Optimal observational planning of local GPS networks: assessing an analytical method

Abstract: Precision, reliability and cost are the major criteria applied in optimization and design of geodetic networks. The terrestrial networks are being replaced quickly by permanent and campaign Global Positioning System (GPS) networks. These networks must be optimized using the same three criteria. In this article the optimization of the observational plan of local GPS networks (Second Order Design (SOD)) is considered using the precision criterion. This study is limited to the selection of optimal numbers and the best distribution of the non-trivial baselines throughout the network. This objective is accomplished based on the SOD solution through the analytical method in operational research by the means of quadratic programming algorithm. This presented method is tested on a real GPS network and appears to be a useful technique in terms of cost reduction in the field work by the provided observational plan and optimal distribution of the baselines throughout the network. Results indicate that weights of almost 36% of the baselines are negligible when compared to the weights of the rest of the baselines; therefore, they could be eliminated from the observational plan, resulting in a 36% saving in the fieldwork cost.

Keywords: Optimization, Precision of Geodetic Networks, Quadratic Programming, Static and Relative GPS Positioning

1 Introduction

During the last decades GPS has been firmly established as a primary tool for the measurements of high-precision geodetic networks. In order to have a high quality GPS network with low cost, it has to be optimized according to the quality control measures. Optimization of geodetic positioning/deformation networks is concerned with precision, reliability, sensitivity and economy as design criteria.

The GPS networks have the same optimization criteria similar to the traditional networks such as precision, reliability, sensitivity and cost. The optimization and design of such networks are analogous to the traditional ones, and are categorized into four orders: zero, first, second and the third (Grafarend 1974).

In the zero order design (ZOD) the best datum of the network is chosen, in the first order design (FOD) the precision of observations are assumed to be given, and by changing the position of the network points the best geometrical configuration of the network is found, the second order design (SOD) is dedicated to determine the precision of the observations based on priori defined configuration and the third order design (THOD) is performed by adding or removing observations and or stations.

From the practical point of view a GPS user has no role in the satellites’ constellation and their configuration in space, although schedule planning can assist the user to find the best time to acquire data. Accordingly FOD of integrated constellation and station points is not considered in this article. The ZOD order and datum definition for GPS system is usually complex, since the reference system is defined by the set of global tracking stations used in generating the broadcast or precise ephemerides. However, by no means would this stop a surveyor applying ZOD by selecting one or more stations as fixed points in their networks.

The optimization and design of geodetic networks were mostly developed in the 1970 - 1980s. The first studies were initiated by the pioneers Grafarend (1974), Baarda (1973) and Koch (1982). Koch (1985) adopted the optimization theory of a network by considering quadratic pro-
programming to optimize configuration. Later notable contributions were made by Xu (1989) and Xu and Grafarend (1995) which introduced the multi-objective optimization model.

The SOD problem became rather popular in Geodesy after its initiation by Grafarend (1974). Over the past few decades, many new algorithms have been developed by simulated examples and real applications, where Taylor-Karman structured criterion matrices are applied by Grafarend (1970), Grafarend (1972), Baarda (1973), Cross (1985), Schaffrin (1985) and Schmitt (1980). In their studies the SOD is discussed for terrestrial geodetic networks in terms of achieving maximum accuracy and minimum cost.

Kuang (1993a) presented another approach to SOD leading to a maximum reliability applying linear programming. Kuang (1996) developed multi-objective model for all design orders simultaneously and gave an overview of recent works of that time. Even-Tzur (2002), Amiri-Simkooei (2004), Amiri-Simkooei and Sharifi (2004), Eshagh (2005), Bagherbandi et al. (2009) have had some contributions in this field.

The design problem can be solved using the three, the trial-and-error, analytical and intelligent methods. The latter is named the stochastic optimization method as well (Fouskakis and Draper, 2002). The collection of intelligent optimization methods such as Simulated Annealing (Johnson and Wyatt 1994, Baselga 2011), Genetic Algorithm (Sahabi et al. 2008, Dwivedi and Dikshit 2013), Graph Theory (Kortesis and Dermanis 1987), Artificial Neural Networks (Jwo and Chen 2006) and particle swarm optimization algorithm (PSO) (Yetkin et al. 2011) are state of the art and in constant advance. This category of algorithms is generally inspired by nature with its own advantages and disadvantages.

These three optimization techniques are portrayed as a flowchart in Fig. 1, in which the optimization methods are divided in two groups, classical and intelligent methods (Doma 2013).

In the trial-and-error method, the user postulates a solution upon which the design criteria are computed. Should either of the criteria not be actuated, a new solution is postulated, and the criteria are recomputed.

