Corrigendum

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An inverse problem in corrosion detection: Stability estimates

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Abstract: In this note we correct the proof of [2, Theorem 2.1].

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Unless otherwise stated, $\Omega$ is a $C^\infty$ bounded domain of $\mathbb{R}^2$ so that its boundary $\Gamma$ is the union of two disjoint closed subsets with nonempty interior, $\Gamma = \Gamma_1 \cup \Gamma_2$.

We considered in [2] the stability issue for the problem of determining the boundary coefficient $q$, appearing in the BVP

$$\begin{cases}
\Delta u = 0 & \text{in } \Omega, \\
\partial_\nu u + qu = 0 & \text{on } \Gamma_1, \\
\partial_\nu u = f & \text{on } \Gamma_2,
\end{cases} \quad (1)$$

from the boundary measurement $u|_{\gamma_2}$, where $\gamma_2$ is an open subset of $\Gamma_2$.

Our proof of [2, Theorem 2.1] is partially incorrect. We rectify here this proof. We precisely establish a stability estimate of logarithmic type for the inverse problem described above. Contrary to the result announced in [2, Theorem 2.1], we do not know whether Lipschitz stability, even around a particular unknown coefficient, is true. Note that Lipschitz stability around an arbitrary unknown boundary coefficient is false in general as the following counter-example shows. Let $\Omega = \{ \frac{1}{2} < |x| < 1 \}$, $\Gamma_1 = \{|x| = \frac{1}{2}\}$, $\Gamma_2 = \{|x| = 1\}$ and let, in polar coordinates $(r, \theta)$,

$$u = 1 + \ln r,$$

$$u_k = u + 2^{-k}k^{-2}(r^k + r^{-k}) \cos(k\theta), \quad k \geq 1.$$ 

By straightforward computations we check that $u$ and $u_k$ are the solutions of the BVP (1) respectively when

$$q = \frac{2}{1 - \ln 2},$$

$$q_k = \frac{2 + k^{-1}(2^{-2k+1} - 2) \sin(k\theta)}{1 - \ln 2 + k^{-2}(2^{-2n} + 1) \sin(k\theta)}, \quad k \geq 1,$$

and $f = 1$.

By simple calculations, we get $\|u - u_k\|_{L^2(\Gamma_2)} = O(2^{-k}k^{-2})$, while $\|q - q_k\|_{L^2(\Gamma_1)} = O(k^{-1})$.

To our knowledge, the only case where Lipschitz stability holds is when $q$ is assumed to be a priori piecewise constant. We refer to [6] for more details.

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Throughout, the unit ball of a Banach space $X$ is denoted by $B_X$ and
\[
L^p_K(D) = \{ h \in L^p(D) : \text{supp}(h) \subset K \}, \quad 1 \leq p \leq \infty.
\]
The characteristic function of a set $A$ is denoted by $\chi_A$.

Fix $q_0 \in L^\infty(\Gamma_1)$ nonnegative and nonidentically equal to zero and let $f \in L^2(\Gamma_2)$ be nonidentically equal to zero. Denote by $u_0 \in H^{3/2}(\Omega)$ the solution of the BVP
\[
\begin{aligned}
\Delta u &= 0 \quad \text{in } \Omega, \\
\partial_\nu u + q_0 u &= 0 \quad \text{on } \Gamma_1, \\
\partial_\nu u &= f \quad \text{on } \Gamma_2.
\end{aligned}
\]

As it is observed in [2],
\[
\Gamma_0 = \{ x \in \Gamma_1 : u_0(x) \neq 0 \}
\]
is an open dense subset of $\Gamma_1$.

For $(\varphi_1, \varphi_2) \in L^2(\Gamma_1) \oplus L^2(\Gamma_2)$, define $L(\varphi_1, \varphi_2) := y$, where $y \in H^{3/2}(\Omega)$ is the unique weak solution of the BVP
\[
\begin{aligned}
\Delta y &= 0 \quad \text{in } \Omega, \\
\partial_\nu y + q_0 y &= \varphi_1 \quad \text{on } \Gamma_1, \\
\partial_\nu y &= \varphi_2 \quad \text{on } \Gamma_2.
\end{aligned}
\]

