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Distributed non-singular dislocation technique for cracks in strain gradient elasticity

Abstract: The mode III fracture analysis of a cracked graded plane in the framework of classical, first strain gradient, and second strain gradient elasticity is presented in this paper. Solutions to the problem of screw dislocation in graded materials are available in the literature. These solutions include various frameworks such as classical elasticity, and the first strain and second strain gradient elasticity theories. One of the applications of dislocations is the analysis of a cracked medium through distributed dislocation technique. In this article, this technique is used for the mode III fracture analysis of a graded medium in classical elasticity, which results in a system of Cauchy singular integral equations for multiple interacting cracks. Furthermore, the technique is modified for gradient elasticity. Owing to the regularization of the classical singularity, a system of non-singular integral equations is obtained in gradient elasticity. A plane with one crack is studied, and the stress distribution in classical elasticity is compared with those in gradient elasticity theories. The effects of the internal lengths, introduced in gradient elasticity theories, are investigated. Additionally, a plane with two cracks is studied to elaborate the interactions of multiple cracks in both the classical and gradient theories.

Keywords: antiplane; crack; distributed dislocations; graded material; strain gradient elasticity.

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1 Introduction

In applications where the size of microstructural elements is not negligible compared with the size of the studied components, the size of the microstructural elements affects the macroscopic behavior. These effects can become predominant for some special practical problems in mechanics and of high importance for structural design, such as in the case of fracture [1].

The generalized continuum elasticity theories were introduced at the beginning of the 19th century first by the brothers Cosserat; about half a century later, they were revisited and revised by the founders of modern continuum mechanics (Toupin, Rivlin, Mindlin, Eringen, and others), as reviewed by Aifantis [2]. In classical elasticity, the strain energy is simply assumed to be a function of strain. Owing to the lack of an intrinsic length scale in classical elasticity, it represents a scale-free continuum theory. In contrast, generalized continuum elasticity theories enrich classical elasticity with additional material characteristic lengths to describe the scale effects resulting from the microstructures [3].

In strain gradient elasticity, the strain energy is generalized and is not simply a function of strain but may also depend on the gradient of strain. These contributions introduce internal lengths to consider scale effects. In the first strain gradient elasticity, the strain energy is assumed to be a quadratic function in terms of strain and first-order gradient strain; in the second strain gradient elasticity, the strain energy is a function of strain, first-order gradient strain, and second-order gradient strain. In generalized elasticity, hyperstress tensors such as double (couple) stress (in the first strain gradient elasticity) and triple stress tensors (in the second strain gradient elasticity) are defined to parameterize the higher-order theory. However, owing to many material constants in these theories, there used to be a gap between theories and experimental investigations.

Analytical solutions for stress/strain fields in dislocations have been a topic of many investigations both in classical and gradient elasticity. In fracture mechanics, considering different approaches in the formulation of the strain gradient theories, singular and non-singular stress fields are predicted ahead of the crack tip. Paulino et al. [4], Chan et al. [5], and Gourgiotis and Georgiadis [6] predicted the singular stresses at the crack tip for the gradient theory. However, Gutkin and Aifantis [7] elaborated that in the framework of gradient elasticity, the elastic singularity is eliminated from dislocation lines. Additionally, Lazar and Maugin [8] reported the same behavior for a
class of strain gradient and a stress gradient theory. Thus, the improved stress and strain fields present no singularity in the core region unlike the classical solutions. Currently, these approaches are under debate to determine correct boundary conditions and the compatibility conditions of gradient elasticity. Recently, Lazar and Polyzos [9] studied the physical meaning and the interpretation of the non-singular stress fields of cracks of mode I and mode III. They investigated whether these solutions satisfy the equilibrium, boundary, and compatibility conditions. This discussion should be considered together with the articles by Aifantis and coworkers [10–14], along with a “Commentary” by Aifantis that was submitted to the International Journal of Solids and Structures as a companion to Ref. [9], to address the concerns raised there (see also related comment in Section 6 of this article).

The gradient theory considered in the work of Lazar and Maugin [8] contained double stresses (hyperstresses), and in the case of screw dislocation these double stresses were still singular at the vicinity of the dislocation. As the first strain gradient elasticity theory succeeded in eliminating the stress singularity of the dislocation, the higher-order gradient elasticity theories efficiently eliminate the singularity of the hyperstresses. Lazar et al. [15] presented the dislocations in the second gradient elasticity. They obtained non-singular expressions for double stresses as well as triple stresses produced by a straight screw dislocation. It should be mentioned that owing to different approaches among the gradient theories available in the literature, singular and non-singular solutions have been reported for gradient frameworks. For instance, the approach by Gutkin and Aifantis [7] eliminated the singularity, whereas in Georgiadis [16] the stress is infinite at the crack tip.

Askes and Aifantis [11] have provided a review on the gradient theories, and have applied the finite element method to the gradient elastic theories. Their numerical approach depicts the elimination of the singularity in the vicinity of the crack tip in gradient frameworks. As Askes and Aifantis [11] have mentioned, for static analysis of a cracked structure, the strain gradient theory is sufficient, whereas for dynamic analysis, the strain gradient should incorporate acceleration gradients for a consistent solution. It should be mentioned that the gradient approach is related to Eringen’s non-local theory [14].

