lies in $\mathcal{T}^{\leq L}_{\geq Q}$. Now consider $N_P(Q) O_{LQ}$ which is a subgroup of $N_L(Q)$. Since O_{LQ} lies in every Sylow p-subgroup of $N_L(Q)$ and $P \in \mathcal{T}^{\leq L}_{\geq Q}$, it follows that the nontrivial elements in $Z(P) \cap Z(S) \cap Z(S_L)$ also lie in $Z(N_P(Q) O_{LQ})$ and the subgroup $N_P(Q) O_{LQ}$ is indeed in $\mathcal{T}^{\leq L}_{\geq Q}$ (that is $O_{LQ} \leq S_L$, with S_L the Sylow p-subgroup which contains P and satisfies the required condition from 3.9).

Step 5: There is an equivariant homotopy equivalence between $\mathcal{D}(G)^T$ and $\mathcal{D}(H)$.

Combine the first four steps of the proof to obtain the chain of $N_G(T)$-homotopy equivalences:

$$\mathcal{D}(G)^T \simeq \widehat{\mathcal{S}}(G)^{\leq C}_{> O_C} \simeq \mathcal{T} \simeq \mathcal{T}^{\leq L} \simeq \widehat{\mathcal{S}}(H).$$

Recall that H has parabolic characteristic p and therefore $\widehat{\mathcal{S}}(H)$ and $\mathcal{D}(H)$ are H-homotopy equivalent, by Proposition 2.12 (iii)–(iv). The proof in [29] of this homotopy equivalence can actually be seen to yield equivariance under any automorphism of H, so that here we have equivariance under $N_G(T)/O_C$, or under $N_G(T)$ with O_C acting trivially. □

4 Some modular representation theory

On the group algebra of a group of parabolic characteristic p

Let G be a finite group and let k be a field of characteristic p, which is a splitting field for all the subgroups of G. Denote by kG the group algebra of G over the field of coefficients k. We are interested in the blocks of kG and in particular their defect groups. For details on the background material used in this section the interested reader is referred to [1], or for a more succinct exposition to [7].

Although not all the results from this section are needed in the later proofs, the information given here provides valuable insight into the properties of the group algebras of those classes of groups considered in the next two sections.

4.1. If a finite group G has characteristic p, then $C_G(O_p(G)) \leq O_p(G)$ and kG has only one block [7, Proposition 6.2.2].

It is now easy to prove that:

Lemma 4.2. *Let G be a finite group of local characteristic p. Then every nonprincipal block of* kG *has defect zero.*

Proof. If B is a block of kG with a nontrivial defect group D, then by Brauer's First Main Theorem there is a corresponding block b of k$N_G(D)$. Since G has local characteristic p, the subgroup $N_G(D)$ has characteristic p and thus the only block is the principal block, by 4.1. Then Brauer's Third Main Theorem implies that B is the principal block of kG. □

Lemma 4.3. *Assume that a finite group G has parabolic characteristic p. If D is a defect group of a nonprincipal block of* kG*, then D is purely noncentral.*

Proof. Let B be a block of kG with a nontrivial defect group D, and let b denote its Brauer correspondent in k$N_G(D)$. If D contains a p-central element, then according to 2.11 (iv), $N_G(D)$ has characteristic p. Thus b is the principal block of k$N_G(D)$, and B is the principal block of kG. □

4.4. If G has parabolic characteristic p, the only distinguished p-radical subgroup which is a defect group of $\mathrm{k}G$ is the Sylow p-subgroup of G, and the only block with this defect group is the principal block of $\mathrm{k}G$.

Next, we relate the defect groups of blocks of $\mathrm{k}G$ to the defect groups of blocks of the group algebra $\mathrm{k}C_G(t)$ of the centralizer of an element t of order p.

Lemma 4.5. *Let G be a finite group and let D be a defect group of a block of* $\mathrm{k}G$. *If $T \leq Z(D)$, then D is a defect group of a block of* $\mathrm{k}C_G(T)$.

Proof. Let C denote $TC_G(T) = C_G(T)$. Since $T \leq Z(D)$ it follows that

$$DC_G(D) \leq N_G(D) \cap C \leq N_G(D).$$

By Brauer's Extended First Main Theorem, $\mathrm{k}N_G(D)$ has a block B with defect group D, and $\mathrm{k}DC_G(D)$ has a block b with defect group D, with $b^{N_G(D)} = B$.

The Brauer correspondents $b^{N_C(D)}$ and $b^{N_G(D)}$ are defined, and $b^{N_G(D)} = (b^{N_C(D)})^{N_G(D)}$. There is a defect group D' of $b^{N_C(D)}$ containing the defect group D of b, and up to conjugacy D' is contained in the defect group D of $b^{N_G(D)}$; see [1, Lemma 14.1 (1)]. Thus $D' = D$ and so $\mathrm{k}N_C(D)$ has a block $b^{N_C(D)}$ with defect group D. Another application of Brauer's First Main Theorem to $N_C(D) \leq C$ gives that $\mathrm{k}C$ has a block b^C with defect group D. □

4.6. If t is an element of order p and $T := \langle t \rangle$, the index of $C_G(t)$ in $N_G(T)$ is relatively prime to p. Since $DC_G(D) \leq C_G(t) \trianglelefteq N_G(T)$, an application of [1, Theorem 15.1 (2), (4), (5)] tells us that the defect groups of blocks of $\mathrm{k}N_G(T)$ are the same as the defect groups of the corresponding blocks of $\mathrm{k}C_G(t)$.

Lemma 4.5 does not give any result regarding a defect group for a block of $\mathrm{k}G$, when a defect group for a block of $\mathrm{k}C_G(t)$ is given. However, the following consequence of [33, Theorem 5.5.21] is useful.

4.7. Let t be an element of order p in G and set $C = C_G(t)$. Let b be a block of $\mathrm{k}C$ with defect group Q. The group $T = \langle t \rangle$ is normal in C, and since Q is a p-radical subgroup of C, it follows that $T \trianglelefteq Q$ and $C_G(Q) \leq C$. Thus b^G is defined. Then for a suitable defect group D of b^G, $Z(D) \leq Z(Q) \leq Q \leq D$ and $Q = D \cap C$.

We record the following direct consequence of the above discussion:

Proposition 4.8. *Let G be a finite group of parabolic characteristic p, and let D be a defect group of a nonprincipal block of* $\mathrm{k}G$. *For any $t \in Z(D)$, the subgroup $O_p(C_G(t))$ is purely noncentral.*

Proof. The subgroup D is purely noncentral by Lemma 4.3, and according to Lemma 4.5 the subgroup D is also a defect group of a block of $\mathrm{k}C_G(t)$. Thus we have $O_p(C_G(t)) \le D$. □

The group ring of $\mathbf{O_C.H.K}$

Assume that C is a finite group of the form $C = O_C.H.K$, with $O_C = O_p(C)$. Set $\widetilde{H} = C/O_C = H.K$. We also assume that $K = \widetilde{H}/H$ is either a p-group or a p'-group, and that H is either a group of Lie type in characteristic p or a group of local characteristic p. These cases will be significant to the study of those groups in which we are interested.

Lemma 4.9. *Let $C = O_C.H.K$ with $O_C = O_p(C)$. Assume that K is a p'-group and that H has local characteristic p. The defect groups of the blocks of* $\mathrm{k}C$ *are either equal to O_C or are Sylow p-subgroups of C.*

Proof. Let B be a block of $\mathrm{k}C$ with defect group D. Since O_C is a normal p-subgroup of C, $O_C \le D$, and there exists a block $\widetilde{b}$ of $\widetilde{H}$ dominated by B (or contained in B) with a defect group equal to D/O_C [34, Theorem 9.9 (b)]. Next, there exists a block b of $\mathrm{k}H$ which is covered by the block $\widetilde{b}$. If D' is a defect group of b, then there exists a defect group $\widetilde{D}$ of $\widetilde{b}$ such that $D' = \widetilde{D} \cap H$ [1, Theorem 15.1 (2)]. Since K is a p'-group, we must have $D' = \widetilde{D}$, which up to conjugacy equals D/O_C. We have assumed that H has local characteristic p, and all of the nonprincipal blocks of $\mathrm{k}H$ have defect zero. So either $D = O_C$ or $D/O_C \in \mathrm{Syl}_p(H)$ and $D \in \mathrm{Syl}_p(C)$. □

