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Hypercentrally embedded subgroups of polycyclic-by-finite groups

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Abstract. A subgroup of a polycyclic-by-finite group is hypercentrally embedded if and only if its projections into the finite quotients have this property.

1 Introduction

Let the group G be polycyclic-by-finite. Following [2], a subgroup S of G is said to be *hypercentrally embedded in* G if S/S_G (equivalently S^G/S_G) lies in the *hypercenter* $\bar{\zeta}(G/S_G)$, the last term of the upper central series of G/S_G. Here S^G and S_G denote the *join* and the *intersection* of all conjugates—the *normal closure* and the *core*—of S in G, respectively. Hypercentral embedding is an embedding property of subgroups which lies between normality and subnormality: normal subgroups are hypercentrally embedded and hypercentrally embedded subgroups are subnormal (since S^G/S_G is in particular nilpotent for such an S). It is not hard to show that if S is a hypercentrally embedded subgroup of a group G, then so are SN/N in G/N and $S \cap U$ in U whenever $N \trianglelefteq G$ and $U \leq G$. Recently hypercentral embedding has awakened interest, since the *permutable* (or *quasinormal*) subgroups of a *finite* group enjoy this property (see [2, 3, 10]).

In the present article we study the hypercentral embedding of subgroups in *polycyclic-by-finite* groups. These groups are distinguished among the *residually finite groups* by having relevant properties determined by their finite quotients. Presumably the first remark in this direction was established by K. A. Hirsch, who proved in 1946 that a nonnilpotent polycyclic group G must have a nonnilpotent finite quotient ([5], see [8] or [12, p. 154, 5.4.18]). This means that the nilpotency of a polycyclic (-by-finite) group is equivalent to the nilpotency of all its finite images. Later it was shown that this result is even true in all finitely generated soluble-by-finite groups (see [8, p. 77] or [12, p. 477, 15.5.3]).

In the subsequent decades, refining Hirsch's line of investigation, several other authors have observed that, besides the nilpotency, other properties can be read off from the finite quotients when a group is polycyclic-by-finite. For example, in 1957, R. Baer [1] showed that *supersolubility* is such a property.

Conjugacy separability of a polycyclic-by-finite group (see Formanek [4]) states that two elements $x, y \in G$ are conjugate in G if and only if for every $N \trianglelefteq G$ with $|G/N| < \infty$, there is a $g \in G$ such that $yN = x^g N$.

Permutability of two subgroups $H, K \leq G$ can be tested in the same way:

$$HK = KH \iff HKN = KHN$$

holds for every $N \trianglelefteq G$ of finite index. This result goes back to Lennox and Wilson (1977), [9].

Some basic embedding properties of subgroups in polycyclic-by-finite groups can be detected from their images in the finite quotients: A well-known theorem of Mal'cev [11] states that every subgroup S of a polycyclic-by-finite group is closed in the profinite topology. Hence S is normal in G if and only if its image in every finite quotient is normal. Finally, in 1966, Kegel [6] proved that the subnormality of a subgroup S of G is equivalent to the subnormality of SN/N in G/N for all $N \trianglelefteq G$ with $|G/N| < \infty$.

The purpose of our article is to show that for polycyclic-by-finite groups hypercentral embedding belongs to the list of properties that can be detected in the finite images of the group. We have:

Theorem 1.1. *Let S be a subgroup of the polycyclic-by-finite group G. Then S is hypercentrally embedded in G if and only if SN/N is hypercentrally embedded in G/N for every $N \trianglelefteq G$ such that $|G/N| < \infty$.*

This theorem allows us to prove that the characterizations of the hypercentral embedding of subgroups in finite groups given in [2] extend to polycyclic-by-finite groups, thus answering positively a question raised on page 436 of the cited article:

Theorem 1.2. *For a subgroup S of a polycyclic-by-finite group G, the following assertions are equivalent:*