One of the applications of the dislocations is the fracture analysis of materials. In classical elasticity, it has been proved that the cracked material with specific loading may be modeled by properly distributing the dislocations in the material [17]. The distributed dislocation technique (DDT) has been largely used in classical elasticity for static [18] and elastodynamic analyses [19]. Mousavi and Paavola [20, 21] have also generalized this technique for the analysis of piezoelectric and magneto-electro-elastic materials. These investigations show that the DDT is useful in the interaction analysis of multiple straight and curved cracks. In these cases, the technique results in a system of Cauchy singular integral equations, which should be solved.

In the framework of generalized elasticity, the DDT has been applied to a few problems. In the couple stress elasticity, Gourgiotis and Georgiadis [22, 23] used the DDT for the analysis of cracks in modes I, II, and III. They studied the dislocation and disclination in the homogeneous couple stress elasticity. In this case, the stress and hyperstresses possess singularity and hypersingularity. They proposed a technique to solve the hypersingular integral equations obtained in the DDT.

The present work discusses the feasible methods for the solution of the dislocation in gradient elasticity and provides an example of the solution of screw dislocation in a Casal’s continuum. Furthermore, the dislocation solutions presented by Gutkin and Aifantis [7], Lazar [24], and Davoudi et al. [25] are applied in the DDT to analyze a cracked medium. The generalized DDT formulation covers classical, first gradient, and second gradient elasticity. The stress distribution in a cracked graded plane under antiplane loading is determined for the classical and gradient theories. The effect of material gradation and gradient parameters are discussed. To the best of the authors’ knowledge, the analysis of the interactions of multiple cracks in gradient elasticity is not available in the literature. To reveal the capability of the DDT for the analysis of various configurations of cracks, two arrangements of cracks, including one crack and two cracks, are presented.

2 Solution to screw dislocation (classical elasticity and strain gradient elasticity)

In this section, the solutions to the screw dislocation in the framework of the classical, first strain gradient, and second strain gradient theories are presented for graded materials. Gutkin and Aifantis [7] presented the screw dislocation in an isotropic medium in first strain gradient elasticity. A screw dislocation in a simplified but straightforward version of second strain gradient elasticity was the topic of another work by Lazar et al. [15]. Lazar [24] presented the stress field of a screw dislocation in a
medium exponentially graded in the y-direction in the framework of the classical elasticity as well as the first strain gradient elasticity. Davoudi et al. [25] used the same approach to analyze a screw dislocation in the plane of graded materials in the second strain gradient elasticity. In the following subsections, the stress fields induced by the dislocation are given.

The gradation direction of the graded plane is assumed to be parallel to the y-axis. A screw dislocation is imposed at the origin of the infinite (x, y)-plane with Burgers vector (0, 0, b). The line of the dislocation is assumed to be perpendicular to the gradation direction of the graded plane.

2.1 Classical solution

In classical continuum mechanics, the strain-energy density is

\[ W = \frac{1}{2} \lambda(y) E_{x}E_{y} + \mu(y) E_{x}E_{y}, \]

while \((\lambda, \mu)=\lambda(y, \mu(y)) \exp(2y\gamma)\) are Lame moduli for a non-homogeneous material graded in the y-direction \((\gamma)\) is the gradation parameter, and \(E_{y}\) is the elastic strain tensor. The analysis of the screw dislocation results in a problem of antiplane shear with the only non-vanishing component of displacement \(w_{0}(x, y)\) satisfying the displacement equilibrium equation

\[ \left( \nabla^{2} + 2a \frac{\partial}{\partial y} \right) w_{0} = 0, \]

with the conditions

\[ E_{x}^{0}(x, 0^{+}) = E_{x}^{0}(x, 0^{-}), \]

\[ w_{0}(x, 0^{+}) - w_{0}(x, 0^{-}) = b_{2} H(-x), \]

where \(E_{x}^{0} = \frac{1}{2} \frac{\partial w_{0}}{\partial y}\) is the classical elastic strain and \(H(...)\) is the Heaviside step function.

Using the method of stress functions [24] or the method of Fourier transform [25], the above partial differential equation (PDE) results in the following displacement, and Cauchy stress field

\[ \sigma_{x}^{0} = b_{2} \frac{\mu(y) e^{y\gamma}}{2\pi} K_{1}(\gamma r), \]

\[ \sigma_{y}^{0} = b_{2} \frac{\mu(y) e^{y\gamma}}{2\pi} K_{2}(\gamma r), \]

where \(K_{n}\) denotes the modified Bessel function of the second kind with order \(n\) and \(r = \sqrt{x^2 + y^2}\). These stress components possess a Cauchy \((1/r)\) as well as logarithmic singularities. The displacement formulation and the Fourier transform used by Davoudi et al. [25] additionally is capable of the determination of the displacement field as

\[ w_{0} = \frac{b_{2}}{2 \pi} e^{y\gamma} \int_{0}^{\infty} \frac{\sin(\pi x s)}{s^{2} + \alpha^2} e^{\frac{x^2 + y^2}{s}} ds \text{ sgn}(y) \]

\[ b_{2} \frac{e^{y\gamma}}{2 \pi} \int_{0}^{\infty} \frac{\sin(\pi x s)}{s} e^{\frac{x^2 + y^2}{s}} ds \cdot \frac{1}{2} H(-y) \]

It is worth mentioning that the screw dislocation in a graded layer has been reported by Fotuhi and Fariborz [18]. It is obvious that the presence of the strip boundary conditions increases the mathematical effort needed to solve the resulting PDE and will result in complicated integral expressions.