An application of Green’s formula leads to
\[
\int_\Omega |\nabla y|^2 \, dx + \int_{\Gamma_1} q_0 y^2 \, d\sigma = \int_{\Gamma_1} \varphi_1 y \, d\sigma + \int_{\Gamma_2} \varphi_2 y \, d\sigma \leq \| (\varphi_1, \varphi_2) \|_{L^2(\Gamma_1) \oplus L^2(\Gamma_2)} \| y \|_{H^{1}(\Omega)}.
\]

Using that
\[
h \to \left( \int_\Omega |\nabla h|^2 \, dx + \int_{\Gamma_1} q_0 h^2 \, d\sigma \right)^{1/2}
\]
defines an equivalent norm on $H^{1}(\Omega)$, we derive from (3) that
\[
\| y \|_{H^{1}(\Omega)} \leq \kappa_0 \|(\varphi_1, \varphi_2)\|_{L^2(\Gamma_1) \oplus L^2(\Gamma_2)}
\]
for some constant $\kappa_0$ depending only on $\Omega$, $q_0$ and $f$.

As $y$ is also the solution of the BVP
\[
\begin{aligned}
\Delta y &= 0 \quad \text{in } \Omega, \\
\partial_\nu y + (1 - q_0) y + \varphi_1 &= 0 \quad \text{on } \Gamma_1, \\
\partial_\nu y &= \varphi_2 \quad \text{on } \Gamma_2,
\end{aligned}
\]
we get from the usual a priori estimates for nonhomogenous BVPs (see [5]) that there exists a constant $\kappa_1$, depending only on $\Omega$, $q_0$ and $f$, so that
\[
\| y \|_{H^{3/2}(\Omega)} \leq \kappa_1 \|(\varphi_1, \varphi_2)\|_{L^2(\Gamma_1) \oplus L^2(\Gamma_2)}.
\]
In other words, we have proved that $L \in \mathcal{P}(L^2(\Gamma_1) \oplus L^2(\Gamma_2), H^{3/2}(\Omega))$ and
\[
\| L \| := \| L \|_{\mathcal{P}(L^2(\Gamma_1) \oplus L^2(\Gamma_2), H^{3/2}(\Omega))} \leq \kappa_1.
\]

For $q \in L^2(\Gamma_1)$, define the operator $H_q$ as follows:
\[
H_q : H^{3/2}(\Omega) \to H^{3/2}(\Omega), \quad H_q(u) = L(-qu|_{\Gamma_1}, 0).
\]

If $\kappa$ is the norm of the trace operator
\[
h \in H^{3/2}(\Omega) \to h|_{\Gamma_1} \in C(\Gamma_1),
\]
then
\[ \|H_q\|_{\mathcal{B}(H^{1/2}(\Omega))} \leq \kappa \|q\|_{L^2(\Gamma_1)}. \]
Whence, for any \( q \in \mathcal{U} = (2\kappa \|L\|)^{-1}B_{L^2(\Gamma)}, I - H_q \) is invertible and
\[ \|(I - H_q)^{-1}\|_{\mathcal{B}(H^{1/2}(\Omega))} \leq 2, \quad q \in \mathcal{U}. \]

Define, for \( q \in \mathcal{U} \) and \((\varphi_1, \varphi_2) \in L^2(\Gamma_1) \otimes L^2(\Gamma_2)\),
\[ u_q(\varphi_1, \varphi_2) = (I - H_q)^{-1}L(\varphi_1, \varphi_2). \]
In light of the identity
\[ u_q(\varphi_1, \varphi_2) = L(-qu_{\Gamma_1} + \varphi_1, \varphi_2), \]
we derive that \( u_q(\varphi_1, \varphi_2) \in H^{3/2}(\Omega) \) is the solution of the BVP
\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \ \Omega, \\
\partial_\nu u + (q_0 + q)u &= \varphi_1 \quad \text{on} \ \Gamma_1, \\
\partial_\nu u &= \varphi_2 \quad \text{on} \ \Gamma_2.
\end{align*}
\]
Note that according to (4),
\[ \|u_q(\varphi_1, \varphi_2)\|_{H^{3/2}(\Omega)} \leq 2\kappa_1 \|\varphi_1, \varphi_2\|_{L^2(\Gamma_1) \otimes L^2(\Gamma_2)}. \]