(a) *S is hypercentrally embedded in G.*

(b) *S is permutable with every pronormal subgroup of G.*

(c) *For every $N \trianglelefteq G$ with $|G/N| < \infty$, SN/N is permutable with every pronormal subgroup of G/N.*

(d) *For every $N \trianglelefteq G$ with $|G/N| < \infty$, SN/N is permutable with every Sylow subgroup and every Sylow normalizer of G/N.*

(e) *For every $N \trianglelefteq G$ with $|G/N| < \infty$, SN/N is hypercentrally embedded in G/N.*

We recall here that a subgroup P is *pronormal* in G if for every $g \in G$, P^g is conjugate to P in $\langle P, P^g \rangle$. Clearly normal subgroups are pronormal and it is easy to see and well known that a subgroup P which is simultaneously pronormal and subnormal in G has to be normal.

We also mention in this context a result from [3] and [7]: if S is a permutable subgroup of G, then the factor S^G/S_G, besides being contained in the hypercenter of G/S_G, is finite. The latter of course must no longer be expected for our S in Theorem 1.2, since S could be an arbitrary subgroup of any finitely generated nilpotent group, which may contain a core-free infinite subgroup.

A further remark which is worth mentioning is that Theorem 1.1 does not hold for finitely generated metabelian groups of finite rank. This can be seen by considering the group $G = \langle x, a \mid a^x = a^2 \rangle$.

2 Proof of the theorems

Let us first recall some routine facts, which we shall apply tacitly: A subgroup H of a polycyclic-by-finite group G is contained in the hypercenter $\bar{\zeta}(G)$ if and only if

$$[H, {}_n G] = [H, G, \dots, G] = [\dots[[H, \underbrace{G], G], \dots, G}_{n \text{ times}}] = 1$$

for some natural number n.

A factor H/K, where $H, K \trianglelefteq G$ with $K \leq H$, lies in the hypercenter of G/K if and only if $[H, {}_n G] \leq K$ for some n. If, in addition, $H_1, K_1 \trianglelefteq G$ are such that $K \leq K_1 < H_1 \leq H$ and H_1/K_1 is a chief factor of G, then H_1/K_1 is G-central, i.e. $[H_1, G] \leq K_1$.

The following result is a consequence of the well-known fact that a polycyclic-by-finite group contains a (possibly trivial) torsion-free normal subgroup of finite index (see [8]):

Lemma. *An infinite polycyclic-by-finite group contains a torsion-free abelian normal subgroup $A > 1$.*

Proof of Theorem 1.1. It is clear that only the sufficiency of the condition has to be verified. Let the group G be polycyclic-by-finite and suppose that the subgroup S has the property that SN/N is hypercentrally embedded in G/N for all $N \trianglelefteq G$ such that $|G/N| < \infty$. To show that S is hypercentrally embedded in G, we may assume $S_G = 1$ and we have to produce a natural number n such that $[S^G, {}_n G] = 1$.

We prove the assertion by induction on the *Hirsch length* of G, i.e. on the well-defined number of infinite cyclic factors in a subnormal series of G whose factors are finite or cyclic. Of course we may assume that G is not finite, so that its Hirsch length is positive.

By the above lemma, there is a torsion-free abelian normal subgroup $A > 1$ of G, which we may assume to be chosen of minimal rank. This guarantees that A is (by conjugation) a rationally irreducible G-module, i.e. $\{1\}$ is the only subgroup of A which is normal in G and of infinite index in A.

In the sequel, we shall several times (i.e. for several choices of X) make use of the following general remark: if $1 < X \leq A$ and $X \trianglelefteq G$, then G/X has smaller Hirsch length than G. Since the hypothesis of the theorem is inherited by G/X, we have that SX/X is hypercentrally embedded in G/X by the induction hypothesis. This means that

$$\frac{(SX/X)^{G/X}}{(SX/X)_{G/X}}$$

is contained in the hypercenter of

$$\frac{G/X}{(SX/X)_{G/X}}.$$

Here we point out that[1]

