Solutions to the ellipsoidal Clairaut constant and the inverse geodetic problem by numerical integration

Research article

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Abstract:
We derive computational formulas for determining the Clairaut constant, i.e. the cosine of the maximum latitude of the geodesic arc, from two given points on the oblate ellipsoid of revolution. In all cases the Clairaut constant is unique. The inverse geodetic problem on the ellipsoid is to determine the geodesic arc between and the azimuths of the arc at the given points. We present the solution for the fixed Clairaut constant. If the given points are not (nearly) antipodal, each azimuth and location of the geodesic is unique, while for the fixed points in the “antipodal region”, roughly within 36°±2 from the antipode, there are two geodesics mirrored in the equator and with complementary azimuths at each point. In the special case with the given points located at the poles of the ellipsoid, all meridians are geodesics. The special role played by the Clairaut constant and the numerical integration make this method different from others available in the literature.

Keywords:
clairaut constant • geodesic • inverse geodetic problem

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1. Introduction

The geodesic is of utmost interest in the classical adjustments of triangulation networks on the ellipsoid. However, it is useful also today in geographic information systems, e.g. for ships and airplane navigation. Also, precise application of the U.N.’s Convention on the Law of the Sea requires geodesic computations, e.g. for delimitation between adjacent states facing the sea.

Each geodesic has a unique Clairaut constant c, defined by the sine of the azimuth at the equator or cosine of the maximum latitude of the geodesic (see (3) below). Although the solution for this constant from known positions on the ellipsoid were discussed also by Helmert (1880, p. 329), Moritz (1959), Schmidt (2006a) and (2006b), Sjöberg (2006a), (2006b) and (2007), and Sjöberg and Shirazian (2012), the solution will now be treated more in detail for various cases.

Of particular interest will be the case when the given points with reduced latitudes β 1 and β 2 and longitudes λ 1 and λ 2 are located nearly or exactly antipodal, i.e. when the following condition is satisfied (Helmert 1880, p. 329; see also Appendix A.1)

\[
\frac{(\pi - \lambda_2 + \lambda_1)^{2/3} \cos(\beta_1)^{2/3}}{\pi \cos \beta_1} < \left(f \right)^{2/3} + O(f^2)
\]

where \(f\) is the flattening of the ellipsoid given by

\[
f = \frac{a - b}{a} = 1 - \sqrt{1 - e^2} \approx \frac{a^2}{2} \quad (1b)
\]

Here \(e\) is the first eccentricity of the ellipsoid. According to Helmert (ibid.), the geodesic is ambiguous for geodesics satisfying this inequality. A relevant question is whether this statement is true also for the Clairaut constant.

Once the Clairaut constant is known or determined for the geodesic between two given points, the solution to the inverse...
geodetic problem (i.e. to find the length of the geodesic arc between and its azimuths at the end points) can be solved for any pair of points on the ellipsoid. Hence, in contrast to well-known procedures, like those of Rainsford (1955) and Vincenty (1975), our solution of the inverse geodetic problem on the ellipsoid relies on that c has first been fixed, and the solutions include numerical integrations (instead of series expansions).

The article is organized as follows: In Ch. 2 we present the basic equations needed further on; Chs. 3 and 4 derive the geodesic arc length and the Clairaut constant in the general case with known points with \((\beta_1, \beta_2) \neq (0, 0)\) and not satisfying (1a), and Ch. 5 treats the general case with \(\beta_1 = \beta_2 = 0\). Chapter 6 considers the special cases, satisfying (1) with \((\beta_1, \beta_2) \neq (0, 0)\). The in-equality is derived in Appendix A.1. Ch. 7 solves the geodesic arc and the azimuths at the end points for a known Clairaut constant, and the conclusions are summarized in Ch. 8.

2. Basic formulas

The geodesic is the curve on an ellipsoid that provides the shortest distance between two specified points. Each geodesic on the ellipsoid is characterized by a unique Clairaut constant, which is given by Clairaut’s equation

\[ c = \cos \beta \sin \alpha. \]  

(2)

Here \(\alpha\) and \(\beta\) are the geodetic azimuth (counted from the north positive to the east) and reduced latitude at any point on the geodesic. The constant thus equals the sine of the azimuths \(\alpha_0\) and \(\pi - \alpha_0\) at the equator for the ascending and descending geodesics, respectively, and the cosines of \(\beta = \pm \beta_0\) at the northern/southern vertices (the northern/southernmost points of the geodesic), i.e.

\[ c = \sin \alpha_0 = \sin (\pi - \alpha_0) = \cos \beta_0 = \cos (-\beta_0). \]

(3)

Once the Clairaut constant for the geodesic that runs through two points on the ellipsoid is known (or has been fixed), the solutions to the longitude difference and the geodesic arc length between the points can be presented as definite integrals of the reduced latitude between the points. Each integral is divided to the closed form integral on the sphere, and an integral correction (denoted \(d\lambda\) and \(ds\) for the longitude and arc length, respectively) of the order of the eccentricity of the ellipsoid squared. The correction can be determined by standard formulas for numerical integration. This will be our strategy to solve the inverse geodetic problem on the ellipsoid (see Sect. 6).