Set \( u_q = u_q(0, f) \). That is, \( u_q \) is the solution of the BVP
\[
\begin{align*}
\Delta u &= 0 \quad \text{in} \ \Omega, \\
\partial_\nu u + (q_0 + q)u &= \varphi_1 \quad \text{on} \ \Gamma_1, \\
\partial_\nu u &= f \quad \text{on} \ \Gamma_2.
\end{align*}
\]
Observe that (5) yields
\[ \|u_q\|_{H^{3/2}(\Omega)} \leq 2\kappa_1 \|\|_{L^2(\Gamma_2)}. \]

Let \( K \) be a compact subset of \( \Gamma_0 \) with nonempty interior so that \( \Gamma_1 \setminus K \neq \emptyset \). We can mimic the proof of [2, Proposition 2.1] to show that the mapping
\[ \Phi : q \in \mathcal{U} \cap L^2_K(\Gamma_1) \mapsto \chi_{\Gamma_1}[\partial_\nu u_q]|K] \in L^2_K(\Gamma_1) \]
is continuously Fréchet differentiable and \( \Phi'(0) = N \). Here, for \( p \in L^2_K(\Gamma_1) \), \( Np = \chi_{\Gamma_1}[\partial_\nu v]|K] \), where \( v_p \) is the solution of the BVP
\[
\begin{align*}
\Delta v &= 0 \quad \text{in} \ \Omega, \\
\partial_\nu v + q_0 v &= -p \quad \text{on} \ \Gamma_1, \\
\partial_\nu v &= 0 \quad \text{on} \ \Gamma_2.
\end{align*}
\]
Similarly to the proof of [2, Lemma 2.1], we prove that \( N \) is an isomorphism. Therefore, by the Implicit Function Theorem, there exists \( \tilde{\mathcal{U}} \subset \mathcal{U} \) so that \( \Phi^{-1} \) is Lipschitz continuous on \( \tilde{\mathcal{V}} = \Phi(\tilde{\mathcal{U}} \cap L^2_K(\Gamma_1)) \) with Lipschitz constant less than or equal to
\[ 2\|N^{-1}\|. \]
That is,
\[ \|q_1 - q_2\|_{L^2(\Gamma_1)} \leq 2\|N^{-1}\|\|\partial_\nu u_{q_1} - \partial_\nu u_{q_2}\|_{L^2(K)}, \quad q_1, q_2 \in \tilde{\mathcal{U}} \cap L^2_K(\Gamma_1). \]

Let \( k \) be a positive integer, \( s \in \mathbb{R}, 1 \leq r \leq \infty \) and consider the vector space
\[ B_{s,r}(\mathbb{R}^k) := \{ w \in \mathcal{S}'(\mathbb{R}^k) : (1 + |\xi|^2)^{s/2} \hat{w} \in L^r(\mathbb{R}^k) \}, \]
where \( \mathcal{S}'(\mathbb{R}^k) \) is the space of tempered distributions on \( \mathbb{R}^k \) and \( \hat{w} \) is the Fourier transform of \( w \). Equipped with the norm
\[ \|w\|_{B_{s,r}(\mathbb{R}^k)} := \|(1 + |\xi|^2)^{s/2} \hat{w}\|_{L^r(\mathbb{R}^k)}, \]
\( B_{s,r}(\mathbb{R}^k) \) is a Banach space. Note that \( B_{s,2}(\mathbb{R}^k) \) is merely the Sobolev space \( H^s(\mathbb{R}^k) \). Using local charts and a partition of unity, we construct \( B_{s,2}(\Gamma_1) \) from \( B_{s,2}(\mathbb{R}) \) similarly as \( H^s(\Gamma_1) \) is built from \( H^s(\mathbb{R}) \).
Fix $m > 0$. If $f \in H^{3/2}(\Gamma_2)$ and $q \in mB_{3/2}(\Gamma_1)$, then by [1, Theorem 2.3], $u_q \in H^3(\Omega)$ and
\[
\|u_q\|_{H^3(\Omega)} \leq C_0. \tag{7}
\]
Here and henceforth, $C_0$ is a constant depending only on $\Omega, f$ and $m$. In dimension two, $H^3(\Omega)$ is continuously embedded in $C^2(\Omega)$. Whence, estimate (7) entails
\[
\|u_q\|_{C^2(\Omega)} \leq C_0.
\]
Let
\[
\Psi(\rho) = |\ln \rho|^{1/2} + \rho, \quad \rho > 0,
\]
be extended by continuity at 0 by setting $\Psi(0) = 0$.