Lemma 4.10. *Let $C = O_C.H.K$ with $O_C = O_p(C)$. Assume that K is a p-group and that H is a finite simple group of Lie type in characteristic p. Then* $\mathrm{k}C$ *has at most two blocks, and if a nonprincipal block exists, it has defect group of the form $D = O_C.K$.*

Proof. Note that $\mathrm{k}H$ has exactly two blocks, the principal block b_0 and a block b_1 of defect zero (containing the Steinberg module). These are both invariant under the action of $\widetilde{H}$. Since K is a p-group, each block of $\mathrm{k}H$ is covered by a unique block of $\widetilde{H}$ (see [34, Corollary 9.6]). So $\mathrm{k}\widetilde{H}$ has exactly two blocks, the principal block $\widetilde{b}_0$ (covering b_0) and one nonprincipal block $\widetilde{b}_1$ (covering b_1). The defect group $\widetilde{D}$ of the nonprincipal block $\widetilde{b}_1$ satisfies $\widetilde{D} \cap H = 1$ and has order equal to the order of K (see [1, Theorem 15.1 (2), (4)]). Thus $\widetilde{D}$ is isomorphic to K. (This shows $\widetilde{H} = H : K$ is a split extension.) Finally, since O_C is a normal p-subgroup of C, each block B of $\mathrm{k}C$ dominates some block of $\mathrm{k}\widetilde{H}$ and each block of $\mathrm{k}\widetilde{H}$ is dominated by exactly one block of $\mathrm{k}C$. Thus either $\mathrm{k}C$ has one block only (dominating both blocks of $\mathrm{k}\widetilde{H}$), or $\mathrm{k}C$ has two blocks, the principal block

B_0 and one nonprincipal block B_1 with defect group D. We have $O_C \leq D$ and $D/O_C = \widetilde{D} \simeq K$. □

Lemma 4.11. *Let $C = O_C.H.K$ with $O_C = O_p(C)$. Assume K is a p-group and that H is a finite simple group of Lie type in characteristic p. Denote $L = O_C.H$, and assume that L is the generalized Fitting subgroup of C. Then* kC *has exactly two blocks.*

Proof. Note that C has one component (a quasisimple group) E with center $Z(E)$ a p-group and $E/Z(E) = H$. Elements of the component commute with elements of O_C (see [4, 31.6 (2)]), so that $C/C_C(O_C)$ is a p-group. This implies that there is a bijection between the blocks of kC and the blocks of k$\widetilde{H}$ (see [34, Theorem 9.10]). □

5 The vertices of the indecomposable summands of the reduced Lefschetz module

In this section we determine the vertices of the indecomposable summands of the reduced Lefschetz module associated to the complex of p-centric and p-radical subgroups, denoted by $\mathcal{D}(G)$, in a finite group of parabolic characteristic p which also has certain p-local properties. We start by reviewing a few basic facts on Lefschetz modules.

The Green ring (or representation ring) consists of formal differences of isomorphism classes of finitely generated kG-modules. The ring structure is given by direct sums and k-tensor products. By the Krull–Schmidt Theorem, the Green ring is a free abelian group with basis the classes of indecomposable modules.

The Grothendieck ring (or character ring) has relations given by extensions. It is the quotient of the Green ring by the ideal generated by elements of the form $M_2 - M_1 - M_3$ where $0 \to M_1 \to M_2 \to M_3 \to 0$ is a short exact sequence of kG-modules. The Grothendieck ring is a free abelian group with basis the classes of simple modules since each element can be decomposed into sums of simple factors for a composition series.

When a group G acts admissibly on a simplicial complex Δ, we can construct the *Lefschetz module* by taking the alternating sum of the k-vector spaces spanned by the chain groups. To obtain the *reduced Lefschetz module*, subtract the trivial one dimensional representation. The n^{th} chain group $C_n(\Delta; \mathrm{k})$ is a permutation module and a k-module with oriented simplices as generators. Thus the reduced Lefschetz module is a virtual module, an element of the Green ring:

$$\widetilde{L}_G(\Delta; \mathrm{k}) = \sum_{\sigma \in \Delta/G} (-1)^{|\sigma|} \mathrm{Ind}_{G_\sigma}^G(\mathrm{k}) - \mathrm{k}.$$

The reduced Lefschetz module associated to a subgroup complex Δ is not in general projective. A theorem of Thévenaz [49, Theorem 2.1] says that the set of vertices for its indecomposable summands is a subset of the collection of p-subgroups $Q \leq G$ for which the fixed point set Δ^Q is not mod-p acyclic.

Information about fixed point sets leads to details about the vertices of indecomposable summands of this virtual module. The following theorem is due to Robinson [37, in proof of Corollary 3.2]; also see [41, Lemma 1].

Theorem 5.1 (Robinson). *The number of indecomposable summands of* $\widetilde{L}_G(\Delta; \mathrm{k})$ *with vertex* Q *equals the number of indecomposable summands of* $\widetilde{L}_{N_G(Q)}(\Delta^Q; \mathrm{k})$ *with vertex* Q.

In the next three results, notably in Theorem 5.4, we combine Robinson's theorem with our working assumptions on the p-local structure of the group, in order to derive explicit information on the vertices of the reduced Lefschetz module $\widetilde{L}_G(\mathcal{D}(G); \mathrm{k})$.

Lemma 5.2. *Assume* G *is a finite group of parabolic characteristic* p. *Suppose that* t *is an element of order* p *in* G *such that* $O_p(C_G(t))$ *contains* p*-central elements. Then no vertex of the reduced Lefschetz module* $\widetilde{L}_G(\mathcal{D}(G); \mathrm{k})$ *contains a conjugate of* t.

Proof. Recall that according to Theorem 3.3, the fixed point set $\mathcal{D}(G)^t$ is contractible. Hence $\mathcal{D}(G)^t$ is mod-p acyclic, and an application of Smith theory (see [12, Theorem VII.10.5 (b)]) gives that $\mathcal{D}(G)^Q$ is mod-p acyclic for any p-subgroup Q containing t. Therefore $\widetilde{L}_{N_G(Q)}(\mathcal{D}(G)^Q; \mathrm{k}) = 0$. Then Theorem 5.1 implies that the reduced Lefschetz module $\widetilde{L}_G(\mathcal{D}(G); \mathrm{k})$ has no indecomposable summand with vertex Q containing a conjugate of t. □

Proposition 5.3. *Let* G *be a finite group of parabolic characteristic* p*, and let* Q *be a vertex of an indecomposable summand of the reduced Lefschetz module* $\widetilde{L}_G(\mathcal{D}(G); \mathrm{k})$. *Assume that* t *is an element of order* p *in the center* $Z(Q)$ *such that the conditions* (N1)*,* (N2) *and* (N3) *from Theorem* 3.6 *hold for* $C = C_G(t)$. *Then* Q *is a vertex of an indecomposable summand of* $\widetilde{L}_C(\mathcal{D}(H); \mathrm{k})$.

Proof. By Robinson's Theorem 5.1, Q is a vertex of a summand of the Lefschetz module $\widetilde{L}_{N_G(Q)}(\mathcal{D}(G)^Q; \mathrm{k})$. Consider the restriction of this Lefschetz module to the subgroup $C \cap N_G(Q)$, obtaining the Lefschetz module $\widetilde{L}_{N_C(Q)}(\mathcal{D}(G)^Q; \mathrm{k})$. There exists an indecomposable summand of $\widetilde{L}_{N_C(Q)}(\mathcal{D}(G)^Q; \mathrm{k})$ with vertex Q ([1, Lemma 9.5]). As $\mathcal{D}(G)^Q = (\mathcal{D}(G)^t)^Q$, another application of Theorem 5.1 yields a summand of $\widetilde{L}_C(\mathcal{D}(G)^t; \mathrm{k})$. Theorem 3.6 says that $\mathcal{D}(G)^t$ is equivariantly homotopy equivalent to $\mathcal{D}(H)$, therefore $\widetilde{L}_C(\mathcal{D}(G)^t;\mathrm{k})$ and $\widetilde{L}_C(\mathcal{D}(H);\mathrm{k})$ are isomorphic virtual $\mathrm{k}C$-modules. □

Note that t can be replaced in the above proof by any subgroup $T \leq Z(Q)$, where $C = TC_G(T)$. Also C can be replaced by $N_G(T)$.