2.2 First strain gradient elasticity

In the first strain gradient elasticity, the strain-energy density has the following form

\[ W = \frac{1}{2} \lambda(y) E_{x}E_{y} + \mu(y) E_{x}E_{y}, \]

\[ + I(\frac{1}{2} \lambda(y) E_{x, k}E_{y,k} + \mu(y) E_{x,k}E_{y,k}), \]

where \(I>0\) is the gradient parameter.

In gradient elasticity, several methods are introduced to solve the dislocation and crack problem. One method is to determine the solution to the dislocation directly, i.e., without the aid of the classical solution. In this method, a higher-order PDE (fourth order) is solved with a generalized condition of the screw dislocation. Gourgiotis and Georgiadis [22] have used this method in the couple stress elasticity. As a consequence of the higher-order theory, additional conditions are needed to solve the higher-order PDE.

The second method is to obtain the solution of the dislocation with the aid of the classical solution. This technique is quite common in gradient elasticity [2]. Lazar [24] solved the problem of a screw dislocation in a graded material within the first strain gradient elasticity. He provided the Cauchy-like stress field of the screw dislocation in a graded material for first gradient elasticity as

\[ \sigma_{x}^{0} = \frac{a_{0}}{a} \frac{\mu_{x}e^{y\gamma}}{2\pi} y \left[ \frac{1 + 1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \]

\[ \sigma_{y}^{0} = \frac{a_{0}}{a} \frac{\mu_{y}e^{y\gamma}}{2\pi} x \left[ \frac{1 + 1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \]

\[ \sigma_{z}^{0} = \frac{a_{0}}{a} \frac{\mu_{z}e^{y\gamma}}{2\pi} \left[ \frac{1 + 1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \left[ \frac{1 + y^2}{r} \right] \]

(7)
Furthermore, Sciarrà and Vidoli [26] have presented asymptotic solutions for antiplane and inplane opening modes. This method might be used for the solution of the dislocations.

It is worth noting that it is a challenging task to determine the correct form of the higher-order condition for the dislocation, as there are different suggestions available in the literature [4, 16, 22, 23, 27]. The first two references provided the higher-order condition for the solution of the crack problem directly, whereas the second two references implied the higher-order condition for the dislocation problem. Polizzotto [28] provided a thorough discussion about non-standard boundary conditions in gradient elasticity. To shed more light on the selection of the proper higher-order condition, an example of this method is carried out and presented in Appendix A. This appendix shows a selection of higher-order conditions and the solution in Casal’s continuum.

As mentioned earlier, the classical solution of the screw dislocation in mediums other than the infinite plane is of interest to some authors, e.g., Fotuhi and Fariborz [18]. They have dealt with the problem of a screw dislocation in a graded strip. As was expected, in comparison with infinite plane [24], the presence of the strip boundaries resulted in complicated expressions for the field equations. This solution might be generalized to the framework of gradient elasticity.

### 2.3 Second strain gradient elasticity

In the framework of the second strain gradient elasticity, the strain-energy density has the following form

\[
W = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} + \frac{1}{2} l^2 C_{ijkl} E_{mn,k} E_{ij,l} + \frac{1}{2} \eta^2 C_{ijkl} E_{mn,k} E_{ij,l},
\]

(8)

where \(l\) and \(l'\) are internal lengths, and \(C_{ijkl}\) is the stiffness tensor of the form

\[
C_{ijkl} = \lambda(y) \delta_{ij} \delta_{kl} + \mu(y) (\delta_{ik} \delta_{jl} + \delta_{jl} \delta_{ik}).
\]

(9)

The lower-order elastic-like stresses and higher-order double stresses are given by the expressions [25]

\[
\sigma^y = \sigma^y_0 + b_1 \mu e^{\gamma} \frac{1}{2 \pi} c_1^2 c_2^2 \left[ -c_1^2 Y K_0(\kappa_r) + c_2^2 Y K_0(\kappa_r) + c_1^2 K_0(\kappa_r) - c_2^2 K_0(\kappa_r) \right],
\]

(10)

where \(c_1'^2 + c_2'^2 = l'^2\), \(c_1 c_2' = l\), and \(\kappa = \sqrt{\gamma^2 + 1/c_1^2}\), and \(l, l'\) are internal lengths. It should be mentioned that in the second-order strain gradient theory, expressions for double stresses are non-singular [25] while they are singular in the first-order strain gradient theory.