Let us assume that the longitude difference between two points on the ellipsoid \(P_i (\beta_i, \lambda_i); i = 1, 2\) is \(0 \leq \lambda_2 - \lambda_1 = \Delta \lambda \leq \pi\), and that the reduced latitudes are related such that \(-\beta_0 \leq \beta_1 < \beta_2 \leq \beta_0\) or \(\beta_0 \leq \beta_1 > \beta_2 \geq -\beta_0\), where \(\beta_0 > 0\) denotes the latitude of the northern vertex of the geodesic connecting the two points. If \(\Delta \lambda = 0\) or \(\pi\), the geodesic runs along the meridian, implying that \(c = 0\). If \(0 < \Delta \lambda < \pi\), it holds that \(0 < c < 1\) and (Sjöberg 2006a; Sjöberg and Shirazian 2011)

\[ \pm \Delta \lambda = c \int_{\beta_1}^{\beta_2} \sqrt{1 - e^2 \cos^2 \beta} \frac{d\beta}{\cos \beta} = D\lambda + d\lambda \quad (4a) \]

where

\[ D\lambda = c \int_{\beta_1}^{\beta_2} \frac{d\beta}{\cos \beta \sqrt{\cos^2 \beta - c^2}} = \left[ \arcsin \frac{t_1}{t_0} \right]^{i=2}_{i=1} \quad (4b) \]

and

\[ d\lambda = c \int_{\beta_1}^{\beta_2} \frac{1 - e^2 \cos^2 \beta - 1}{\sqrt{\cos^2 \beta - c^2}} \frac{d\beta}{\cos \beta} \quad (4c) \]

where \(t_i = \tan \beta_i\), and \(e\) is the first eccentricity of the ellipsoid.

If the known points are given on adjacent sides of the northern vertex, one obtains

\[ D\lambda = c \int_{\beta_1}^{\beta_0} \frac{d\beta}{\cos \beta \sqrt{\cos^2 \beta - c^2}} + c \int_{\beta_0}^{\beta_2} \frac{d\beta}{\cos \beta \sqrt{\cos^2 \beta - c^2}} = \]

\[ = \pi - \arcsin \frac{t_1}{t_0} - \arcsin \frac{t_2}{t_0} \quad (5a) \]

and

\[ D\lambda = c \int_{\beta_1}^{\beta_0} \frac{\sqrt{1 - e^2 \cos^2 \beta} - 1}{\cos \beta} \frac{d\beta}{\sqrt{\cos^2 \beta - c^2}} + c \int_{\beta_0}^{\beta_2} \frac{\sqrt{1 - e^2 \cos^2 \beta} - 1}{\cos \beta} \frac{d\beta}{\sqrt{\cos^2 \beta - c^2}} \quad (5b) \]

and in case of adjacent sides of the southern vertex:

\[ D\lambda = c \int_{-\beta_1}^{\beta_0} \frac{d\beta}{\cos \beta \sqrt{\cos^2 \beta - c^2}} + c \int_{-\beta_0}^{\beta_2} \frac{d\beta}{\cos \beta \sqrt{\cos^2 \beta - c^2}} = \]

\[ = \pi + \arcsin \frac{t_1}{t_0} + \arcsin \frac{t_2}{t_0} \quad (5c) \]

and

\[ D\lambda = c \int_{-\beta_1}^{\beta_0} \frac{\sqrt{1 - e^2 \cos^2 \beta} - 1}{\cos \beta} \frac{d\beta}{\sqrt{\cos^2 \beta - c^2}} + c \int_{-\beta_0}^{\beta_2} \frac{\sqrt{1 - e^2 \cos^2 \beta} - 1}{\cos \beta} \frac{d\beta}{\sqrt{\cos^2 \beta - c^2}} \quad (5d) \]

In (5a) and (5c) we have used that \(\arcsin(1) = \pi/2\).

Applying (4a) and (4b) the longitude difference between the southern and northern vertices becomes:

\[ \Delta \lambda = c \int_{\beta_0}^{\beta_0} \frac{1 - e^2 \cos^2 \beta}{\cos \beta \sqrt{\cos^2 \beta - c^2}} \frac{d\beta}{\cos \beta} = \pi + d\lambda, \quad (6a) \]
where
\[ d\lambda = 2c e^2 \int_{\beta_1}^{\beta_2} \frac{1 - e^2 \cos^2 \beta - 1}{\sqrt{\cos^2 \beta - c^2}} \cos \beta \, d\beta \approx \frac{ce^2 \pi}{2}. \] (6b)

This shows that the longitude difference between the northern and southern vertices depends on \( e \) equals \( \pi \) only for \( e = 0 \) and is otherwise somewhat shorter than \( \pi \). This implies that for \( e \neq 0 \), the geodesic does not have a closed envelope around the Earth, but its period is somewhat shorter than \( 2\pi \).

3. The geodesic arc length

In the first case above, corresponding to (4a) – (4c), the arc length along the geodesic between the known points is given by (Sjöberg 2006)

\[ \pm s = a \int_{\beta_1}^{\beta_2} \sqrt{1 - e^2 \cos^2 \beta} \cos \beta \, d\beta = \]
\[ = a \left[ \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}} \right]_{\beta_1}^{\beta_2} + ds \] (7a)

where
\[ ds = a \int_{\beta_1}^{\beta_2} \frac{1 - e^2 \cos^2 \beta}{\sqrt{\cos^2 \beta - c^2}} \cos \beta \, d\beta \approx \]
\[ \approx \frac{ae^2}{2} \int_{\beta_1}^{\beta_2} \frac{1 - x^2}{\sqrt{1 - x^2}} \, dx. \] (7b)

In case \( P_1 \) and \( P_2 \) are located on adjacent sides of the northern vertex, the equation becomes

\[ s = a \int_{\beta_1}^{\beta_2} \sqrt{1 - e^2 \cos^2 \beta} \cos \beta \, d\beta = \]
\[ + a \int_{\beta_1}^{\beta_2} \frac{1 - e^2 \cos^2 \beta}{\sqrt{\cos^2 \beta - c^2}} \cos \beta \, d\beta = \]
\[ = a \pi - a \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}} - a \arcsin \frac{\sin \beta_2}{\sqrt{1 - c^2}} + ds \] (8a)

where
\[ ds = a \sum_{i=1}^{2} \int_{\beta_i}^{\beta_2} \frac{1 - e^2 \cos^2 \beta}{\sqrt{\cos^2 \beta - c^2}} \cos \beta \, d\beta. \] (8b)

In case of the known points located on adjacent sides of the southern vertex the arc length becomes

\[ s = a \int_{\beta_1}^{\beta_2} \sqrt{1 - e^2 \cos^2 \beta} \cos \beta \, d\beta = \]
\[ + a \int_{\beta_1}^{\beta_2} \frac{1 - e^2 \cos^2 \beta}{\sqrt{\cos^2 \beta - c^2}} \cos \beta \, d\beta = \]
\[ = a \pi + a \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}} + a \arcsin \frac{\sin \beta_2}{\sqrt{1 - c^2}} + ds \] (8c)

These formulas will be useful in the derivations that follow.