Theorem 5.4. *Let G be a finite group of parabolic characteristic p. Let T be a p-subgroup of G and set $C = TC_G(T)$. Assume that the following conditions hold:*

(i) *$C = O_C.H.K$ where $O_C = O_p(C)$ and $L = O_C.H$ is the generalized Fitting subgroup of C,*

(ii) *the group $H = L/O_C$ is a finite simple group of Lie type in characteristic p,*

(iii) *there exists a triple of Sylow p-subgroups (S_L, S_C, S) with $Z(S) \cap S_L \neq 1$.*

Then T is a vertex of an indecomposable summand of the reduced Lefschetz module $\widetilde{L}_G(\mathcal{D}(G);\mathrm{k})$ if and only if:

(a) *$T = O_C$ and T is purely noncentral,*

(b) *$K = C/L$ is a p'-group,*

(c) *the index of C in $N_G(T)$ is relatively prime to p.*

Under these conditions, there will exist a unique indecomposable summand of $\widetilde{L}_G(\mathcal{D}(G);\mathrm{k})$ with vertex T, which will lie in a block of $\mathrm{k}G$ with defect group T.

Proof. Theorem 5.1 tells us that T is a vertex of a summand of $\widetilde{L}_G(\mathcal{D}(G);\mathrm{k})$ if and only if T is a vertex of a summand of $\widetilde{L}_{N_G(T)}(\mathcal{D}(G)^T;\mathrm{k})$. By Lemma 5.2, if O_C contains p-central elements, T is not a vertex; thus O_C is purely noncentral.

The generalized Fitting subgroup L is characteristic in C and so is normal in $N_G(T)$. Denote the quotient group by $K' = N_G(T)/L$. Note that K' will be a p'-group if and only if K is a p'-group and the index of C in $N_G(T)$ is relatively prime to p. We have $N_G(T) = O_C.H.K'$. The group C has one component E with center $Z(E)$ a p-group and $E/Z(E) = H$. Elements of O_C commute with elements of E.

Theorem 3.6 tells us that the fixed point set $\mathcal{D}(G)^T$ is $N_G(T)$-homotopy equivalent to the Tits building Δ for the Lie group H. Thus their reduced Lefschetz modules are isomorphic virtual $\mathrm{k}N_G(T)$-modules. Set $M = \widetilde{L}_{N_G(T)}(\Delta;\mathrm{k})$. The subgroup O_C acts trivially on M (by definition of the action of $N_G(T)$ on Δ) and so M is the inflation of a module $\widetilde{M}$ over the group $\widetilde{H}' = N_G(T)/O_C = H.K'$. The restriction of $\widetilde{M}$ to the Lie group H is up to a sign the Steinberg module St_H, which is irreducible. This implies that M and $\widetilde{M}$ are also irreducible modules.

The restriction M' of M to $L = O_C.H$ is also irreducible, being the inflation of $\pm\mathrm{St}_H$. The module $\widetilde{M}$ is referred to as an extended Steinberg module; see [43] and [32].

Next, kH has only two blocks, the principal block b_0 and a defect zero block b_1 containing the Steinberg module. Then kL has two blocks also, the principal block B_0 and a block B_1 which contains the module M', using that E centralizes O_C and [34, Theorem 9.10]. The block B_1 dominates (or contains) the block b_1. The normal p-subgroup O_C is contained in the vertex Q_1 of the irreducible module M' (see [33, Theorem 4.7.8]). The defect group D_1 of the block B_1 contains O_C (and also Q_1), and the quotient D_1/O_C equals 1, the defect group of the block b_1 [34, Theorem 9.9 (b)]. Therefore $Q_1 = D_1 = O_C$.

Let B be the block of k$N_G(T)$ which contains the module M. Note that B covers the block B_1. The defect group D of B satisfies $D \cap L = D_1 = O_C$ by [1, Theorem 15.1 (2)]. If Q is the vertex of M, then $O_C \leq Q$ and $O_C \leq Q \cap L \leq D \cap L = O_C$. Also the quotient $S' = Q/(Q \cap L) = Q/O_C$ is a Sylow p-subgroup of $K' = N_G(T)/L$ by [1, Lemma 9.8]. Since $Q = O_C.S'$ is a subgroup of the p-group D, the quotient map $N_G(T) \to K'$ applied to D has kernel O_C and image S'. Therefore $Q = D = O_C.S'$.

Since $T \leq O_C$, we have $T = Q$ if and only if $T = O_C$ and $S' = 1$. In this situation, K' is a p'-group. Under these conditions, there will be a unique summand of the reduced Lefschetz module $\widetilde{L}_G(\mathcal{D}(G);\mathrm{k})$ with vertex T, and this summand will lie in a block with defect group T because of the relationship between the Brauer correspondence and the Green correspondence. □

5.5. Work of Mühlherr and Schmid [32, Theorem C and Lemma 7] concerning the action of an automorphism of H on the associated building Δ can be applied to $H \trianglelefteq \widetilde{H}'$. If the value of the rational character χ of the extended Steinberg module $\widetilde{M}$ evaluated on an element $g \in \widetilde{H}'$ is nonzero, then the fixed point set Δ^g is a Moufang complex. Also, $\chi(g) \neq 0$ if and only if the p-part of g is conjugate to an element of the defect group S' of the block of k$\widetilde{H}'$ containing $\widetilde{M}$. If $g \in S'$, then Δ^g is a building for the group $O^{p'}(C_H(g))$. Since O_C acts trivially on Δ, we can phrase this result in terms of the character of the inflated extended Steinberg module M, with fixed point set Δ^g, a building when $g \in D = O_C.S'$.

It follows that $\mathcal{D}(G)^D$ is homotopy equivalent to the building $\Delta^D = \Delta^{S'}$ for $O_C.O^{p'}(C_H(S'))$. As noted in the proof of Lemma 4.10 the extension $H : S'$ is split and S' is a subgroup of $\widetilde{H}'$. This homotopy is equivariant with respect to $N_G(T) \cap N_G(D)$, which contains the group $DC_G(D)$. To obtain a homotopy equivariant with respect to $N_G(D)$, we require more information about $DC_G(D)$ so that we can apply Theorem 3.6 to $DC_G(D)$.

Proposition 5.6. *Assume that H is a normal subgroup of $N_G(T)$ with complement a group D containing $O_C = O_p(TC_G(T))$. Denote $S' = D/O_C$, and assume that $D = O_C : S'$ is a split extension. Then $DC_G(D) = D.C_H(S')$.*

Proof. Note that $N_G(T) = H : D = (O_C \times H) : S'$. As $T \le O_C \le D$, we have $C_G(D) \le C_G(T) \le N_G(T)$, so that $C_G(D) = C_N(D)$, denoting $N = N_G(T)$. An element of $C_N(D)$ can be written uniquely as a product xy with $x \in H$ and $y \in D$, such that for all $z \in D$ we have $xy = z(xy)z^{-1} = (zxz^{-1})(zyz^{-1})$. As $zxz^{-1} \in H$ and $zyz^{-1} \in D$, we must have $zxz^{-1} = x$ and so $x \in C_H(D)$. Since elements of H commute with elements of O_C, $C_H(D) = C_H(S')$. It now follows that $DC_G(D) = DC_H(D) = D.C_H(S')$. □

5.7. In the above situation, we can write $DC_G(D)$ in the form $D.H_D.K_D$, with $H_D = O^{p'}(C_H(S'))$ and $K_D = C_H(S')/H_D$. Usually, when H is a finite simple group of Lie type, the extension $H : S'$ is split and H_D will also be a finite simple group of Lie type. Then H_D will be a component of $DC_G(D)$ and will be normal in $N_G(D)$. As noted in 5.5, under the conditions of Theorem 5.4, $\mathcal{D}(G)^D$ will be homotopy equivalent to a building and thus not contractible, so D must be purely noncentral. Finally, K_D is a p'-group, and the conditions for Theorem 3.6 will be satisfied.