### 3 DDT for classical and gradient elasticity

The DDT is a method to analyze a medium containing multiple cracks. In this technique, the dislocations are distributed in the locations of the cracks and the stress field, and fracture parameters such as stress intensity factors (SIFs) are determined for the cracked medium [17]. Stress fields of dislocations usually contain singularities that result in singular integral equations in the DDT. This is usually the case in classical elasticity. In the framework of the gradient elasticity, the singularity of the stress (and hyperstress) components is different from classical elasticity. For instance, Gourgiotis and Georgiadis [22] have given the solution for a screw dislocation in the couple stress theory and have solved governing integral equations in DDT with cubic singularity.

As mentioned earlier, a great difference in the generalized elasticity is the non-singular stress fields, which will alleviate the solution of the resulting integral equations. In this section, the DDT is modified for the class of graded materials in the framework of strain gradient elasticity. The stress fields presented in the previous section [Eqs. (4), (7), and (10)] are used in this section in DDT.

The graded plane, with a varying shear modulus \(\mu = \mu_0 \exp(2\gamma y)\), is assumed to contain a screw dislocation situated at a point with coordinates \((\eta, \zeta)\). The line of the dislocation is assumed to be parallel to the \(x\)-axis:

\[
\mu(y) = \mu_0 \exp(2\gamma y).
\]

(11)

The stress components at a point with coordinates \((x, y)\) due to the dislocation at \((\eta, \zeta)\) may be obtained by making the conversions \(x \rightarrow (x-\eta), y \rightarrow y-\zeta\) in Eqs. (4), (7), and (10).

The movable orthogonal coordinate system \(n, t\) is chosen such that the origin may move on the crack while the \(t\)-axis remains tangent to the crack surface. The anti-plane traction on the surface of the \(k\)th crack, in terms of stress components in the Cartesian coordinates \(x, y\), becomes

\[
\sigma_n^n (x_n, y_n) = \sigma_n^n (\theta_k) \cos(\theta_k) \sigma_n^n (\theta_k),
\]

\[
k \in \{1, 2, \ldots, N\}, \quad i = 0, 1, 2,
\]

(12)
where $\theta_k(s) = \tan^{-1}( \beta_k(s)/\alpha_k(s) )$ is the angle between the $x$- and $t$-axes and prime denotes differentiation with respect to the argument. Additionally, $i = 0, 1, 2$ denote the components in the classical, first strain gradient, and second strain gradient elasticity, respectively. A crack is constructed by a continuous distribution of dislocations. Suppose dislocations with unknown density $B_j$ are distributed on the infinitesimal segment $\int (\alpha_j^2 + \beta_j^2) dt$ at the surface of the $j$th crack where $-1 \leq s \leq 1$ and $k \in \{1, 2, ..., N\}$, $i = 0, 1, 2$, and prime denotes differentiation with respect to the argument. Additionally, $\nu = \frac{1}{2}$ (classical elasticity) and non-singular integral equations with respect to the argument. Furthermore, the displacement field is single valued out of crack surfaces. Thus, the dislocation density for an embedded crack is subjected to the following closure requirement:

$$\int_{-1}^{1} \sqrt{[\alpha_j'(t)]^2 + [\beta_j'(t)]^2} B_j(t) dt = 0,$$

(19)

where $x_{ij} = \alpha_i \alpha_j$, $y_{ij} = \beta_i \beta_j$, and $r_{ij} = \sqrt{x_{ij}^2 + y_{ij}^2}$. Equation (13) consists of Cauchy singular integral equations for $i = 0$ (classical elasticity) and non-singular integral equations for $i = 1, 2$ (strain gradient elasticity).

Consider a plane with the following loadings:

$$\sigma_{yy} = \sigma_0, \sigma_{xx} = 0.$$

(15)

The graded plane in the absence of cracks under this loading is in a state of pure antiplane shear with the following stress field:

$$\sigma_{yy}(x, y) = \sigma_0.$$

(16)

Substituting the above stress field in Eq. (12), the traction in the location of cracks in the plane in the absence of cracks yields

$$\sigma_{yy}(x_j, y_j) = \sigma_0 \cos(\theta_j), \quad k \in \{1, 2, ..., N\}.$$

(17)

By virtue of Bueckner superposition principle, the traction in Eq. (17) with the opposite sign should be substituted in the left-hand side of Eq. (13).

Employing the definition of dislocation density function, the equation for the crack opening displacement across the $j$th crack becomes

$$w_j(s) = \int_{-1}^{1} \sqrt{[\alpha_j'(t)]^2 + [\beta_j'(t)]^2} B_j(t) dt.$$

(18)