4. The Clairaut constant in the general case with \( (\beta_1, \beta_2) \neq (0, 0) \)

The general case assumes that (1a) is not satisfied. Assuming also that \( (\beta_1, \beta_2) \neq (0, 0) \) Proposition 4.1 below provides the solution for the Clairaut constant.

**Proposition 4.1.**

If (1a) is not satisfied and \( (\beta_1, \beta_2) \neq (0, 0) \), then

\[ \cos D\lambda = \cos (A_1 - A_2) = \cos A_1 \cos A_2 + \sin A_1 \sin A_2 \] (9a)

where
\[ \sin A_i = t_i / t_0, t_i = \tan \beta_i; i = 1, 2 \text{ and } t_0 = \cos \beta_0 \] (9b)

Hence,
\[ \cos D\lambda - \frac{t_1 t_2}{t_0^2} = \sqrt{1 - \left( \frac{t_1}{t_0} \right)^2} \sqrt{1 - \left( \frac{t_2}{t_0} \right)^2} \] (10a)

which expression after squaring and a few manipulations yield
\[ t_0^2 \left[ (t_0^2 \sin^2 D\lambda + 2t_1 t_2 \cos D\lambda - t_1^2 - t_2^2) \right] = 0. \] (10b)

Here \( t_0 = 0 \) is not a solution, because \( (\beta_1, \beta_2) \neq (0, 0) \). Hence, if \( D\lambda \neq \pi \), the only solutions for \( t_0 \) becomes:

\[ t_0 = \frac{\sqrt{t_1^2 + t_2^2 - 2t_1 t_2 \cos D\lambda}}{\sin D\lambda}, \] (11)
and the solution for $c$ can be written

$$c = \frac{1}{\sqrt{1 + t_0^2}} = \frac{\sin D\lambda}{\sqrt{\sin^2(D\lambda/2) + t_1^2 + t_2^2 - 2t_1t_2 \cos D\lambda}}.$$  

(12)

ii) Consider now the case where $P_1$ and $P_2$ are located on different sides of the vertex. Then (5a) and (5c) yield the following equations

$$D\lambda = \pi \mp (A_1 + A_2),$$

(13a)

or

$$\cos D\lambda = - \cos (A_1 + A_2),$$

(13b)

where $A_1$ are the same as above and the minus/plus sign in (13a) refers to the northern/southern vertex, respectively. Carrying out manipulations similar to those for (9a), one arrives once again at (11) and (12).

\[ \textbf{Corollary 4.1.} \]

Alternative forms for the Clairaut constant are

$$c = \frac{\cos (D\lambda/2)}{\sqrt{\cos^2(D\lambda/2) + (t_1 - t_2)/(4 \sin (D\lambda/2))^2 + t_1t_2}},$$

(14a)

and

$$c = \frac{\sin (D\lambda/2)}{\sqrt{\sin^2(D\lambda/2) + (t_1 + t_2)/(4 \cos (D\lambda/2))^2 - t_1t_2}}.$$  

(14b)

The proof follows by inserting the formulas

$$\sin D\lambda = 2 \sin \frac{D\lambda}{2} \cos \frac{D\lambda}{2}, 1 - \cos D\lambda = 2 \sin^2 \frac{D\lambda}{2}$$

and

$$1 + \cos D\lambda = 2 \cos^2 \frac{D\lambda}{2}$$

(15)

into (12).

\[ \textbf{Corollary 4.2.} \]

If $B_2 = -B_1$, then

$$c = \frac{\cos (D\lambda/2)}{\sqrt{\cos^2(D\lambda/2) + t_1^2}}.$$  

(16)

The proof follows directly from (14a).

\[ \textbf{Corollary 4.3.} \]

If $B_2 = B_1$, and given points are located on adjacent sides of the vertex, then

$$c = \frac{\cos (D\lambda/2)}{\sqrt{\cos^2(D\lambda/2) + t_1^2}}.$$  

(17)

The proof follows directly from (14a). The application of the proposition requires that $D\lambda$ is known. This is achieved from the relation

$$D\lambda = \Delta \lambda - d\lambda,$$

(18)

where $d\lambda$, of order $e_\epsilon$, must be iterated together with the solution of $c$ from the proposition. $d\lambda$ can either be numerically integrated directly from (4b), (5b) and (5d), or from the refined formulas of Appendix A.2.