$$(SX/X)^{G/X} = S^G X/X \quad \text{and} \quad (SX/X)_{G/X} = (SX)_G/X.$$

Therefore, the G-isomorphism

$$\frac{(SX/X)^{G/X}}{(SX/X)_{G/X}} = \frac{S^G X/X}{(SX)_G/X} \overset{G}{\cong} \frac{S^G X}{(SX)_G}$$

shows that $S^G X/(SX)_G$ lies in the hypercenter of $G/(SX)_G$. If we have in addition $X \leq S^G$, then

$$S^G/(SX)_G \text{ lies in the hypercenter of } G/(SX)_G. \qquad (*)$$

With these preparations we can now prove the following items, which will lead to the construction of our number n such that $[S^G, {}_n G] = 1$.

(i) *We may assume* $S^G \cap A > 1$. If $S^G \cap A = 1$, then $S^G A = S^G \times A$, so that S^G and $S^G A/A$ are G-isomorphic. Since $S_G = 1$, we also have that SA/A must be core-free in G/A in this case, i.e. $(SA)_G = A$. By induction, SA/A is hypercentrally embedded in G/A, and we see that $S^G A/A$ lies in the hypercenter of G/A. Therefore $S^G \overset{G}{\cong} S^G A/A$ is contained in the hypercenter of G and we are done. Thus, putting $B = S^G \cap A$, we have $1 < B \trianglelefteq G$ and $S \cap A = S \cap B$. Also B is G-rationally irreducible. Moreover, $B \leq S^G$.

[1] Warning: In general $(SX)_G > S_G X$. The core of SX/X in G/X is not $S_G X/X$.

(ii) *For every prime p we have that $B \not\leq SB^p$.* The assertion is equivalent to saying that $(S \cap B)B^p < B$. Indeed suppose $(S \cap B)B^p = B$. Then

$$B/B^p = (S \cap B)B^p/B^p \cong S \cap B/S \cap B^p,$$

so that $S \cap B$ has an elementary abelian p-quotient isomorphic to B/B^p. This means that the rank of $S \cap B$ equals the rank of B. We conclude that $S \cap B$ must have finite index in B, say $|B : S \cap B| = k$. Now $B^k \leq S \cap B$ and since $B^k \trianglelefteq G$ and $(S \cap B)_G = 1$, we see that $B^k = 1$ and B is finite, i.e. $B = 1$, contradicting item (i).

(iii) *B lies in the center of G.* We have $1 < B^p \trianglelefteq G$ for any prime p. Applying $(*)$ with $X = B^p$, we see that $S^G/(SB^p)_G$ lies in the hypercenter of $G/(SB^p)_G$. We abbreviate $K_p = (SB^p)_G$, so that S^G/K_p lies in the hypercenter of G/K_p. Since $B \not\leq SB^p$ by (ii), we have also $B \not\leq K_p$. Therefore $BK_p > K_p$ and $B > B \cap K_p$. Hence we may choose $N_p \trianglelefteq G$ such that $B \cap K_p \leq N_p < B$ and B/N_p is a chief factor of G. Now

$$\begin{aligned} BK_p/N_pK_p = B(N_pK_p)/N_pK_p &\overset{G}{\cong} B/B \cap N_pK_p \\ &= B/(B \cap K_p)N_p = B/N_p, \end{aligned}$$

so BK_p/N_pK_p is a chief factor of G isomorphic to B/N_p. Since $BK_p \leq S^G$ and S^G/K_p lies in the hypercenter of G/K_p, this means that BK_p/N_pK_p must be a *central* chief factor of G. Thus, also B/N_p is G-central, i.e. $[B, G] \leq N_p$. Since B/N_p is non-trivial for all p, we see that $\bigcap_p N_p$ is of infinite index in B. Of course $\bigcap_p N_p \trianglelefteq G$, so we conclude that $\bigcap_p N_p = 1$, as B is G-rationally irreducible. Therefore $[B, G] = 1$ and B lies in the center of G.