This step is repeated until it meets the optimized network (Cross 1985). The analytical method offers specific algorithms for the solution of particular design, used in describing a method that solves a particular design problem by a unique series of mathematical steps (Kuang 1996). In this method the nonlinear matrix function describing the quality of the network is linearized around an initial point using Taylor series. Then by adopting the available optimization methods such as linear or quadratic programming, some corrections to the initial values are made in such a way that the network becomes optimal.

The multi-objective analytical method can be implemented to design geodetic networks as it gives a simultaneous optimal solution for the configuration and precision of the observables according to their given criteria and the limiting constraints of the locations of some stations and required precision of the deformation parameters (Amiri-Simkooei 2001 and Mehrabi 2002).

Kuang (1993a) presented another approach to SOD leading to a maximum reliability applying linear programming. Kuang (1996) developed multi-objective model for all design orders simultaneously and gave an overview of recent works of that time. Even-Tzur (2002), Amiri-Simkooei (2004), Amiri-Simkooei and Sharifi (2004), Eshagh (2005), Bagherbandi et al. (2009) have had some contributions in this field.

The trial-and-error method, the user postulates a solution upon which the design criteria are computed. Should either of the criteria not be actuated, a new solution is postulated, and the criteria are recomputed. This step is repeated until it meets the optimized network (Cross 1985). The analytical method offers specific algorithms for the solution of particular design, used in describing a method that solves a particular design problem by a unique series of mathematical steps (Kuang 1996). In this method the nonlinear matrix function describing the quality of the network is linearized around an initial point using Taylor series. Then by adopting the available optimization methods such as linear or quadratic programming, some corrections to the initial values are made in such a way that the network becomes optimal.

The determination of the schedule of sessions, consisting of temporarily placing GPS receivers sequentially at pre-chosen points, is a sample case of GPS network optimization, especially in large networks, considered by Dare and Saleh (2000). Even-Tzur (2002) suggested an applicable algorithm based on the sensitivity analysis of the network for the selection of the optimal GPS vectors which is subject to a postulated velocity field.


Amiri-Simkooei et al. (2012) indicated that the optimization of GPS networks is conceptually comparable to the design of classical geodetic networks; therefore, in their article the Geometrical Dilution Of Precision (GDOP) is applied as a measure of precision to solve the best configuration of a simple simulated network analytically. They showed that if when performing point positioning with four satellites, the first three satellites are located at an
equal elevation angle and azimuths of 0, 120, and 240 degrees, a quite clear result has been obtained, namely, the best location of the fourth satellite will be the zenith.

In practice, this situation is not so simple, because non-stationary configuration of satellites and terrestrial stations are to be considered. The complex nature of the FOD in practice is the main reason for considering only SOD. Indeed, the configuration of GPS networks is composed of moving satellites in space and ground stations on the Earth; therefore, the network configuration composed of these facilities is essentially dynamic and continuous (Xue et al. 2013). An optimal network design should yield a precise, reliable and cost effective survey plan. For GPS networks, the number of baselines to be measured and thereby the number of sessions is an important cost criterion (Yetkin and Berber 2012).

To achieve high precision positioning with GPS, observations should be gathered through multiple frequency receivers in static and relative positioning methods simultaneously for all or the most of independent baselines of the network; however observing all possible baselines is not necessary, because it will increase the cost of the fieldwork. The number and distribution of the baselines throughout the network may be optimized in such a manner that postulated criteria are met. A greater number of baselines may increase the precision and reliability of the network, but it leads to having an expensive network; therefore, the selection of optimal baselines is crucial in GPS network planning. Before the installation of bench marks and gathering of survey data, a GPS network must be designed to meet some quality criteria.

An attempt is made here to reach a GPS observational plan with high precision and low cost simultaneously, by SOD. Indeed, the purpose is to show how the quadratic programming can be applied to perform the SOD in local GPS networks and reveal its influences on reducing the cost of the campaign.

Modeling and linearization of GPS measurements produce a design matrix which is independent of the positions of the network points. This property which is common to GPS and to ordinary leveling makes it possible to implement powerful mathematical tools for the solution of the SOD problem. The efficacy of this method is verified by applying an example of a real GPS network intuitively where all of its baselines are measured independently.

2 GPS network optimal design

The precision of GPS relative positioning depends on the distribution of the tracked satellites in space and the quality of the observations. In this article it is assumed that satellite configuration, GPS network configuration, and the type of GPS receiver instrument are known in order to reveal the influence of the number and distribution of the baselines on the precision of the network stations.

The design strategy we follow in this contribution has no effect on the planning for carrying out the observations. To achieve better precision of GPS positioning, it should be noted that the best time of fieldwork base on sky plot visibility and Dilution of Precision (DOP)values must be determined in the planning stage.