5. Solutions with $(B_1, B_2) = 0$

If the known points are located on the equator, (4a), (5a) and (5b) can be modified to

$$\Delta \lambda = D\lambda + d\lambda,$$

(19a)

where

$$D\lambda = 2c \int_{0}^{e_0} \frac{d\beta}{\sqrt{\cos^2 \beta - c^2 \cos \beta}} = \pi - 2 \arcsin \frac{t_0}{e_0},$$

(19b)

and

$$d\lambda = 2c \int_{0}^{e_0} \frac{\sqrt{1 - e^2 \cos^2 \beta} - 1}{\sqrt{\cos^2 \beta - 1}} d\beta.$$  

(19c)

These equations are valid as long as $e_0 > 0$ (or $e_0 < 0$). This includes the limiting case $\Delta \lambda = \pi$ whence the geodesic runs along the meridian with $c = 0$. On the other hand, for small $\Delta \lambda$ the geodesic runs along the equator (i.e. $c = 1$), and the above equations do not hold. Hence, if $\Delta \lambda$ is close to $\pi$, (18) yields that $D\lambda = \pi$, and $c$ should therefore vary between 0 and 1, and there must also be a limiting longitude difference $e$ away from $\pi$ for which $c$ approaches 1 and $\pi(0 \to 0)$. According to (1), discussed by Helmert (1881, p.319), $e \approx \pi f$, and Schmidt (2006a), (2006b) and Sjöberg (2006b) and (2007) presented approximate values for $e$ to about 30' of arc. Here we will derive an exact solution for $c$ in two ways.

\[ \textbf{Proposition 5.1.} \]

$$c = \pi \left( 1 - \sqrt{1 - e^2} \right) = \pi f$$
Proof. Let us assume that \( \Delta \lambda \) differs only by a small value \( \hat{\epsilon} \) from \( \pi \), i.e.

\[
\hat{\epsilon} = \pi - \Delta \lambda = -d\lambda.
\] (20)

Using the substitutions of (20),

\[
f(\beta) = \frac{c}{\sqrt{\cos^2 \beta - c^2 \cos \beta}},
\]

\[
F(\beta) = \int f(\beta) \, d\beta = \arcsin \left( \frac{\tan \beta}{t_0} \right)
\] (21a)

and

\[
G(\beta) = \sqrt{1 - e^2 \cos^2 \beta} - 1,
\]

\[
g(\beta) = \frac{dG(\beta)}{d\beta} = e^2 \frac{\sin \beta \cos \beta}{\sqrt{1 - e^2 \cos^2 \beta}}.
\] (21b)

(19c) can be evaluated by partial integration to

\[
-\hat{\epsilon} = 2F(\beta) G(\beta)\bigg|_{\beta=0}^{\beta=\beta_0} - 2 \int_{\beta=0}^{\beta=\beta_0} F(\beta) \, g(\beta) \, d\beta,
\] (22a)

or

\[
-\hat{\epsilon} = 2 \left[ \arcsin \left( \frac{\tan \beta}{t_0} \right) \left\{ \sqrt{1 - e^2 \cos^2 \beta} - 1 \right\} \right]_{\beta=0}^{\beta=\beta_0}
- 2 \int_{\beta=0}^{\beta=\beta_0} \arcsin \left( \frac{\tan \beta}{t_0} \right) \, g(\beta) \, d\beta,
\] (22b)

As the integrand in (22b) is always finite, the integral vanishes as \( \beta_0 \) approaches zero. Hence,

\[
\hat{\epsilon} = \lim_{\beta_0 \to 0} \arcsin \left( \frac{\tan \beta}{t_0} \right) \left\{ \sqrt{1 - e^2 \cos^2 \beta} - 1 \right\}
= \pi f, \text{ as } \beta_0 \to 0,
\] (23)

and the proposition is proved.

An alternative proof of Proposition 5.1

Proof. The arc-length along the equator from \( \rho_1 \) to \( \rho_2 \) is \( a\Delta \lambda = a(\lambda_2 - \lambda_1) \). The arc length of the geodesic is obtained by (7a) and (7b), yielding

\[
s = 2a \int_{\beta=0}^{\beta=\beta_0} \sqrt{\frac{1 - e^2 \cos^2 \beta}{\cos^2 \beta - c^2}} \cos \beta \, d\beta = a \pi + ds,
\] (24a)

where

\[
ds = 2a \int_{\beta=0}^{\beta=\beta_0} \sqrt{\frac{1 - e^2 \cos^2 \beta}{\cos^2 \beta - c^2}} - 1 \cos \beta \, d\beta.
\] (24b)

By partial integration, with \( f(\beta) \) and its integral \( F(\beta) \) [as in (18) and (19a)], as well as

\[
H(\beta) = \frac{\sqrt{1 - e^2 \cos^2 \beta} - 1}{\cos \beta - e},
\]

one obtains from (24b):

\[
ds = 2a \frac{\pi}{c} \int_{\beta=0}^{\beta=\beta_0} F(\beta) \, g(\beta) \, d\beta.
\] (26)

In the limit \( \beta_0 \to 0 \), \( c \) goes to unity, the last integral vanishes and

\[
ds \to -a \varepsilon
\] (27)

where \( \varepsilon \) was defined in the proposition. Also, as in the limit the length of the geodesic between the given points is the same as the length along the equator, it holds that

\[
a\Delta \lambda = a\pi - a\varepsilon,
\] (28)

Hence, this is the maximum longitude difference at which the geodesic is along the equator.

**Corollary 5.1.**

\[
\varepsilon = \pi f = \frac{\pi e^2}{1 + \sqrt{1 - e^2}}
\]

**Proof.** The corollary follows immediately upon multiplying the right side of the proposition by \( \left( 1 + \sqrt{1 - e^2} \right) / \left( 1 + \sqrt{1 - e^2} \right) \).

Using \( e' = 0.00669438002290 \) according to Geodetic System 1980 (Moritz 2000, p. 131), one obtains \( \varepsilon = 36'12'' .6213 \), which result differs by about 1 arc second with Schmidt (2006a) and (2006b), obtained by a numerical integration procedure. The various cases for the Clairaut constant with the known points located on the equator can now be summarized in the following proposition.