Proposition 5.8. *Assume $N = N_G(T) = O_C.(H : S')$ and assume that $C_H(S')$ is a simple group. Also assume that $D = O_C.S'$ is purely noncentral in G. Then $DC_G(D) = D.C_H(S')$.*

Proof. Note that $DC_G(D) = DC_N(D) \trianglelefteq N_N(D)$, and under the quotient map $N \to H : S'$, the image of $N_N(D)$ equals $N_{H:S'}(S') = S'N_H(S')$. Next, take the quotient modulo S' to obtain $N_H(S')$. The image of $DC_G(D)$ under the composition of these two quotient maps is a normal subgroup of the simple group $C_H(S')$. It is impossible for $C_G(D) \le D$ to be purely noncentral, since if we have $D \le S \in \mathrm{Syl}_p(G)$, then $Z(S) \le C_G(D)$. Therefore $DC_G(D) = D.C_H(S')$. □

Proposition 5.9. *Let G be a finite group of parabolic characteristic p. Let T be a p-subgroup of G and set $C = TC_G(T)$. Assume that:*

(i) *Conditions* (N1), (N2) *and* (N3) *hold for C. In particular $C = O_C.H.K$ with $O_C := O_p(C)$.*

(ii) *The subgroup $L = O_C.H$ is the generalized Fitting subgroup of C.*

Then $O_C\,C_G(O_C)$ is of the form $O_C.H.\overline{K}$ for some subgroup $\overline{K} \le K$, and conditions (N1), (N2) *and* (N3) *are satisfied for $O_C\,C_G(O_C)$.*

Proof. The group C has one component E with center $Z(E)$ a p-group and $E/Z(E) = H$. We have $E \leq C_G(O_C)$, and thus $L = O_C.H \leq O_C\, C_G(O_C)$. In order to prove that (N2) holds for $O_C\, C_G(O_C)$, we have to show that L is a normal subgroup of $N := N_G(O_C)$. Observe that it suffices to show that E is the only component of N, since in this case L is the product of two normal subgroups of N, namely O_C and E. Let $E(N)$ denote the layer of N, the central product of all the components of N. Since $E(N)$ centralizes the Fitting subgroup of N, which contains O_C, it follows that $E(N) \leq O_C\, C_G(O_C) \leq C$. Thus $E(N) = E$. Define $\overline{K}$ to be the image of $O_C\, C_G(O_C)$ under the quotient map $C \to C/L = K$. Thus conditions (N1) and (N2) are satisfied for $O_C\, C_G(O_C) = O_C.H.\overline{K}$. Condition (N3) also holds since any Sylow p-subgroup S_{O_C} of $O_C\, C_G(O_C)$ can be extended to a Sylow p-subgroup S_C of $TC_G(T)$, and we have $S_L = S_C \cap L$ is a Sylow p-subgroup of L. Then using 3.7 there exists a Sylow p-subgroup S of G extending S_C such that $Z(S) \cap S_L \neq 1$. □

The reduced Lefschetz module $\widetilde{L}_G(\mathcal{C}(G); \mathrm{k})$ was studied by Sawabe [41] in the more general context when G is any finite group and $\mathcal{C}(G)$ is a collection of p-subgroups of G which is closed under p-overgroups. Sawabe applied his results to the collection $\mathcal{C}(G)$ of p-centric subgroups of G, which is equivariantly homotopy equivalent to $\mathcal{D}(G)$. The collection $\widetilde{\mathcal{S}}(G)$ also satisfies this condition by 2.11 (i), and for G of parabolic characteristic p is equivariantly homotopy equivalent to $\mathcal{D}(G)$ by Theorem 2.12 (iii–iv). We give below two of Sawabe's results.

Proposition 5.10 (Sawabe). *Assume that G is a finite group and let $\mathcal{C}(G)$ denote a collection of p-subgroups of G that is closed under taking p-overgroups. Take V in $\mathcal{B}(G) \setminus \mathcal{C}(G)$ of maximal order. Then the following hold:*

(i) [41, Proposition 4] *For any p-subgroup Q of order greater than that of V, the fixed point set $\mathcal{C}(G)^Q$ is $N_G(Q)$-contractible. In particular Q is not a vertex of $\widetilde{L}_G(\mathcal{C}(G); \mathrm{k})$.*

(ii) [42] *The group V is a vertex of an indecomposable summand of $\widetilde{L}_G(\mathcal{C}(G); \mathrm{k})$ if and only if $\widetilde{L}_{N_G(V)}(\mathcal{C}(G)^V; \mathrm{k}) \neq 0$.*

We end this section with a result which asserts that under our assumptions the defect groups of the nonprincipal blocks of kG are vertices for the nonprojective indecomposable summands of the reduced Lefschetz module associated to the collection of p-centric and p-radical subgroups of G.

Theorem 5.11. *Let G be a finite group of parabolic characteristic p. Let T be a p-subgroup of G and set $C = TC_G(T)$. Assume the following hold:*

(i) *Conditions* (N1), (N2) *and* (N3) *hold for C. Thus we have $C = O_C.H.K$ and $N_G(T) = O_C.H.K'$.*

(ii) *H is a finite group of Lie type in characteristic p.*

(iii) *The defect group D of the block B for $\mathrm{k}N_G(T)$ which contains the inflated extended Steinberg module of H is a noncentric p-radical subgroup of G of maximal order.*

Then D is a vertex of an indecomposable summand of the reduced Lefschetz module $\widetilde{L}_G(\mathcal{D}(G);\mathrm{k})$, and if this summand lies in a nonprincipal block, the defect group of this block equals D.

Proof. Theorem 3.6 tells us that $\mathcal{D}(G)^T$ is equivariantly homotopy equivalent to the building Δ for the Lie group H. Thus $\mathcal{D}(G)^D$ is homotopy equivalent to Δ^D, a building by [32], see also 5.5. This latter homotopy is only equivariant with respect to $N_G(T) \cap N_G(D)$, but all we need is an ordinary homotopy to conclude that the reduced Lefschetz module $\widetilde{L}_{N_G(D)}(\mathcal{D}(G)^D;\mathrm{k})$ is nonzero. Thus there is at least one indecomposable summand of $\widetilde{L}_G(\mathcal{D}(G);\mathrm{k})$ with vertex D by Sawabe's Proposition 5.10 (ii). The defect group D_0 of the block containing this summand must contain D up to G-conjugacy, and if this block is not the principal block, the defect group D_0 is purely noncentral by Lemma 4.3. If D is a proper subgroup of D_0, then D_0 is p-centric (note D_0 is p-radical, and D was assumed to be a maximal noncentric p-radical subgroup). But for any Sylow p-subgroup S of G with $D_0 \leq S$, we have $Z(S) \leq C_G(D_0)$ and thus $Z(S) \leq D_0$ since $Z(D_0)$ is the unique Sylow subgroup of $C_G(D_0)$. This contradicts D_0 being purely noncentral. Thus $D = D_0$. □

With additional information, including $DC_G(D) = D.C_H(S')$ as in Propositions 5.6 and 5.8, we would be able to obtain a homotopy equivalence, equivariant with respect to $N_G(D)$, so that $\widetilde{L}_{N_G(D)}(\mathcal{D}(G)^D;\mathrm{k})$ would be the irreducible inflated extended Steinberg module for the Lie group $O^{p'}(C_H(S'))$. Then there would be exactly one indecomposable summand of $\widetilde{L}_G(\mathcal{D}(G);\mathrm{k})$ with vertex D.

6 Applications: Sporadic simple groups of parabolic characteristic p

In this final section we consider a few examples. We discuss how our results apply to the sporadic simple groups of parabolic characteristic p. We start with even characteristic, after which we discuss examples in characteristic 3.

Sporadic simple groups of parabolic characteristic 2

Among the 26 sporadic groups nine have local characteristic 2; these are M_{11}, M_{22}, M_{23}, M_{24}, J_1, J_3, J_4, Co_2 and Th. The twelve groups given in Table 1 have parabolic characteristic 2 and contain noncentral involutions whose centralizers have component type. The remaining five groups have 2-central involutions with centralizer of component type; these are Co_3, Fi_{23}, McL, O′N and Ly.

Notation 6.1. In Table 1 below we gather information regarding the sporadic simple groups of parabolic characteristic 2. The notation is as follows. We let t be a noncentral involution in G. In the second column, the centralizer $C_G(t)$ is given. It is of the form $C_G(t) = O_C.H.K$ (cf. Notation 3.4) with $O_C = O_2(C_G(t))$, where H denotes either the unique component of $C_G(t)$, or a central quotient, or an extension of the component. Also H is a group of Lie type (described in the Atlas [13] notation) for the first nine groups in the table. For the remaining three groups H is a simple group of parabolic characteristic 2. We are studying $\mathcal{D}_2(G)$, the collection of 2-centric and 2-radical subgroups, or equivalently the distinguished 2-radical subgroups, see Definition 2.10. In the fourth column we describe the fixed point set $\mathcal{D}_2(G)^t$. This is usually contractible (homotopy equivalent to a point), or homotopy equivalent to a building, for which we use a standard notation [38, p. 213] also used for Coxeter groups. For the last three groups, $\mathcal{D}_2(G)^t$ is not homotopy equivalent to a building, but it is homotopy equivalent to $\mathcal{D}_2(H)$ (as proved in Theorem 3.6). Columns 5, 6 and 7 contain information on a vertex V of a nonprojective indecomposable summand of $\widetilde{L}_{C_G(t)}(\mathcal{D}_2(G)^t; \mathbb{F}_2)$. Finally, H_V denotes the Lie type component of $VC_G(V)$ and the column $\mathcal{D}_2(G)^V$ gives the type of the corresponding building associated to this fixed point set under the action of V.