Let $\gamma_2$ be a nonempty open subset of $\Gamma_2$. According to [3, Proposition 2.7], there exists a constant $C > 0$, depending only on $\Omega, f, m$ and $\gamma_2$, so that
\[
\|\partial_\nu u_{q_1} - \partial_\nu u_{q_2}\|_{L^q(K)} \leq C\Psi(\|u_{q_1} - u_{q_2}\|_{H^3(\Omega)}). \tag{8}
\]
Set
\[
\mathcal{D}_m = mB_{3/2}(\Gamma_1) \cap \overline{\Omega} \cap L^2(\Gamma_1).
\]
Note that $\mathcal{D}_m \neq \emptyset$ if $m$ is chosen sufficiently large.

We can now combine (6) and (8) in order to obtain
\[
\|q_1 - q_2\|_{L^2(\Gamma_1)} \leq C\Psi(\|u_{q_1} - u_{q_2}\|_{H^3(\Omega)}), \quad q_1, q_2 \in \mathcal{D}_m.
\]

We sum up our analysis in the following theorem, where we use the fact that $H^{3/2}(\Gamma_2)$ is continuously embedded in $C^2(\Gamma_2)$.

**Theorem 1.** Let $f \in H^{3/2}(\Gamma_2), f \neq 0, 0 \leq q_0 \in L^{\infty}(\Gamma_1), q_0 \neq 0$, let $K$ be a compact subset of $\Gamma_0$, given by (2), with nonempty interior so that $\Gamma_1 \setminus K \neq \emptyset$ and let $\gamma_2$ be a nonempty open subset of $\Gamma_2$. There exists a neighborhood $U$ of $q_0$ in $L^2(\Gamma_1)$, depending on $f$, $\Omega$ and $K$ with the property that, if $m > 0$ is chosen in such a way that
\[
\mathcal{D}_m = mB_{3/2}(\Gamma_1) \cap \overline{U} \cap L^2(\Gamma_1) \neq \emptyset,
\]
we find a constant $C > 0$, depending on $f$, $\Omega$, $q_0$, $K$ and $\gamma_2$, so that
\[
\|q_1 - q_2\|_{L^2(\Gamma_1)} \leq C\Psi(\|u_{q_1} - u_{q_2}\|_{H^3(\Omega)}), \quad q_1, q_2 \in \mathcal{D}_m.
\]

Observe that, as in [2], the last theorem can be extended to the case where $\partial \Gamma_1 \cap \partial \Gamma_2 \neq \emptyset$. Also, for the most general case, in dimensions two and three, we can prove a stability estimate of triple logarithmic type (see [3, Theorem 4.9]).

**Remark 1.** Note that, in general, $\Gamma_0$ given by (2) is strictly contained in $\Gamma_1$ for an arbitrary $q_0$. However, we can construct an example of $q_0$ for which $\Gamma_0 = \Gamma_1$. To this end, fix $0 < \alpha < 1$ and, for $0 \leq f \in C^{2,\alpha}(\Gamma_2)$, denote by $w(f) \in C^{2,\alpha}(\overline{\Omega})$ the solution of the BVP

\[
\begin{align*}
\Delta w &= 0 & \text{in } \Omega, \\
\partial_\nu w &= 0 & \text{on } \Gamma_1, \\
\partial_\nu w &= f & \text{on } \Gamma_2.
\end{align*}
\]

According to strong maximum principle’s and Hopf’s lemma (see for instance [4]), $\partial_\nu w < 0$ on $\Gamma_1$. Let $q_0 = -\partial_\nu w(f)|_{\Gamma_1}(> 0)$ and set $u_0 = 1 + w$. Then it is straightforward to check that $u_0$ is the unique solution of the BVP

\[
\begin{align*}
\Delta u &= 0 & \text{in } \Omega, \\
\partial_\nu u + q_0 u &= 0 & \text{on } \Gamma_1, \\
\partial_\nu u &= f & \text{on } \Gamma_2.
\end{align*}
\]

We see that for this particular choice of $q_0$, we have $\Gamma_0 = \Gamma_1$. 

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References