**Proposition 5.2.**

Let \( \beta_1 = \beta_2 = 0 \).

a) If \( \Delta \lambda \leq \pi (1 - f) \), then \( c = 1 \).
b) If \( \varpi (1 - F) \leq \Delta \lambda < \pi \), then
\[
c = \frac{2 (1 - \Delta \lambda / \pi)}{e^2 [1 + q (\beta_0)]},
\tag{29a}
\]
where
\[
qu (\beta_0) = \frac{2}{\pi} \int_0^{\beta_0} \frac{\cos \beta}{\sqrt{\cos^2 \beta - c^2}} \left[ \frac{1}{1 + \sqrt{1 - e^2 \cos^2 \beta}} \right] d\beta \approx \frac{e^2 c^2}{4} \tag{29b}
\]
c) If \( \Delta \lambda = \pi \), then \( c = 0 \).

**Proof.** The proofs in cases a) and c) were already discussed above. Case b) is proved as follows: For a finite \( \beta_0 \), with \( \Delta \lambda \) close to \( \pi \), one obtains from (18) that \( \Delta \lambda = \pi - \hat{\epsilon} \), where \( \hat{\epsilon} \) is given by (16c). Thus one obtains:
\[
\pi - \Delta \lambda = \hat{\epsilon} = 2c \int_0^{\beta_0} \frac{1 - \sqrt{1 - e^2 \cos^2 \beta}}{\cos^2 \beta - c^2} d\beta
\tag{30}
\]
Here \( \hat{\epsilon} \) can be rewritten as
\[
\hat{\epsilon} = 2c \int_0^{\beta_0} \frac{\cos \beta}{\sqrt{\cos^2 \beta - c^2}} \left[ \frac{d\beta}{1 + \sqrt{1 - e^2 \cos^2 \beta}} \right] = \hat{\epsilon}_1 + \hat{\epsilon}_2
\tag{31a}
\]
where
\[
\hat{\epsilon}_1 = ce^2 \int_0^{\beta_0} \frac{\cos \beta}{\sqrt{\cos^2 \beta - c^2}} d\beta = ce^2 \arcsin(1) = ce^2 \pi / 2
\tag{31b}
\]
and
\[
\hat{\epsilon}_2 = \frac{ce^2 \pi}{2} q (\beta_0) = \frac{ce^2}{2} \int_0^{\beta_0} \frac{d\beta}{\sqrt{\cos^2 \beta - c^2}} \left[ \frac{1 - \sqrt{1 - e^2 \cos^2 \beta}}{1 + \sqrt{1 - e^2 \cos^2 \beta}} \right] \approx \frac{ce^2}{4} \int_0^{\beta_0} \frac{\cos^3 \beta d\beta}{\sqrt{\cos^2 \beta - c^2}} \approx \frac{ce^4 \pi}{8} \tag{31c}
\]
and thereby Case b) of the proposition follows. Note that cases b) and c) of the proposition satisfy (1a), and the two ambiguous geodesics run on opposite sides of the equator. The solution for Case b) needs to be iterated to convergence. In all these cases the Clairaut constant is unique. \( \square \)

6. Solutions for nearly antipodal points with \( (\beta_1, \beta_2) \neq (0, 0) \)

In the previous section we showed that for the given points located on the equator close to their antipodes, the solution to Clairaut’s constant is unique, but the geodesic is not, as there are two solutions symmetrical around the equator. The question we ask now is what happens to the solution if \( \beta_1 \) and \( \beta_2 \) are slightly removed from the equator? If \( \beta_1 + \beta_2 = x \) and \( \pi - \Delta \lambda \) are both of order \( e^2 \), Prop. 1 becomes
\[
c \approx \frac{\pi - \Delta \lambda + d\lambda}{\sqrt{\pi - \Delta \lambda + d\lambda}^2 (1 + \beta_1^2) + x^2}, \tag{32a}
\]
where all terms are of order \( e^2 \), and an iterative solution for \( c \) would not converge well. From (6b) and (A23) we notice that
\[
d\lambda \approx -e^2 \pi c / 2, \text{ and by slightly reformulating (32a) one obtains}
\]
\[
c_{k+1} \approx \frac{1}{\sqrt{1 + \beta_1^2} + [x/(\pi - \Delta \lambda + d\lambda)]^2}, \tag{32b}
\]
where, from (6b) or (A23),
\[
d\lambda_k \approx -e^2 \pi c_k / 2. \tag{32c}
\]

As we let \( c_k \) increase from 0 to 1 in the last equation, \( c_{k+1} \) in (32b) will be monotonically decreasing, implying that \( c_{k+1} = c_k \) will be a unique solution for \( c \) (to order \( e^2 \)). (The above reasoning does not hold for \( c = 0 \), in which case \( D\lambda = \pi \) and (32a) is not correct.) In order to obtain an exact solution for \( c \) we will now consider \( D\lambda \) for the known points located on adjacent sides of the northern and southern vertices, respectively. (Note that only these cases are of interest for nearly antipodal points.) In this case \( D\lambda \) can be written
\[
D\lambda = \pi - \arcsin \left( \frac{t_1}{\beta_0} + \arcsin \frac{t_2}{\beta_0} \right), \tag{33}
\]
and from (A15a) and (??) we obtain
\[
- d\lambda = D\lambda - \Delta \lambda = - \frac{e^2 c}{2} \left[ \pi - \arcsin \frac{s_1}{s_0} + \arcsin \frac{s_2}{s_0} \right] + e^2 h (\beta_0, \beta_1, \beta_2), \tag{34}
\]
and, after inserting (33),
\[
c = \frac{2}{e^2} \frac{\pi - \Delta \lambda - \arcsin \frac{t_1}{\beta_0} + \arcsin \frac{t_2}{\beta_0}}{\arcsin \frac{s_1}{s_0} + \arcsin \frac{s_2}{s_0} + e^2 h (\beta_0, \beta_1, \beta_2)}, \tag{35a}
\]
where (see (A2) for details)

\[
h (\beta_0, \beta_1, \beta_2) \approx \frac{3 - c^2}{2} \left( \pi - \arcsin \frac{s_1}{s_0} + \arcsin \frac{s_2}{s_0} \right) 
\]

\[
\mp \left( \frac{s_1}{2} \sqrt{s_0^2 - s_1^2} + \frac{s_2}{2} \sqrt{s_0^2 - s_2^2} \right).
\]  \tag{35b}