The main sources of information are the Atlas [13], Chapter 5 in the third volume of the classification of the finite simple groups [16] and Landrock's paper [25] on the nonprincipal 2-blocks of the sporadic simple groups. Additional references are provided in the furthest right column of the table.

The following proposition is a direct consequence of our results from Theorems 3.6 and 5.4 combined with the information from Table 1.

Proposition 6.2. *Let G be a sporadic simple group of parabolic characteristic* 2 *and let $\widetilde{L}_G$ denote the reduced Lefschetz module for the complex of* 2*-radical and* 2*-centric subgroups in G, over $\mathbb{F}_2$.*

(i) *Let G be one of the groups M_{12}, J_2,* HS, Ru, Suz, Fi_{22}, He, Co_1 *or* BM. *The virtual module $\widetilde{L}_G$ has precisely one indecomposable nonprojective summand M, and this summand lies in a nonprincipal block of $\mathbb{F}_2G$ and has vertex V equal to the defect group of this block. The module M is the Green correspondent of the inflated extended Steinberg module for the building given in column $\mathcal{D}_2(G)^V$ of Table* 1.

(ii) *If G is one of the groups* Fi_{24}', HN *or* M, *then the reduced Lefschetz module* $\widetilde{L}_G$ *has at least one indecomposable nonprojective summand, which lies in a nonprincipal block of* $\mathbb{F}_2G$ *and has vertex* V *equal to the defect group of this block.*

6.3. In every case, the vertex V referred to in the proposition is in the unique conjugacy class of maximal noncentric 2-radical subgroups of G, as suggested by the work of Sawabe; see Proposition 5.10 (i). For example, when G is either HN or M, there are five conjugacy classes of noncentric 2-radical subgroups, with the semidihedral group SD_{16} maximal; see [58, Table VI] for HN and [60, Table 2] for M. In both of these cases[2], there are inclusions $2A \leq 2A^2 \leq D_8 \leq \mathrm{SD}_{16}$ and also $2A \leq Q_8 \leq \mathrm{SD}_{16}$. For $G = \mathrm{BM}$, there are four conjugacy classes of noncentric 2-radical subgroups, with inclusions $2A \leq 2A_2C_1 \leq D_8$ and $2C^3 \leq D_8$; see [60, Table 1]. For $G = \mathrm{Fi}_{24}'$, there are four classes of noncentric 2-radical subgroups, including two classes of elementary abelian subgroups $2A^2$ of rank 2, with inclusions $2A \leq 2A_I^2 \leq D_8$ and $2A \leq 2A_{II}^2 \leq D_8$; see [24, Table 2]. Regarding the remaining groups from Table 1: J_2, Ru and Fi_{22} have one conjugacy class of noncentric 2-radical subgroups each; M_{12} and HS have two classes of such subgroups each and Co_1, Suz and He have three classes each. It is easy to check that the poset of noncentric 2-radical subgroups has up to G-conjugacy a unique maximal member in each case. Although not all of the details are given, the relevant references for each sporadic simple group can be found in Chapter 7 of the book by Benson and Smith [9].

6.4. The three cases in Table 1 in which the fixed point set is contractible are verified using Theorem 3.3 and Lemma 5.2 since O_C contains central elements. The following information is available in the Atlas [13]. For $t = 2C$ in $G = \mathrm{Fi}_{22}$, there is a purely central elementary abelian 2-group $2B^4$ of rank 4 in the subgroup $2^5 = 2A_6B_{15}C_{10}$ of 2^{5+8}. For $t = 2C$ in $G = \mathrm{Co}_1$, the elementary abelian $2^{11} = 2A_{759}C_{1288}$ contains central elements of class $2A$. The normalizer is $N(2^{11}) = 2^{11} : M_{24}$, and the subgroup $M_{12}.2 \leq M_{24}$ has index 1288. And for $t = 2D$ in $G = \mathrm{BM}$, the elementary abelian subgroup 2^9 of $2^9.2^{16}$ contains a purely central $2B^8$ of rank 8. The normalizer is $N(2B^8) = 2^9.2^{16}.\mathrm{Sp}_8(2)$. The lift of the group 2^9 in $2.\mathrm{BM} \leq M$ is a group 2^{10} referred to as an "ark" in the work of Meierfrankenfeld and Shpectorov [30, 31]. The notation $O_8^+(2)$ in the centralizer $C(2D) = 2^9.2^{16}.O_8^+(2).2$ is the Atlas notation for the simple group, also denoted by $\Omega_8^+(2)$, so that

$$O_8^+(2).2 = \mathrm{SO}_8^+(2).$$

2 Recall the Atlas [13] notation described in the Introduction.

6.5. For those cases in Table 1 in which the fixed point set $\mathcal{D}_2(G)^t$ is not contractible, and also for the fixed point sets of the form $\mathcal{D}_2(G)^V$ in every case, we will be able to apply Theorem 3.6 to show that this fixed point set is equivariantly homotopy equivalent to the complex $\mathcal{D}_2(H)$ of 2-centric 2-radical subgroups of the group H associated to $C_G(t)$, or respectively to $\mathcal{D}_2(H_V)$ for the group H_V associated to $VC_G(V)$. So the next step in the proof of Proposition 6.2 is to verify that the hypothesis (N3) for Theorem 3.6 holds for these groups. We first recall this condition. Let T be a p-subgroup of G. If $TC_G(T) = C = O_C.H.K = L.K$, then (N3) asserts that there exists a triple of Sylow p-subgroups $S_L \trianglelefteq S_C \leq S$ of L, C and G respectively, such that $Z(S) \cap S_L \neq 1$; in particular the group S_L is distinguished in G. Note this condition is trivially satisfied when K is a group of odd order, since then $S_L = S_C$. This happens in the following four cases: when G equals J_2, Ru or M, and for $t = 2A$ in $G = \mathrm{Fi}_{22}$. It also occurs when $C = VC_G(V)$ for every group V from the fifth column of Table 1.

We now prove a lemma which yields condition (N3) in five additional cases from Table 1.

Lemma 6.6. *Let G be one of the following groups: M_{12},* HS*,* Suz*,* He*, or* BM*. Let t be a noncentral involution in G (where for $G =$* BM *we assume t is of class $2A$), and let X be the component of the centralizer $C_G(t)$. Let $\overline{X} = X$ except in the case $G =$* HS*, where $\overline{X} = S_6 \leq \mathrm{Aut}(A_6)$. Then the 2-central involutions of G which lie in $C_G(t)$ actually lie in $\overline{X}$.*

Proof. The first three groups (and also J_2 and Ru) are discussed in [28, Lemma 2.3]. For $G =$ HS, this result is Lemma 1.7 of Aschbacher [5, p. 24]. For $G =$ Suz, see the paragraph on p. 456 before Lemma 1 of Yoshiara [57]. For $G = M_{12}$, the centralizer $C_G(t) = 2 \times S_5$ has five conjugacy classes of involutions, fusing to two classes in M_{12}. The involutions in A_5 are squares of elements of order 4, which are 2-central in M_{12} by the Atlas [13]. The other four classes in $C_G(t)$ are noncentral in M_{12}. There exists a purely noncentral elementary abelian 2^2 in M_{12}, so the involutions in $S_5 \setminus A_5$ are noncentral. The 2-central involutions are closed under nontrivial commuting products; the class multiplication coefficient is zero [9, p. 107, 162]. Thus the diagonal involutions in $2 \times A_5$ are noncentral in M_{12}. Next, for $G =$ He, the lemma follows from [22, p. 277–278]. Finally, for $t = 2A$ in $G =$ BM, the result follows from Segev [44, 2.5.2 and 3.18]. □

Note that if $L = C_G(O_C)$, then whenever $S_L \leq S$, we have $Z(S) \leq L$. This argument shows that condition (N3) is satisfied for G equal to Suz, He, also for $t = 2A$ in Fi_{22}, $t = 2B$ in Co_1 and for $t = 2C$ in BM. Condition (N3) holds for

$G = \mathrm{HN}$ by a description of the groups S_L and S_C given in [21, (L) on p. 128], where it is shown that $Z(S_C) = 2^2 \leq S_L$.