In SOD order of terrestrial networks, the weight of the observations are optimized; while in GPS networks the observational weights are known and are related to the surveying receiver and the time interval of observations. Here, the number of baselines and their distribution on the network should be decided upon.

The internal and external reliability of the network are directly correlated to the degree of freedom (DOF) of the network, which is the summation of the redundancy number of individual observations. The DOF of each baseline of the GPS network is mutually related to the solution method of a baseline, e.g. single, double or triple differences in GPS observations. The number of satellites, time of tracking satellites, log rate recording data (time interval of recording data), number of cycle slips etc. are another affecting factors. Therefore, it is not possible to calculate the DOF of the baselines at the design stage. In other words, reliability of combined configuration of network stations and satellites cannot be the subject of the design and optimization of the GPS networks in traditional manner, except for the simple and special situations. However, when components of baselines are assumed as indirect observations (Eq. 2), the redundancy numbers and reliability measures of GPS networks can be easily calculated, similar to their calculation in conventional networks.

2.1 Mathematical model

The mathematical model for a least squares solution of a GPS network is:

\[ \mathbf{l} + \mathbf{v} = f(\mathbf{x}) \]  

where \( \mathbf{l}, \mathbf{v} \) and \( \mathbf{x} \) are the vector of observations, the residuals and the unknowns, respectively. This model will be linear if a three dimensional Cartesian coordinate system
is used; where each baseline of the network adds a $3 \times 1$
vector into observations vector. If $i, j$ are indices of the two
stations, then this baseline is expressed as:

$$
\begin{pmatrix}
\Delta x_{ij} \\
\Delta y_{ij} \\
\Delta z_{ij}
\end{pmatrix} =
\begin{pmatrix}
x_j - x_i \\
y_j - y_i \\
z_j - z_i
\end{pmatrix}
$$

(2)

therefore, the design matrix of the $i, j$ baseline vector is as
follows:

$$
A = \begin{bmatrix}
\frac{\partial f(x)}{\partial x}
\end{bmatrix} =
\begin{bmatrix}
-1 & 0 & 0 & 1 & 0 & 0 \\
0 & -1 & 0 & 0 & 1 & 0 \\
0 & 0 & -1 & 0 & 0 & 1
\end{bmatrix}
$$

(3)

In Eq. (3), $A$ is the design matrix of the GPS baseline observables. Each baseline adds three rows into the configuration matrix. Observation equations of GPS baseline measurements produce a design matrix which is independent of the positions of the network points, see the coefficient matrix in Eq. (3). As two datum defects of the orientation and scale of the network are resolved by considering the baseline observations it is essential to just constrain the origin of the coordinate system. Indeed, baselines measurements in GPS networks can be assumed to have an orientation and scale, therefore the size of the datum defect in a GPS network is three, since the definition of origin is missing.

Observation equations relating to GPS networks could raise issues with respect to singularity and rank-deficiency whenever the datum of the network is not completely defined. There is, therefore, a need to define the datum in order to solve for the positioning parameters. Datum definition can be accomplished by a careful selection of some constraints added to the observation equation model. This could be accomplished while fixing one or more reference stations during the adjustment procedure. When only one reference station is held fixed a datum referred to as a minimum constraint solution is established; if more than one station is fixed, the datum will be referred to as an over-constrained solution (Koch 1999, Kashani et al. 2003).

In the inner constraint adjustment the origin of the system is defined by fixing the coordinate of the centroid of the network. The datum of the network is defined as:

$$
Dx = 0
$$

(4)

where $D$ is the datum matrix of GPS networks, which spans the null spaces of design matrix of the parametric models. $m$ is the number of network points used in definition of centroid of the network, expressed as:

$$
D = \begin{bmatrix}
1 & 0 & 0 & 1 & 0 & 0 & \ldots & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 1 & 0 & \ldots & 0 & 1 & 0 \\
0 & 0 & 1 & 0 & 0 & 1 & \ldots & 0 & 0 & 1
\end{bmatrix}_{3 \times 3m}
$$

(5)

The variance-covariance matrix of the estimated coordinates is given (assuming the priori variance factor $\sigma^2_0 = 1$
in the design stage) as follows (Kuang 1996 p. 114):

$$
C_x = (A^T PA + D^T D)^{-1} - D^T (DD^T DD^T)^{-1} D
$$

(6)