Here the minus and plus signs apply to the cases with northern and southern vertices, respectively. Also here \( c \) must be iterated. As \( D\lambda \) is close to \( \pi \), we may get a starting point for \( c \) by setting \( \beta_2 = -\beta_1 \) in (35a) and omitting \( h (\beta_0, \beta_1, \beta_2) \), which yields

\[
c \approx 2 \frac{\pi - \Delta\lambda}{\pi}.
\]  \tag{36}

(Notice that (35a) and (36) also hold for \( \beta_1 = \beta_2 = 0 \).) Alternatively, an approximation for \( c \) is obtained from the adjustment of (32b). As a result of the iteration both \( c \) and \( d\lambda \) are obtained.

7. Determination of the azimuth and arc length of the geodesic

Once we know the Clairaut constant and the coordinates of the end points of the geodesic arc we may determine the azimuths of the geodesic at the end points from (2):

\[
\sin \alpha_l = c/\cos \beta_l \Rightarrow \\
\alpha_l = \arcsin (c/\cos \beta_l) \text{ and } \alpha_0 = \pi - \arcsin (c/\cos \beta_l).
\]  \tag{37}

There are obviously two solutions to (2), and the correct azimuth depends on whether the arc length is related with the northern or southern vertex or both. The northern vertex applies if \( \beta_1 + \beta_2 > 0 \), otherwise the southern vertex applies. Only if the given points have the same latitudes but with opposite signs, both azimuths of (37) are correct, and there are two equally long geodesic arcs. This includes also the case when both given points are within the AR and have latitudes zero.

The arc length can be determined by the following three steps by taking advantage of computations on the auxiliary sphere (Sjöberg and Shirazian 2012; cf. Helmert 1880, Ch. 5 and Bessel 1826):

1. Determine the longitude difference \( \Delta\lambda \) on the auxiliary sphere as

\[
\Delta\lambda = \lambda_2 - \lambda_1.
\]

2. The geocentric angle \( \psi \) between the given points located on the same and adjacent sides of a vertex are given by (A19a) and (A19b), respectively:

\[
\psi = \theta_2 - \theta_1 = \arcsin \frac{\sin \beta_2}{\sqrt{1 - c^2}} - \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}}
\]  \tag{39}

and

\[
\psi = \pi - \arcsin \frac{\sin \beta_2}{\sqrt{1 - c^2}} + \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}}.
\]  \tag{40}

3. From Sjöberg and Shirazian (2012) one obtains

\[
s = \frac{1}{8} \left[ 1 + \frac{k^2}{2} \right] \psi + \frac{k^2}{4} \sin 2\gamma_1 \sin 2\gamma - \frac{\sin^4 \gamma}{1 + \frac{1 + k^2}{2} \sin^2 \gamma} d\theta,
\]  \tag{41}

where

\[
b = a \sqrt{1 - e^2}, k^2 = e^2 \left( 1 - e^2 \right),
\]

\[
\gamma_1 = \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}} \text{ and } \gamma = \gamma (\theta) = \gamma_1 + \theta.
\]  \tag{42}

Here the integral (of order \( e^4 \)) is most conveniently solved by numerical integration.

8. Concluding remarks

This study presents computational formulas for determining the Clairaut constant from two given points on the oblate ellipsoid of revolution. If the given points are not in the “antipodal region (AR)” governed by (1), the iterative formula of Proposition 4.1 solves the constant. If the given points are in the AR, the alternative iterative procedures described in Prop. 5.1 (for given points on the equator) and in Sect. 6 (for all other cases with nearly antipodal points). In all cases the Clairaut constant is unique. Once the Clairaut constant is determined, the length of the geodesic between and its azimuths at the fixed points are easily determined. Outside the AR the location of the geodesic is unique, while for the fixed points within the AR there are generally dual geodesics mirrored in the equator, i.e. the azimuths are complementary [arcsin (c/\cos \beta_l) and \( \pi - \arcsin (c/\cos \beta_l) \)] at each point. The only exception in the AR is for the fixed points located at the poles of the ellipsoid, in which case all meridians are geodesics.

Our technique to solve the inverse geodetic problem differs from other solutions found in the literature mainly from the central role played by the Clairaut constant. Once the constant is fixed, the solution is straightforward. In this way we avoid the difficulties to find the solutions in the AR, as reported e.g. in Vincenty (1975). Notable is also that the small corrections from spherical to ellipsoidal solutions for longitude difference and geodesic distance are provided by standard numerical integrations instead of being utilized from specially derived truncated series in ellipsoidal flattening or eccentricity.
Below we derive the envelope. According to Proposition 4.1, will fail in this region, where $D\lambda$ is equal to or near to $\pi$. Below we derive the envelope.

**Lemma A.1.**

Arclength

$$\arcsin(t/t_0) = \arctan \left( \frac{cs_i}{\sqrt{c_i^2 - c^2}} \right). \quad (A1a)$$

where

$$t = \tan \beta_i, \ i = \sin \beta_i, \ c_i = \cos \beta_i, \ t_0 = \tan \beta_0. \quad (A1b)$$

**Proof.** From $\int \frac{dx}{\sqrt{c^2 - y^2}} = \arcsin \frac{y}{\sqrt{c^2 - y^2}}$ with $y = t_i$ and $a = t_0$ the lemma follows.