Furthermore, the displacement field is single valued out of crack surfaces. Thus, the dislocation density for an embedded crack is subjected to the following closure requirement:

$$\int_{-1}^{1} \sqrt{[\alpha_j'(t)]^2 + [\beta_j'(t)]^2} B_j(t) dt = 0,$$

(19)

where $i = 0, 1, 2,$ and $j \in \{1, 2, ..., N\}$ is the crack number. To obtain the dislocation density for cracks, the integral Eqs. (13) and (19) are to be solved simultaneously. In a cracked plane, the dislocation density obtained from Eqs. (13) and (19) can be used to determine the stress distribution in the plane. To evaluate the stress distribution on a curve such as $\alpha_0(s), \beta_0(s)$, these coordinates and the obtained
densities from Eqs. (13) and (19) should be substituted into Eq. (13), which yields

\[\sigma_{\alpha\beta}(\alpha_0(s), \beta_0(s)) = \sum_{j=1}^{n} \int_{0}^{l} K(s, t, 0, j, 0) \sqrt{[\alpha_j(t)]^2 + [\beta_j(t)]^2} B_j(t) dt, \]

\[-1 \leq s \leq 1, i=0,1,2 \tag{20}\]

which should be superposed with stress fields (16). The system of integral Eqs. (13) and (19) is an ill-posed problem. In the case of classical elasticity, this system of equations is Cauchy singular, whereas in the cases of gradient elasticity, it is non-singular. In the following section, proper methods are selected to solve the singular and non-singular equations.

4 Solution of the integral equations

After a proper discretization of the integral equations (both the systems of singular and non-singular equations) together with its respective constraint integral equations, one obtains two sets of discrete linear systems of equations. Generally, such problems are difficult to solve accurately. A cure is to solve through the singular value decomposition (SVD) [29–31], which can handle such problems and thus obtain more stable/robust solutions. Details on the SVD can be found in textbooks on matrix analysis.

In the case of the singular system (classical elasticity), the Gauss-Chebyshev quadrature scheme developed by Erdogan et al. [32] is used to validate the solution obtained through the SVD (Appendix B).

One of the parameters commonly used to characterize the behavior of cracked materials is the SIF. In classical elasticity, it is shown that the stress is Cauchy singular in a graded material and the suitable SIF has been defined. In recent works on gradient elasticity theories, as the singularities of the stress components differ from the linear elastic fracture mechanics (LEFM), the definition of the SIF is modified. Paulino et al. [4] and Chan et al. [5] defined the SIF for hypersingular stress components for a mode-III crack problem. Gourgiotis and Georgiadis [22] used the same definitions as LEFM for a mode II crack problem, as they have singularity in the relevant stress field.

In the present work, the solutions to the dislocation and crack are non-singular for the first and second gradient theories. To analyze the crack tip, the stress distribution due to the presence of the crack is determined. In the following sections, the stress distribution in classical elasticity will be compared with that in gradient elasticity theories.

5 Numerical results and discussion

Two examples are provided to compare the results in classical and gradient elasticity.

5.1 Graded plane with a horizontal straight crack

The medium graded in the y-direction is assumed to be weakened by a horizontal crack (Figure 1A). The plane is under constant remote loading \(\sigma_0=\mu_0\). Two cases of gradation, including \(\gamma a=0.2\) and \(\gamma a=2\), are considered. The internal lengths are assumed to be \(l=0.01a\) and \(l'=l/5\).

The crack configuration is (Figure 1A)

\[\alpha_0(s) = a, -1<s<1, \]
\[\beta_0(s) = 0. \tag{21}\]

Using the DDT, Eqs. (13) and (19) are written for the current configuration. This system of integral equations, after proper discretization, is solved by means of SVD and the dislocation densities are determined for the classical, first gradient, and second gradient theories. To compare the prediction of the stress distribution in these theories, the vicinity of the crack tip on the x-axis with the following parametric form is under consideration:

\[x_0 = \alpha_0(s) = a(s+3)/2, -1<s<1, \]
\[y_0 = \beta_0(s) = 0. \tag{22}\]

Equation (20) may be simplified to determine the stress distribution in the case of one horizontal crack:

\[\sigma_{\alpha\beta}^{\prime}(\alpha_0(s), \beta_0(s)) = a \int_{0}^{l} K(s, t, 0, 1, 1) B_j(t) dt, \]

\[-1 \leq s \leq 1, i=0, 1, 2, \tag{23}\]

where \(\theta_j(s)=0\). Figure 1B shows the variation of the stress component \(\sigma_{\alpha}^{\prime}\), while \(i=0,1,2\) depict the classical, first gradient, and second gradient theories, respectively, versus position. According to the stress distribution in the classical framework \((\sigma_{\alpha}^{\prime})\), the singularity exists in the crack tip, while in the gradient theories, the classical singular behavior is regularized and, as was expected, there is no singularity in the crack tip. \(\sigma_{\alpha}^{\prime}\) and \(\sigma_{\alpha}^{\prime}\) show a peak at \(x=1.01a\). The difference between the stress components in the first and second gradient theories is trivial for \(x>1.01a\), while for \(x<1.01a\) we have \(\sigma_{\alpha}^{\prime}\). To analyze the effect of the gradation parameter \(\gamma\) on the stress components, \(\gamma a=0.2\) and \(\gamma a=2\) are considered. Figure 1C shows that a higher gradation parameter results in higher stress components in both the classical and first

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gradient theories, although in the vicinity of the crack tip, the location of the maximum stress is independent of the gradation parameter ($\gamma$).