Finally, let $G = \mathrm{Fi}_{24}'$, with $C_G(t) = 2{}^{\cdot}\mathrm{Fi}_{22} : 2$ and $L = 2{}^{\cdot}\mathrm{Fi}_{22}$. Let S denote a Sylow 2-subgroup of Fi_{22}, with $S_L = 2{}^{\cdot}S$ and $S_C = 2{}^{\cdot}S : 2$. Let z be an involution in the center $Z(S_C)$ and assume that z is not in S_L. Consider the centralizer $C_G(2^2)$ of the elementary abelian $2^2 = \langle t, z\rangle$, which must contain S_C. But by [54, Table 10], this centralizer is divisible by only 2^{14} (it equals either $2^2 \times O_8^+(2) : 3$ or $2^2 \times 2^6 : U_4(2)$). This contradiction implies that the subgroup $\Omega_1(Z(S_C))$ containing all involutions of the center $Z(S_C)$ must be a subgroup of S_L. This implies that condition (N3) holds.

6.7. In what follows we determine those 2-subgroups which are candidates for the vertices of $\widetilde{L}_G$. Note that Proposition 5.3 above implies that we need only consider those 2-subgroups Q which are vertices of the reduced Lefschetz module $\widetilde{L}_{C_G(t)}(\mathcal{D}_2(G)^t; \mathbb{F}_2)$ for the action of the centralizer $C_G(t)$, with t an involution in $Z(Q)$, on the complex $\mathcal{D}_2(H)$ for H described in the fourth column of Table 1. This yields those groups given in column five, denoted by V.

Let us consider the first nine groups in Table 1. Theorem 5.4 applies to both $C = C_G(t)$ and $C = VC_G(V)$ in every case. Observe that for $G = \mathrm{BM}$, the group $V = 2^2 = 2A_1C_2$ is not a vertex of a summand of the reduced Lefschetz module for G since the centralizer of V has index 2 in the normalizer of V.

In the remaining three cases, Theorem 5.4 does not apply if we let $C = C_G(t)$. The subgroup $C_G(t)$ is of the form $O_C.H.K$ but H is not a group of Lie type. We can, however, apply Theorem 5.4 to the group $C = VC_G(V)$ from the sixth column of the table. Specifically, $V = D_8$ in $G = \mathrm{Fi}_{24}'$ and $V = \mathrm{SD}_{16}$ in $G = \mathrm{HN}$ and $G = M$, obtaining the existence of summands with vertex equal to the dihedral or semidihedral group. For $G = M$ it is possible that $T = 2A$ is a vertex of a summand of the reduced Lefschetz module (if BM has a projective summand in its reduced Lefschetz module). For $G = \mathrm{HN}$, we do not know the vertices for the action of $2{}^{\cdot}\mathrm{HS} : 2$ on the complex $\mathcal{D}_2(\mathrm{HS})$. Finally, we remark that for $G = \mathrm{Fi}_{24}'$ and for $G = \mathrm{HN}$ the group $V = 2^2$ is not a vertex, since the centralizer of V has even index in the normalizer of V.

6.8. Our results and the Benson–Grizzard–Smith Conjecture. The book of Benson–Smith [9] describes a 2-local geometry for each of the 26 sporadic finite simple groups. Based on their homotopy types, these 2-local geometries are divided into three (overlapping) lists. On pages 105–108, they list those sporadic groups for which the 2-local geometry is equivariantly homotopy equivalent to the Brown complex $\mathcal{S}_2(G)$ (see [9, (5.9.2)]):

List $\mathcal{S}$: M_{11}, M_{22}, M_{23}, M_{24}, J_1, J_3, J_4, McL, Co_2, Th, Ly, to the Benson complex[3] $\mathcal{E}_2(G)$ (see [9, (5.10.3)]):

List $\mathcal{E}$: M_{12}, J_2, HS, Suz, Co_3, Ru,

or to the complex $\mathcal{D}_2(G) = \mathcal{B}_2^{\mathrm{cen}}(G)$ (see [9, (5.11.3)]):

List $\mathcal{D}$: M_{12}, J_2, Suz, Co_1, Fi_{22}, Fi_{23}, Fi_{24}', HN, B, M, He, Ru, O′N.

The eleven groups in List $\mathcal{S}$ have projective reduced Lefschetz modules; all but one case (the group Ly) appear in [39] and [47].

The six groups in List $\mathcal{E}$ are studied by Grizzard in his thesis [17]. Grizzard computes reduced Lefschetz characters for these six groups, and also for O′N and He. The characters were already known for three of these eight groups, namely M_{12} (see [10, p. 44]), J_2 (see [9, after Theorem 7.7.1]) and HS (Klaus Lux, unpublished).

The results of this paper can be applied to most of the groups in List $\mathcal{D}$. Four of the groups in the List $\mathcal{E}$ also appear in List $\mathcal{D}$, and it seems that HS should also appear in both lists. Note that although HS does not appear in List $\mathcal{D}$, in [9, p. 193] a quotient map is described yielding a homotopy equivalence of the 2-local geometry of HS with $\mathcal{D}_2$(HS).

The sporadic groups of parabolic characteristic 2 given in Table 1 consist of eleven of the thirteen groups in List $\mathcal{D}$, plus HS. Thus Proposition 6.2 applies to the 2-local geometries for these twelve groups. The remaining two groups in List $\mathcal{D}$ are O′N and Fi_{23}, and in both groups all involutions are 2-central. Then our distinguished 2-radical complex equals the Bouc complex of all nontrivial 2-radical subgroups, which is homotopy equivalent to the Brown complex of all nontrivial 2-subgroups. Thus the results in this paper do not apply to the 2-local geometries for the two groups O′N and Fi_{23}.

In a footnote in [9, p. 164], the following comment appears: “For all 15 sporadic groups G in (5.10.3) and (5.11.3) for which the 2-local geometry is not homotopy equivalent to $\mathcal{S}_2(G)$, it seems that the reduced Lefschetz module involves an indecomposable in a suitable nonprincipal 2-block of G of small positive defect”. Grizzard refers to this as the “Nonprincipal Block Observation” [18, Remark 3.1] and conjectures that each sporadic group affording a nonprojective Lefschetz module will have a nonprojective part in a nonprincipal block. Grizzard verifies this conjecture for the eight groups in his thesis. Grizzard also shows that for seven of his groups, the defect group of this nonprincipal block has order equal to the ratio of the 2-part $|G|_2$ of the order of the group and the 2-part of the reduced Euler characteristic of the 2-local geometry. This was also predicted by Steve Smith; see

[3] Corresponds to a certain collection of elementary abelian 2-groups, see [9, (5.10.1)] for a precise definition.

for example [45]. Grizzard observed that this was not true for O′N, which also differed from the other examples by having a nonprojective part of the reduced Lefschetz module lying in the principal block.

Sawabe approaches this latter idea from the viewpoint of vertices of modules, and proves a theorem concerning a lower bound for the p-part of the reduced Euler characteristic of the complex $\mathcal{D}_p(G)$ (see [41, Proposition 7]) in terms of the order of a noncentric p-radical subgroup V of maximal order. An unpublished result of Sawabe [42] also shows that such a maximal noncentric p-radical subgroup V is a vertex of a summand of the reduced Lefschetz module $\widetilde{L}_G(\mathcal{D}(G); \mathrm{k})$ if and only if $\widetilde{L}_W(\mathcal{S}(W); \mathrm{k}) \neq 0$, for $W = N_G(V)/V$. This result can be applied to many groups, including all of those in Table 1, showing the existence of a summand of the reduced Lefschetz module with vertex V. This result also applies to the sporadic group Fi_{23} and the maximal noncentric 2-radical subgroup $V = D_8$ (see [2] for the radical subgroups of Fi_{23}, and for the normalizer $N_G(V) = D_8 \times \mathrm{Sp}_6(2)$). It follows that the dihedral group D_8 is a vertex of a summand of the reduced Lefschetz module for the 2-local geometry of Fi_{23}. But the work of Sawabe does not give any information on the block in which this summand lies. Also, his results yield existence of nonprojective summands for all the groups in Table 1, but do not supply the uniqueness of the nonprojective summand that we find for the first nine groups. Indeed, it has been shown that there are examples where there are more than one nonprojective summand. Grizzard shows that the reduced Lefschetz module for the 2-local geometry of the sporadic group O′N has at least two nonprojective summands, one in the principal block and one in a block of defect three. And for Co_3, there are precisely three nonprojective summands, all lying in a nonprincipal block of defect three; see [28].