**Lemma A.2.**

If $\beta_2 = -\beta_1 - x$ and $x$ is small, it holds to first order that

$$t_2 \approx t_1 - \frac{x}{c_1} \quad (A2)$$

**Proof.** It hold to first order that $t_2 \approx -\frac{-t_i - x}{t_0 - t_1} \approx -t_1 - \frac{x}{c_1}$. 

**Lemma A.3.**

Let both $\pi - D\lambda$ and $x$ defined as above, be of order $e^2$ (implying that the two given points are close to their antipodes). Then it holds (to order $e^2$) that

$$\pi - D\lambda \approx -\frac{x c}{c_1 \sqrt{c_i^2 - c^2}} \quad (A3a)$$

or

$$D\lambda \approx \pi + \frac{x c}{c_1 \sqrt{c_i^2 - c^2}} \quad (A3b)$$

**Proof.** From (5a) follows that

$$I = \sin(\pi - D\lambda) = t_1 \frac{t_i}{t_0} \left[ 1 - \frac{t_i^2}{t_0^2} + \frac{t_i}{t_0} \sqrt{1 - \frac{t_i^2}{t_0^2}} \right]. \quad (A4)$$

and by inserting (A2) one obtains after some first order manipulations that

$$I \approx -\frac{x c}{c_1 \sqrt{c_i^2 - c^2}} \quad (A5)$$

As $\pi - D\lambda$ is small, the lemma follows from $I \approx \pi - D\lambda$ and (A5).

**Note.** As the spherical latitude difference $D\lambda$ must not exceed $\pi$, it follows that $x < 0$. 

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**References**


Schmidt H., 2006a, Berechnung geodätischer Linien auf dem Rotationsellipsoid im Grenzbereich diametraler Endpunkten, ZfV, 124, 121-128.


**Appendix A: APPENDIX**

**A.1 Estimation of the envelope of the Antipodal Region (AR)**

The Antipodal envelope Region (AR) is defined by the curve, where the derivative of $D\lambda$ w.r.t. the Clairaut constant vanishes or is close to zero. This means that an iterative method, using e.g. Newton-Raphson’s method, to determine the Clairaut constant from Proposition 4.1, will fail in this region, where $D\lambda$ is equal to or near to $\pi$. Below we derive the envelope.

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Lemma A.4.

\[ \Delta \lambda \approx \pi + \frac{xc}{c_1 \sqrt{c_1^2 - c^2}} - \frac{e^2 \pi c}{2}. \quad (A6) \]

Proof. From (4a) and (6b) follows that

\[ \Delta \lambda \approx D\lambda - e^2 c \pi/2, \quad (A7) \]

and by inserting \( D\lambda \) from (A3b) the lemma follows. \( \square \)

Proposition A.1.

If the derivative of \( \Delta \lambda \) with respect to \( c \) vanishes, then

\[ (\beta_1 + \beta_2)^{2/3} + \left( \pi - \Delta \lambda \right) c_1^{2/3} \approx \left( \frac{e^2 \pi c}{2} \right)^{2/3}. \quad (A8) \]

Proof. The derivative of (A6) with respect to \( c \) is

\[ \frac{d\Delta \lambda}{dc} \approx \frac{xc_1}{\sqrt{(c_1^2 - c^2)^3}} - \frac{e^2 \pi c}{2}, \quad (A9) \]

and by equating it to zero one obtains the following solutions for the Clairaut constant:

\[ c \approx \pm \sqrt{c_1^2 - \left( \frac{2xc_1}{e^2 \pi} \right)^{2/3}}. \quad (A10) \]

Alternatively, the last equation can be written in the form

\[ x = \frac{e^2 \pi}{2c_1} \sqrt{(c_1^2 - c^2)^3}, \quad (A11) \]

and by inserting it into (A6) one obtains

\[ \Delta \lambda \approx \pi - \frac{e^2 \pi c^3}{2c_1}. \quad (A12) \]

Finally, by inserting (A10) into the last equation and considering that \( -x \) equals \( \beta_1 + \beta_2 \) one obtains the proposition. \( \square \)

Note. The proposition yields exactly the limiting equation for ambiguous solutions of the geodesic presented by Helmert (1881); see (1).

A.2 Longitude increment \( d\lambda \) in the AR

Here we derive different expressions for the longitude difference \( d\lambda \) between the ellipsoidal and spherical longitudes, i.e.

\[ d\lambda = \Delta \lambda - D\lambda, \quad (A13) \]

as functions of \( \beta_1 \) and \( \beta_2 \). The resulting expressions differ for both given points located on one side of the vertex (Case 1) and for the given points located on either side of the northern and southern vertices (Cases 2 and 3, respectively).

Case 1) For both given points located on the same side of the vertex (4c) yields

\[ d\lambda = c \int_{\beta_1}^{\beta_2} \frac{1 - e^2 \cos^2 \beta - 1}{\sqrt{e^2 \cos^2 \beta - c^2}} \frac{d\beta}{\cos \beta} = -c^2 e^2 \int_{\beta_1}^{\beta_2} \frac{\cos \beta}{\sqrt{e^2 \cos^2 \beta - c^2}} \frac{d\beta}{\sqrt{1 + \sqrt{1 - e^2 \cos^2 \beta}}} = -\frac{c^2 e^2}{2} \left[ \arcsin \frac{s_2}{s_0} - \arcsin \frac{s_1}{s_0} + e^2 \int_{s_1}^{s_2} f(x, c) dx \right] \quad (A14a) \]

where

\[ f(x, c) = \frac{1 - x^2}{\sqrt{1 - c^2 - x^2} \left[ 1 + \sqrt{1 + e^2 (1-x^2)} \right]} \approx \frac{1 - x^2}{4 \sqrt{1 - c^2 - x^2}}. \quad (A14b) \]

\[ s_i = \sin \beta_i \text{ and } s_0 = \sqrt{1 - c^2}. \]