Furthermore, to observe the effect of the gradient parameter ($l$) in first strain gradient elasticity, the normalized stress distribution is depicted for $l=0.01a$, 0.02$a$, and 0.05$a$ (Figure 1D). The gradation parameter of the plane is assumed to be $\gamma a=2$.

Comparison of the results in Figure 1D depicts that the gradient parameter in the first gradient elasticity ($l$) affects the peak value and the peak position of the stress component $\sigma_{yz}^1$. Higher values of $l$ result in a lower stress peak and a higher distance between the stress peak and crack tip (Figure 1D).

5.2 Graded plane with two horizontal cracks

In this example, a graded plane under constant remote loading $\sigma_{yz}=\mu_0$ is weakened by two equal-length straight
cracks (Figure 2A). Crack $L_1R_1$ centered at $-3a/2$ and crack $L_2R_2$ centered at $3a/2$ are assumed. The plane is under constant remote loading $\sigma_{yz} = \mu_0$.

Assuming the gradation parameter as $\gamma a = 2$ and the gradient parameters as $l = 0.01a$ and $l' = l/5$, the cracked plane in Figure 2A is analyzed using the DDT and the plane behavior is determined. Considering the symmetry with respect to the $y$-axis, Figure 2B depicts the normalized stress distribution at $(0 < x < a/2)$ on the $x$-axis. Similar to the case of one crack, singularity exists in the prediction of the classical theory, while the gradient strain theory does not include singularity in the crack tip. Furthermore, the first strain gradient theory predicts higher stress than the second strain gradient theory.

As the final case, two gradient parameters, $l' = l/5$ and $l' = l/2$, are considered, while $\gamma a = 2$ and $l = 0.01a$. Figure 2C depicts the effect of the parameter $l'$ on the stress distribution.

Figure 2C shows that in the second gradient theory, higher values of parameter $l'$ results in lower stress components in the vicinity of the crack tip, while predicting higher stress away from the crack tip. Additionally, lower values of $l'$ yield a higher stress peak near the crack tip.

6 Conclusions

The DDT is successfully applied to the dislocation solutions available in the literature. The general formulation is derived for the DDT to analyze the planes with various configurations of cracks in the classical, first strain gradient, and second strain gradient theories. Two examples of cracked graded plane show that the singular behavior of the crack tip is regularized in the gradient strain theories. The elimination of the singularity is an important aspect of such gradient elasticity framework, while in some non-classical frameworks the stress components exhibit singularity [4–6].

Furthermore, the second strain gradient theory predicts generally lower stress components in the vicinity of the crack tip than the first gradient theory. Owing to the generality of the present DDT formulation, it is capable of analyzing any other configuration as well as non-horizontal or curved cracks in the graded plane. This is helpful in the analysis of the interaction of planes containing multiple cracks in the gradient strain theories.

Finally, it is noted that an interesting and rather conclusive discussion on the origin and physical meaning of non-singular crack-tip solutions for both strains and
stresses can be found in a recent article by Aifantis [33]. The article addresses the concerns raised by Lazar and Polyzos [9], and sheds light to some controversial issues that have risen in gradient linear elastic fracture mechanics.

Appendix A: Screw dislocation in graded plane

In classical elasticity, the constitutive equation for graded materials is identical to the one for homogeneous materials. Contrary to this fact, in gradient elasticity, owing to the interaction of material gradation and the non-local effect of strain gradient, it has been shown that the constitutive equation for the graded material is different from that for homogeneous material [34]. For functionally graded materials, the corresponding constitutive equations in the case of antiplane shear in a Casal’s continuum [34] are

\[
\begin{align*}
\sigma_{xx} &= \sigma_{yy} = \sigma_{zz} = 0, \quad \sigma_{xy} = 0, \\
\sigma_{yx} &= 2G(y)(\epsilon_{xx} + \epsilon_{yy}) - 2l'^2(\partial_y G(y))(\partial_y \epsilon_{xx}) \\
&\quad + \mu_{xx}(\partial_x G(y))(\partial_x \epsilon_{xx}) \\
\sigma_{yz} &= 2G(y)(\epsilon_{xz} + \epsilon_{yz}) - 2l'^2(\partial_y G(y))(\partial_y \epsilon_{xz}) \\
&\quad + \mu_{xz}(\partial_x G(y))(\partial_x \epsilon_{xz}) \\
\mu_{yy} &= 2G(y)l'^2\epsilon_{xx}, \\
\mu_{yz} &= 2G(y)l'^2\epsilon_{xz}, \\
\mu_{xz} &= 2G(y)(l'^2\epsilon_{xz} - l'\epsilon_{yz}),
\end{align*}
\]

where \( G, l, \) and \( l' \) are the shear modulus, volume strain gradient term, and surface energy strain gradient term, respectively. \( \sigma_{ij} \) and \( \mu_{ijk} \) are the total stresses and couple stresses, respectively. In the present discussion, the terms \( l \) and \( l' \) are assumed to be constant for graded materials, whereas modulus \( G \) is a function of \( y \). Additionally, the non-trivial strains are

\[
\epsilon_{xx} = \frac{1}{2} \frac{\partial w}{\partial x}, \quad \epsilon_{yy} = \frac{1}{2} \frac{\partial w}{\partial y}, \quad \epsilon_{xz} = \frac{1}{2} \frac{\partial w}{\partial z}.
\]