To summarize, the twelve sporadic simple groups of parabolic characteristic 2 in Table 1 are most of the groups of List $\mathcal{D}$, those for which the 2-local geometry is equivariantly homotopy equivalent to the complex of 2-centric and 2-radical subgroups. Then the geometry and this complex have isomorphic reduced Lefschetz modules. The theorems of Section 3 allow us to describe the homotopy types of fixed point sets, and the results from Section 5 yield the vertices for most of the nonprojective indecomposable summands of the reduced Lefschetz module. These summands lie in blocks whose defect groups equal the corresponding vertices. Nine of these sporadic groups have precisely one nonprojective summand. We verify the "Nonprincipal Block Conjecture" for the groups in Table 1. In terms of these examples, our work can be thought of as an extension of the results of Sawabe and Grizzard, in the sense that we are able to provide further details on the nonprojective part of the reduced Lefschetz module. The information provided in Proposition 6.2 (and in Proposition 6.10 below) appears to be the best available in the literature for these sporadic groups.

Sporadic simple groups of parabolic characteristic $p = 3$

We now discuss the cases when G is a sporadic group of parabolic characteristic 3.

Notation 6.9. The notations from Table 1 are maintained in Table 2 and below. In what follows t will denote a noncentral element of order 3 whose centralizer is of the form $C_G(t) = O_C.H.K$ and where $O_C = O_3(C_G(t))$. Observe that for the first eight examples the group H is a finite group of Lie type, while in the remaining three cases H is a simple group of parabolic characteristic 3.

Proposition 6.10. *Let G be a sporadic simple group of parabolic characteristic* 3 *and let $\widetilde{L}_G$ denote the reduced Lefschetz module for the complex of* 3*-radical and* 3*-centric subgroups in G, over $\mathbb{F}_3$.*

(i) *Let G be one of the groups M_{12}, Co_3, J_3, Co_2, Fi_{22}, Fi_{23}, Fi_{24}' or Th. The virtual module $\widetilde{L}_G$ has precisely one indecomposable nonprojective summand M, and this summand lies in a nonprincipal block of $\mathbb{F}_3G$ and has vertex equal to the defect group of this block. The module M is the Green correspondent of the inflated extended Steinberg module for the building given in the eighth column of Table* 2*.*

(ii) *If G is one of the groups* HN, BM *or M, then the reduced Lefschetz module $\widetilde{L}_G$ has at least one nonprojective indecomposable summand which lies in a nonprincipal block of $\mathbb{F}_3G$.*

Proof. The eight cases in Table 2 in which the fixed point set is contractible are verified using Theorem 3.3 and Lemma 5.2 since O_C contains central elements. For two of these cases, the necessary information is in the Atlas [13]. Let $t = 3B$ in $G = \mathrm{Co}_3$. The elementary abelian 3^5 is of type $3A_{55}B_{66}$, containing central elements of type $3A$. Next, let $t = 3C$ in $G = \mathrm{Fi}_{24}'$. The elementary abelian 3^7 is of type $3A_{378}B_{364}C_{351}$, containing central elements of type $3B$. The normalizer of 3^7 is $N(3^7) = 3^7.O_7(3)$ and the subgroup $2.U_4(3) : 2 \le O_7(3)$ has index 351. The group 3^7 is the natural orthogonal module for $O_7(3)$ with the 364 subgroups of type $3B$ corresponding to the totally isotropic points [24, Section 4.1].

We now quote information from three papers of R. A. Wilson. For $t = 3C$ in $G = \mathrm{Th}$ the elementary abelian 3×3^4 is of type $3B_{40}C_{81}$ with type $3B$ central; see [55, p. 21].

For $t = 3C$ in $G = \mathrm{Fi}_{22}$ the elementary abelian 3^5 is of type $3A_{36}B_{40}C_{45}$, containing central elements of type $3B$; see [53, p. 201]. Note that its normalizer is $N(3^5) = 3^5.U_4(2).2$ and the subgroup $2.(A_4 \times A_4).2 \le U_4(2)$ has index 45.

For $t = 3C$ in $G = \mathrm{Fi}_{23}$ the elementary abelian group 3^6, whose normalizer is $3^6.L_4(3).2$, contains elements of type $3A, 3B$ and $3C$, with $3B$ being 3-central. This follows from the fact that the group 3^6 has an invariant quadratic form defined by putting the norms of elements of classes $3A$, $3B$ and $3C$ equal to -1, 0 and 1 respectively; see [54, Proposition 1.3.3]. This indirectly states that 3-central elements $3B$ occur in 3^6.

For $G = \mathrm{Fi}_{24}'$, with type $3B$ central, see [54, p. 87] for a discussion of the normalizer of a purely central elementary abelian $3B^2$ of rank 2. The normalizer is $N(3B^2) = 3^2.3^4.3^8 : (A_5 \times 2A_4.2)$. The group $3^2.3^4 = 3^6$ has an A_5 action on the 3^4-factor with orbits of size 5, 10, 10, 15 on the 40 cyclic subgroups. The corresponding cosets of $3^2 \le 3^2.3^4$ consist of elements of type $3D$, $3A$, $3B$ and $3C$ respectively. This yields a subgroup $C(3D) = 3^2.3^4.3^6.(A_4 \times 2A_4) \le N(3B^2)$. Thus the 3^2 in $C(3D)$ is the purely central $3B^2$.

Lastly, let G be either Fi_{22} or Fi_{23} and let $t = 3D$. The fact that O_C contains central elements of type $3B$ was not immediately evident from the literature, and we would like to thank Ronald Solomon for providing the following argument. Let $N = N_G(\langle t \rangle)$, and denote $Q = O_3(N)$, which is equal to O_C. Then the quotient group N/Q is either $2S_4 = \mathrm{GL}_2(3)$ if $G = \mathrm{Fi}_{22}$, or $2 \times 2S_4$ if $G = \mathrm{Fi}_{23}$. Let $P \in \mathrm{Syl}_3(N)$ be a Sylow 3-subgroup of the normalizer, and extend it to a Sylow 3-subgroup S of G, so that $P \le S \in \mathrm{Syl}_3(G)$. Note that $Q \le P$ is a subgroup of index three. Since $t \in Q \le P \le S$, we have $Z(S) \le C_G(t) \le N$ and also $Z(S) \le C_G(Q)$, so that $Z(S) \le C_N(Q)$. We want to show that $Z(S) \le Q$. Assume that this is not true, so that $P = QZ(S)$ and also under the quotient map $q : N \to N/Q$, the image $q(Z(S))$ is a Sylow 3-subgroup of $\mathrm{GL}_2(3)$. Note that for any $g \in N$, $gZ(S)g^{-1} \le C_N(Q)$, and the images of these conjugates in the quotient group N/Q are the Borel subgroups of $\mathrm{GL}_2(3)$ and thus generate $2A_4 = \mathrm{SL}_2(3)$. Therefore there is an involution $z \in C_N(Q)$ which maps to the central involution of the matrix group in N/Q. Clearly $Q \le C_G(z)$, and $Z(S) \le C_G(z)$. Thus $P \le C_G(z)$ and the order of P divides the order of $C_G(z)$, with $|P| = 3^7$ if $G = \mathrm{Fi}_{22}$ and $|P| = 3^{10}$ if $G = \mathrm{Fi}_{23}$. Each of these Fischer groups has three conjugacy classes $2A$, $2B$ and $2C$ of involutions. In Fi_{22} their centralizers have orders $2^{16}{\cdot}3^6{\cdot}5{\cdot}7{\cdot}11$, $2^{17}{\cdot}3^4{\cdot}5$ and $2^{16}{\cdot}3^3$ respectively. In Fi_{23}, the centralizers of involutions have orders $2^{18}{\cdot}3^9{\cdot}5^2{\cdot}7{\cdot}11{\cdot}13$, $2^{18}{\cdot}3^6{\cdot}5{\cdot}7{\cdot}11$ and $2^{18}{\cdot}3^5{\cdot}5$. This contradicts $P \le C_G(z)$, and therefore $Q = O_C$ contains central elements of type $3B$.

The condition (N3) concerning the triple of Sylow subgroups is trivially satisfied in every case except one, namely $C(3A)$ in Fi_{24}'. In this case the stronger condition is satisfied: every central element of type $3B$ which lies in the centralizer $C(3A) = 3 \times O_8^+(3) : 3$ actually lies in $3 \times O_8^+(3)$. This follows from the discussion in [54, p. 83] about a symplectic form defined using commutators and lifts to the group $3.\mathrm{Fi}_{24}'$ in M. The lift of $3 \times O_8^+(3) : 3$ is $O_8^+(3) : 3^{1+2}$.