(iv) *B is cyclic, $|B/B^p| = p$, $S \cap B = 1$ and $SB = S \times B$.* By (iii), B lies in the center of G, so that every subgroup of B is normal in G. Thus B, being G-rationally irreducible, must be cyclic. Clearly now $|B/B^p| = p$. Also $S \cap B = 1$ since $S \cap B = (S \cap B)_G \leq S_G = 1$. Finally $SB = S \times B$ holds, since B centralizes S.

(v) *We have $\bigcap_p SB^p = S$.* Putting $Y = \bigcap_p SB^p$, we have $S \leq Y \leq SB$. We know that $\bigcap_p B^p = 1$ and $S \cap B = 1$ by (iv). Therefore

$$\begin{aligned} Y = S(Y \cap B) &= S\Big(\Big(\bigcap_p SB^p\Big) \cap B\Big) \\ &= S\Big(\bigcap_p (SB^p \cap B)\Big) = S\Big(\bigcap_p (S \cap B)B^p\Big) \\ &= S\Big(\bigcap_p B^p\Big) = S. \end{aligned}$$

(vi) *We claim that $(SB)_G/(SB^p)_G$ is an elementary abelian p-group.* By (iv) we have $SB = S \times B$. So

$$SB/B^p = SB^p/B^p \times B/B^p.$$

Putting $R = (SB)_G \cap SB^p$, we have $(SB)_G/B^p = R/B^p \times B/B^p$. It follows that R and every conjugate R^g $(g \in G)$ is a normal subgroup of $(SB)_G$ of index p. Thus, $(SB)_G/R_G$ is elementary abelian. But

$$(SB^p)_G \leq \bigcap_{g \in G} R^g = R_G \leq (SB^p)_G,$$

so that $(SB^p)_G = R_G$. Hence $(SB)_G/(SB^p)_G$ is an elementary abelian p-group.

(vii) *Completion of the proof.* By the property $(*)$ applied with $X = B$, we see that $S^G/(SB)_G$ lies in the hypercenter of $G/(SB)_G$. So there is a number k such that $[S^G, {}_k G] \leq (SB)_G$. Also the elementary abelian factor $(SB)_G/(SB^p)_G$ lies in the hypercenter of $G/(SB^p)_G$, since $(SB)_G/(SB^p)_G \leq S^G/(SB^p)_G$ and $S^G/(SB^p)_G$ is hypercentral in $G/(SB^p)_G$.

Since in a polycyclic-by-finite group, the rank of an elementary abelian subgroup (or section) is bounded, there is a number ℓ, which does not depend upon p, such that $[(SB)_G, {}_\ell G] \leq (SB^p)_G$. Therefore also $[S^G, {}_{k+\ell} G] \leq (SB^p)_G$ holds for all p. Now

$$[S^G, {}_{k+\ell} G] \leq \bigcap_p (SB^p)_G \leq \bigcap_p SB^p = S,$$

by (v). Since $[S^G, {}_{k+\ell} G] \trianglelefteq G$ and $S_G = 1$, we conclude that $[S^G, {}_{k+\ell} G] = 1$. Hence $n = k + \ell$ is the number we were looking for. □

Proof of Theorem 1.2. (a) $\Rightarrow$ (b). Suppose, S is hypercentrally embedded in G and let P be a pronormal subgroup of G. To prove that $SP = PS$ we may assume $S_G = 1$. Then S^G lies in the hypercenter of G. Since the center of a group normalizes every subgroup, we have that P is subnormal in $P\bar{\zeta}(G)$ and also in PS^G. Since P is also pronormal in PS^G, we conclude that $P \trianglelefteq PS^G$. Therefore $SP = PS$.

(b) $\Rightarrow$ (c). This is clear as the pronormal subgroups of G/N correspond to the pronormal subgroups of G containing N.

(c) $\Rightarrow$ (d) $\Rightarrow$ (e). We apply the result of [2] to all the finite quotients G/N.

(e) $\Rightarrow$ (a). This is our Theorem 1.1. □

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