The equilibrium equation (in the absence of body forces) reads

\[
\frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0,
\]

By substituting the expressions for stress and strain components [Eqs. (A1) and (A2)] in the equilibrium equation [Eq. (A3)] and assuming the material gradation as

\[
G(y) = \mu_0 e^{\gamma y},
\]

one obtains the following governing partial differential equation (PDE)

\[
\left( 1 - \gamma l' \frac{\partial}{\partial y} + \gamma \epsilon^2 \frac{\partial^2}{\partial y^2} \right) w = 0,
\]

while the two operators in the parenthesis in Eq. (A5) commute. Owing to the higher-order theory, solving the higher-order PDE for the dislocation in gradient elasticity needs more conditions than classical elasticity. Considering the antisymmetry of the problem, only the upper half-plane is considered. In the upper half-plane, the dislocation gives rise to the following boundary conditions:

\[
\begin{align*}
w(x, 0^+) &= \frac{b}{2} H(x), \\
\mu_{yy}(x, 0^+) &= 0,
\end{align*}
\]

where \( H(\ldots) \) is the Heaviside step function. The first condition is from the classical elasticity, while the second condition is for the higher-order theory. The second condition is inferred by a couple of investigations about mode III crack problems in non-local elasticity. An identical condition has been used by Paulino et al. [4] and Vardoulakis et al. [27] for the analysis a mode III crack problem in gradient elasticity theory. Georgiadis [16] has also applied this condition to a mode III crack problem in dipolar gradient elasticity. Gourgiotis and Georgiadis [22] have used a similar boundary condition for the dislocation solution in the couple stress elasticity.

The boundary value problem [Eqs. (A5) and (A6)] is solved with the aid of a complex Fourier transform, which is defined by

\[
f'(s) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{isx} dx.
\]

The inversion of Eq. (A7) is

\[
f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(s) e^{isx} ds.
\]

Using complex Fourier transform, the solution to the PDE [Eqs. (A5) and (A6)] reads

\[
w(x, y) = \frac{b}{4} e^{\gamma y} \rho(\gamma, 0) e^{\left(1 + \sqrt{1 + 4(\rho(\gamma, 0))^2}\right) y/2} \left( 1 + \rho(\gamma, 0) \right) \\
+ \frac{ib}{4\pi} \int_{-\infty}^{\infty} \frac{e^{-\gamma y} \rho(\gamma, s) e^{isx}}{s(1 + \rho(\gamma, s))} e^{isx} ds,
\]

where

\[
\rho(\gamma, s) = \frac{(l'^2 + l'^2)}{(l'^2 + l'^2)}.
\]
The total stresses $\sigma_{zx}$, $\sigma_{zy}$, and the couple stress $\mu_{yyz}$ in view of Eqs. (2), (3), (6), and (19) may be expressed as

$$\sigma_{zx} = g_0 e^{\gamma z} \frac{b_{mz} G_0}{4\pi} \int \left[ \left( 1 + s^2 \lambda_{1z}^2 \right) e^{i \lambda_{1z} (\gamma, \rho, \gamma, s)} \right] e^{-i\omega \gamma} \frac{1}{\rho(\gamma, s)} d\rho,$$

$$\sigma_{zy} = \frac{b_{mz} G_0}{4} \rho(\gamma, 0) e^{(n-1)\lambda_{1z}^2 / 2} \left[ - \frac{(\gamma + \sqrt{\gamma'^2 + 4 / F^2})}{2} - \frac{(\gamma + \sqrt{\gamma'^2 + 4 / F^2})^2}{4} - \frac{F(\gamma + \sqrt{\gamma'^2 + 4 / F^2})}{8} \right],$$

$$\mu_{yyz} = \frac{b_{mz} G_0}{4\pi} e^{\gamma y} \frac{b_{mz} G_0}{4} \rho(\gamma, 0) e^{(n-1)\lambda_{1z}^2 / 2} \left( \frac{(\gamma + \sqrt{\gamma'^2 + 4 / F^2})}{2} - \frac{(\gamma + \sqrt{\gamma'^2 + 4 / F^2})^2}{4} + \frac{F(\gamma + \sqrt{\gamma'^2 + 4 / F^2})}{8} \right).$$

It is observed that $\sigma_{zy}$ and $\mu_{yyz}$ do not have singularity in the vicinity of the dislocation.