Theorems 3.6 and 5.4 now imply the results of this proposition. Note that for $G = \mathrm{HN}, \mathrm{BM}$ or M the reduced Lefschetz module may have other nonprojective summands with vertex a cyclic group of order three. □

G	$C_G(t) = O_C.H.K$	H	$\mathcal{D}_2(G)^t$	V	$VC_G(V)$	$N_G(V)$	H_V	$\mathcal{D}_2(G)^V$	Ref.
M_{12}	$C(2A) = 2 \times A_5.2$	$L_2(4)$	A_1	2^2	$2^2 \times S_3$	$A_4 \times S_3$	$L_2(2)$	A_1	[28]
J_2	$C(2B) = 2^2 \times A_5$	$L_2(4)$	A_1	2^2	$2^2 \times A_5$	$A_4 \times A_5$	$L_2(4)$	A_1	[28]
HS	$C(2B) = 2 \times A_6.2^2$	$\mathrm{Sp}_4(2)$	C_2	2^2	$2^2 \times 5:4$	$2^2 \times 5:4$	$\mathrm{Sz}(2)$	A_1	[28]
Ru	$C(2B) = 2^2 \times \mathrm{Sz}(8)$	$\mathrm{Sz}(8)$	A_1	2^2	$2^2 \times \mathrm{Sz}(8)$	$(2^2 \times \mathrm{Sz}(8)):3$	$\mathrm{Sz}(8)$	A_1	[28]
Suz	$C(2B) = (2^2 \times L_3(4)):2$	$L_3(4)$	A_2	D_8	$D_8 \times 3^2 : Q_8$	$D_8 \times 3^2 : Q_8$	$U_3(2)$	C_1	[28]
Fi_{22}	$C(2A) = 2{\cdot}U_6(2)$ $C(2C) = 2^{5+8} : (S_3 \times 3^2 : 4)$	$U_6(2)$	C_3 point	2	$2{\cdot}U_6(2)$	$2{\cdot}U_6(2)$	$U_6(2)$	C_3	[29]
He	$C(2A) = (2^2 \cdot L_3(4)):2$	$L_3(4)$	A_2	D_8	$D_8 \times L_3(2)$	$D_8 \times L_3(2)$	$L_3(2)$	A_2	[22]
Co_1	$C(2B) = (2^2 \times G_2(4)):2$ $C(2C) = 2^{11} : M_{12}.2$	$G_2(4)$	G_2 point	D_8	$D_8 \times G_2(2)$	$D_8 \times G_2(2)$	$G_2(2)$	G_2	[40]
BM	$C(2A) = 2 \cdot {}^2E_6(2):2$ $C(2C) = (2^2 \times F_4(2)):2$ $C(2D) = 2^9.2^{16}O_8^+(2).2$	${}^2E_6(2)$ $F_4(2)$	F_4 F_4 point	2^2 D_8	$2^2 \times F_4(2)$ $D_8 \times {}^2F_4(2)$	$(2^2 \times F_4(2)).2$ $D_8 \times {}^2F_4(2)$	$F_4(2)$ ${}^2F_4(2)$	F_4 I_2	[60] [44]
Fi_{24}'	$C(2A) = 2{\cdot}\mathrm{Fi}_{22}:2$	Fi_{22}	$\mathcal{D}_2(\mathrm{Fi}_{22})$	D_8 2^2	$D_8 \times \mathrm{Sp}_6(2)$ $2^2 \times O_8^+(2).3$	$D_8 \times \mathrm{Sp}_6(2)$ $(A_4 \times O_8^+(2).3).2$	$\mathrm{Sp}_6(2)$ $O_8^+(2)$	C_3 D_4	[24]
HN	$C(2A) = 2{\cdot}\mathrm{HS}:2$	HS	$\mathcal{D}_2(\mathrm{HS})$	SD_{16} 2^2	$\mathrm{SD}_{16} \times 5:4$ $2^2 \times A_8$	$\mathrm{SD}_{16} \times 5:4$ $(A_4 \times A_8).2$	$\mathrm{Sz}(2)$ $L_4(2)$	A_1 A_3	[58] [35]
M	$C(2A) = 2{\cdot}\mathrm{BM}$	BM	$\mathcal{D}_2(\mathrm{BM})$	SD_{16}	$\mathrm{SD}_{16} \times {}^2F_4(2)$	$\mathrm{SD}_{16} \times {}^2F_4(2)$	${}^2F_4(2)$	I_2	[60]

Table 1. Information needed for $\widetilde{L}_G(\mathcal{D}_2(G); \mathbb{F}_2)$ (see Notation 6.1).

G	$C_G(t) = O_C.H.K$	H	$\mathfrak{D}_3(G)^t$	V	$VC_G(V)$	$N_G(V)$	$\mathfrak{D}_3(G)^V$	Ref.
M_{12}	$C(3B) = 3 \times A_4$	$L_2(3)$	A_1	3	$3 \times A_4$	$S_3 \times A_4$	A_1	[28]
Co_3	$C(3B) = 3^5 : (2 \times A_5)$		point					[29]
	$C(3C) = 3 \times L_2(8) : 3$	$R(3)$	A_1	3	$3 \times R(3)$	$S_3 \times R(3)$	A_1	
J_3	$C(3B) = 3 \times A_6$	$L_2(9)$	A_1	3	$3 \times A_6$	$(3 \times A_6).2$	A_1	[28]
Co_2	$C(3B) = 3 \times \mathrm{Sp}_4(3) : 2$	$\mathrm{Sp}_4(3)$	C_2	3	$3 \times \mathrm{Sp}_4(3) : 2$	$S_3 \times \mathrm{Sp}_4(3) : 2$	C_2	
Fi_{22}	$C(3A) = 3 \times U_4(3) : 2$	$U_4(3)$	C_2	3	$3 \times U_4(3) : 2$	$S_3 \times U_4(3) : 2$	C_2	[24]
	$C(3C) = 3^5 : 2(A_4 \times A_4).2$		point					[53]
	$C(3D) = 3^3.3.3^2 : 2A_4$		point					
Fi_{23}	$C(3A) = 3 \times O_7(3)$	$O_7(3)$	B_3	3	$3 \times O_7(3)$	$S_3 \times O_7(3)$	B_3	[2]
	$C(3C) = 3^6 : (2 \times O_5(3))$		point					[54]
	$C(3D) = [3^9].(2 \times 2A_4)$		point					
Fi'_{24}	$C(3A) = 3 \times O_8^+(3) : 3$	$O_8^+(3)$	D_4	3^2	$3^2 \times G_2(3)$	$(3^2 : 2 \times G_2(3)).2$	G_2	[24]
	$C(3C) = 3^7.2.U_4(3)$		point					[54]
	$C(3D) = 3^2.3^4.3^6.(A_4 \times 2A_4)$		point					
	$C(3E) = 3^2 \times G_2(3)$	$G_2(3)$	G_2	3^2	$3^2 \times G_2(3)$	$(3^2 : 2 \times G_2(3)).2$	G_2	
Th	$C(3A) = 3 \times G_2(3)$	$G_2(3)$	G_2	3	$3 \times G_2(3)$	$(3 \times G_2(3)) : 2$	A_1	[55]
	$C(3C) = 3 \times 3^4 : 2A_6$		point					
HN	$C(3A) = 3 \times A_9$	A_9	$\mathfrak{D}_3(A_9)$	3^2	$3^2 \times A_6$	$(3^2 : 4 \times A_6)2^2$	A_1	[59]
BM	$C(3A) = 3 \times \mathrm{Fi}_{22} : 2$	Fi_{22}	$\mathfrak{D}_3(\mathrm{Fi}_{22})$	3^2	$3^2 \times U_4(3) : 2^2$	$(3^2 : D_8 \times U_4(3) : 2^2).2$	C_2	[59]
M	$C(3A) = 3{\cdot}\mathrm{Fi}'_{24}$	Fi'_{24}	$\mathfrak{D}_3(\mathrm{Fi}'_{24})$	3^{1+2}	$3^{1+2} \times G_2(3)$	$(3^{1+2} : 2^2 \times G_2(3)).2$	G_2	[3]
	$C(3C) = 3 \times \mathrm{Th}$	Th	$\mathfrak{D}_3(\mathrm{Th})$	3^2	$3^2 \times G_2(3)$	$(3^{1+2} : 2 \times G_2(3)).2$	G_2	[56]

Table 2. Information needed for $\widetilde{L}_G(\mathcal{D}_3(G); \mathbb{F}_3)$ (see Notations 6.1 and 6.9).

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