Case 2) Northern vertex: From (5b) one obtains

\[ d\lambda = -\frac{e^2 c}{2} \left[ \left( \pi - \arcsin \frac{s_1}{s_0} - \arcsin \frac{s_2}{s_0} \right) + e^2 h_n(\beta_0, \beta_1, \beta_2) \right], \quad (A15a) \]

where

\[ h_n(\beta_0, \beta_1, \beta_2) = \sum_{i=1}^{2} \int_{s_i}^{s_0} f(x, c) dx \approx \frac{3 - c^2}{2} \left( \pi - \arcsin \frac{s_1}{s_0} - \arcsin \frac{s_2}{s_0} \right) - \frac{s_1}{2} \sqrt{s_0^2 - s_1^2} - \frac{s_2}{2} \sqrt{s_0^2 - s_2^2}. \quad (A15b) \]

Case 3) Southern vertex:

\[ d\lambda = -\frac{e^2 c}{2} \left[ \left( \pi + \arcsin \frac{s_1}{s_0} + \arcsin \frac{s_2}{s_0} \right) + e^2 h_s(\beta_0, \beta_1, \beta_2) \right], \quad (A16a) \]
where
\[ h_1(x, c) = \sum_{i=1}^{n} f(x, c) dx \]  
(A16b)

\[ \frac{3 - c^2}{2} \left( \pi + \arcsin \frac{s_1}{s_0} + \arcsin \frac{s_2}{s_0} \right) + \frac{s_1}{2} \sqrt{s_0^2 - s_1^2} + \frac{s_2}{2} \sqrt{s_0^2 - s_2^2}. \]  
(A16c)

Note. For Case 2 (northern vertex) holds: \( \arcsin \frac{s_1}{s_0} + \arcsin \frac{s_2}{s_0} \geq 0 \). For Case 3 (southern vertex) holds: \( \arcsin \frac{s_1}{s_0} + \arcsin \frac{s_2}{s_0} \leq 0 \). These relations are important for the determination of \( c \). For the different cases for \( d\lambda \) derived above can be unified by taking advantage of the auxiliary sphere. (See the next section.)

A.3 Derivations along the auxiliary sphere

Differentiating \( (7a) \) one obtains the differential arc length along the geodesic:

\[ \pm Ds = a \sqrt{1 - e^2 \cos^2 \beta \left( \cos^2 \beta - c^2 \right)} d\beta. \]  
(A17)

and (for positive \( d\beta \)) the differential arc length on the sphere can be expressed as

\[ DS = \frac{a \cos \beta d\beta}{\sqrt{\cos^2 \beta - c^2}} = a d\theta. \]  
(A18)

where \( \theta \) is the geocentric angle on the great circle of the sphere. Integrating both sides of the last equality of (A18) one obtains

\[ \theta - \theta_1 = \arcsin \frac{\sin \beta}{\sqrt{1 - c^2}} - \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}}. \]  
(A19a)

if the integration points are on the same side of a vertex, and

\[ \theta - \theta_1 = \pi - \left| \arcsin \frac{\sin \beta}{\sqrt{1 - c^2}} + \arcsin \frac{\sin \beta_1}{\sqrt{1 - c^2}} \right|. \]  
(A19b)

if the points are on adjacent sides of the vertex. This implies that

\[ \sin \beta = \sqrt{1 - c^2} \sin \theta \] and \( \sin \beta_1 = \sqrt{1 - c^2} \sin \theta_1 \).  
(A20)

(A17), (A18) and (A20) yield the geodesic arc length

\[ s = a \int_{\theta_1}^{\theta} \sqrt{1 - e^2 \left[ 1 - (1 - c^2) \sin^2 \theta \right]} d\theta = \]

\[ = b \int_{\theta_1}^{\theta_1 + \psi} \frac{\sin^2 \theta}{\sqrt{1 - k^2 \sin^2 \theta}} d\theta = \]

\[ = b \psi + bk^2 \int_{\theta_1}^{\theta_1 + \psi} \frac{\sin^2 \theta}{1 + \sqrt{1 + k^2 \sin^2 \theta}} d\theta, \]  
(A21)

where \( k^2 = \frac{e^2 (1 - c^2)}{1 - e^2 \cos^2 \beta} \) and \( \psi \) is the total geocentric angle between known points, which can be determined by (A19a) and (A19b) with \( \psi = \theta_2 - \theta_1 \).

Similarly, the differential difference between ellipsoidal and geocentric longitudes can be expressed

\[ d(d\lambda) = -c \frac{1 - \sqrt{1 - e^2 \cos^2 \beta}}{\sqrt{1 - e^2 \cos^2 \beta}} \cos \beta d\beta = \]

\[ = -c e^2 \frac{d\theta}{1 + \sqrt{1 - e^2 \cos^2 \theta}} = \]

\[ = -c e^2 \frac{d\theta}{1 + \sqrt{1 - e^2 \sqrt{1 + k^2 \sin^2 \theta}}} = \]

\[ = -c e^2 \frac{1 - \sqrt{1 - e^2 \sqrt{1 + k^2 \sin^2 \theta}}}{1 + \sqrt{1 - e^2 \sqrt{1 + k^2 \sin^2 \theta}}} d\theta. \]  
(A22)

This yields

\[ d\lambda = \]

\[ = -c e^2 \left( 1 - \frac{1 - (1 - c^2) \sin^2 \theta}{1 + \sqrt{1 - e^2 \sqrt{1 + k^2 \sin^2 \theta}}} \right) d\theta, \]  
(A23)

where the integral, of order \( e^2 \psi/2 \), is most conveniently determined by numerical integration.