**Appendix B: Gauss-Chebyshev quadrature (GCQ) scheme for classical elasticity**

In the framework of classical elasticity, the stress fields exhibit square-root singularity at an embedded crack tip, i.e., the kernels in the integral Eq. (13) are singular. Therefore, the dislocation density of embedded cracks is taken as

$$B_i(t) = \frac{g_i^0 (t)}{\sqrt{1-t^2}}, \quad -1 \leq t \leq 1. \quad (B1)$$

Substituting Eqs. (B1) and (17) into the system of Eqs. (13) and (19) results in

$$\psi(n, m, i) = \frac{1}{s(1 - \rho(\gamma, s))} \sum_{i=1}^{3} \lambda_{1z}^i e^{i\lambda_{1z}^i \psi(1,1,1)} \left( \psi(1,1,1) e^{i\lambda_{1z}^i (\gamma, \rho, \gamma, s)} \right) e^{-i\omega \gamma} \frac{1}{\rho(\gamma, s)} d\rho,$$

Substituting Eqs. (B1) and (17) into the system of Eqs. (13) and (19) results in

$$\psi(n, m, i) = \frac{1}{s(1 - \rho(\gamma, s))} \sum_{i=1}^{3} \lambda_{1z}^i e^{i\lambda_{1z}^i \psi(1,1,1)} \left( \psi(1,1,1) e^{i\lambda_{1z}^i (\gamma, \rho, \gamma, s)} \right) e^{-i\omega \gamma} \frac{1}{\rho(\gamma, s)} d\rho.$$
By means of the GCQ scheme developed by Erdogan et al. [32], Eq. (B2) is discretized into

\[
-\sigma_0 \cos(\theta_i) = \sum_{p=1}^{N} \int_{-1}^{1} K_0(s, t, k, j) \sqrt{\left[ \alpha'_p(t) \right]^2 + \left[ \beta'_p(t) \right]^2} \frac{g^0_j(t)}{\sqrt{1-t^2}} \, dt, \\
-1 \leq s \leq 1, k \in \{1, 2, \ldots, N\}, \\
\int_{-1}^{1} \sqrt{\left[ \alpha'_p(t) \right]^2 + \left[ \beta'_p(t) \right]^2} \frac{g^0_j(t)}{\sqrt{1-t^2}} \, dt = 0,
\]

(B2)

The system of \( N \times m \) linear algebraic equations [Eq. (B3)] can be solved to determine the \( N \times m \) unknowns \( g^0_j \). The dislocation density can be obtained by applying \( g^0_j \) to Eq. (B1).

\[
-\sigma_0 \cos(\theta_i) = \sum_{m=1}^{N} K_0(s, t, k, j) \sqrt{\left[ \alpha'_p(t) \right]^2 + \left[ \beta'_p(t) \right]^2} \frac{g^0_j(t)}{\sqrt{1-t^2}} = 0, \quad j \in \{1, 2, \ldots, N\},
\]

in which

\[
s_r = \cos \left( \frac{\pi r}{m} \right), \quad r = 1, 2, \ldots, m-1, \quad t_p = \cos \left( \frac{\pi (2p-1)}{2m} \right), \quad p = 1, 2, \ldots, m.
\]

(B4)

The SIFs for the \( j \)th embedded crack in terms of crack opening displacement is defined as [35]

\[
(k_m)_{x_j} = \sqrt{2} \lim_{\eta_j \to 0} \int_{r_j} \sigma_{z x_j} \left( \sin \frac{\gamma_j}{2} \right), \\
(k_m)_{y_j} = \sqrt{2} \lim_{\eta_j \to 0} \int_{r_j} \sigma_{z y_j} \left( \sin \frac{\gamma_j}{2} \right),
\]

(B5)

where \( \sigma_{z y_j} \) is the tangential stress component with respect to the crack surface (Figure 4).

The SIFs for the \( j \)th embedded crack in terms of crack opening displacement is defined as [35]

\[
(k_m)_{x_j} = \sqrt{2} \lim_{\eta_j \to 0} \int_{r_j} \frac{w_j(s) - w'_j(s)}{\sqrt{r_j}}, \\
(k_m)_{y_j} = \sqrt{2} \lim_{\eta_j \to 0} \int_{r_j} \frac{w_j(s) - w'_j(s)}{\sqrt{r_j}},
\]

(B6)
where the subscripts \( L \) and \( R \) designate the left and right crack tips, respectively, and the geometry of the crack implies

\[
\begin{align*}
L_j^{(s_0)} & = \sqrt{(\alpha_j(s) - \alpha_j(-1))^2 + (\beta_j(s) - \beta_j(-1))^2},

R_j^{(s_0)} & = \sqrt{(\alpha_j(s) - \alpha_j(1))^2 + (\beta_j(s) - \beta_j(1))^2}.
\end{align*}
\]

(B7)

Expanding functions \( \alpha(s) \) and \( \beta(s) \) by Taylor series in the vicinity of points \( s = \pm 1 \) leads to

\[
\begin{align*}
(k_{\alpha})_j & = \frac{\mu_0 y_j}{2} \left[ (\alpha'_j(-1))^2 + (\beta'_j(-1))^2 \right]^{\frac{1}{2}} g_j^{(1)}, \\
(k_{\beta})_j & = \frac{\mu_0 y_j}{2} \left[ (\alpha'_j(1))^2 + (\beta'_j(1))^2 \right]^{\frac{1}{2}} g_j^{(1)}.
\end{align*}
\]

(B8)

The solutions for \( g'(t_0) \) are plugged into Eq. (B8), thereby obtaining the SIFs in the classical elasticity.

References