### 2.2 Gauss-Markov model for GPS network adjustment

To adjust the GPS networks, all of the observed baselines are processed separately between each two reference stations. Then all the processed baselines are used to adjust the network simultaneously. In this approach when the obtained baselines are combined to form a network, the coordinates of stations are estimated. The Gauss-Markov model for 3-Dimensional network adjustment is:

$$
E(\Delta r) = Ax
$$

(7)

$$
D(\Delta r) = C_{Ax}
$$

(8)

where $E$ and $D$ are the mathematical expectation and Dispersion operator, respectively. $A$ is the configuration matrix of the network, and $x$ is the correction vector to the preliminary coordinates of the network points and $\Delta r$ is the observations vector as follows:

$$
\Delta r = (\Delta r_1^T, \Delta r_2^T, \ldots, \Delta r_n^T)
$$

(9)

$$
\Delta r_i = r_i - r_k \quad i = 1, 2, \ldots, n
$$

(10)

where $\Delta r_i$ is the difference between coordinates of the $j, k$ stations and $r_i, r_k$ are coordinates of unknown points $j, k$ expressed in the Cartesian coordinate system as

$$
\Delta r_i = \begin{pmatrix}
x_i \\
y_i \\
z_i
\end{pmatrix} - \begin{pmatrix}
x_k \\
y_k \\
z_k
\end{pmatrix} = \begin{pmatrix}
\Delta x_i \\
\Delta y_i \\
\Delta z_i
\end{pmatrix}
$$

(11)

The stochastic part of Eq. (8) can be achieved from the following rule of thumb (Wells 1986, Lichton 1989):

$$
\sigma^2_i = a^2 + (b. S)^2
$$

(12)

where, $S$ is the length of the baseline. The $a, b$ are the constants known for each receiver. In fact, in order to achieve a realistic design in the optimization process, the GPS receiver standard technical manual is used to calculate the initial weights of the baselines.

At the design stage, the components of $\Delta r_i$ vector can be written as (Kuang 1993b):

$$
\sigma^2_{\Delta x_i} = \sigma^2_{\Delta y_i} = \sigma^2_{\Delta z_i} = a^2 + (b.S_i)^2 = \sigma_i^2
$$

(13)
If the $n$ numbers of baselines in a GPS network are assumed independent, then the variance-covariance matrix of the observations could be expressed by

$$C_{tr} = \text{diag} \left( C_{tr} \right) = \text{blkdiag} \left( \sigma_i^2 I_1, \sigma_i^2 I_1, \ldots, \sigma_i^2 I_1 \right)$$  \hspace{1cm} (14)

where, $I_1$ is a $3 \times 3$ identity matrix, $\sigma_i^2$ is the variance component of each baseline and blkdiag is a block-diagonal structure. The datum of the network is provided by the minimum constraints, the adjusted coordinates and its structure. The datum of the network is provided by the component of each baseline and blkdiag is a block-diagonal matrix of the observations could be expressed by

$$\hat{x} = \left( A^T P A \right)^{-1} A^T P \Delta r$$  \hspace{1cm} (15)

and

$$C_{\hat{x}} = \sigma_0^2 \left( A^T P A \right)^{-1}$$  \hspace{1cm} (16)

where

$$P = \sigma_0^2 C_{tr}^{-1} = \text{diag} \left( p_i I_1 \right) \quad i = 1, 2, \ldots, n$$  \hspace{1cm} (17)

and

$$p_i = \frac{\sigma_0^2}{\sigma_i^2} \quad i = 1, 2, \ldots, n$$  \hspace{1cm} (18)

Where, $P$ is the $3n \times 3n$ weight matrix of the baselines of the network, $p_i$ is a component of $3 \times 1$ $p_i$ vector, in which $p_i$ is a vector of diagonal elements of the weight matrix of $i$-th baseline of the network and $n$ is the number of baselines.

### 2.3 Optimization model

Optimization means maximizing or minimizing an objective function which represents the criterion adopted to define the quality of a geodetic network. The analytical methods offer specific algorithms for the solution of a particular design problem without human intervention. The term analytical design is used to describe a method that solves a particular design problem by a unique series of mathematical steps rather than a trial and error method which needs human intervention.

Operational research is the application of scientific approaches to complex problems arising in the management of an organization’s operations in order to carry out those operations in a more efficient manner (Dare and Saleh 2000). Linear and quadratic programming are some aspects of operational research which have been applied to optimize geodetic problems from past to now (Cross and Thapa 1979, Kuang and Chrzanowski 1992 and Kuang 1993b).

Our objective is optimization of local GPS networks in engineering applications by adopting the Gauss-Markov model. If the datum of the network is provided by inner constraint then the variance-covariance matrix is obtained through Eq. (6); if the datum is provided by minimum constraint, the variance-covariance matrix will be presented in the converted form of Eq. (6) in Eq. (16).

The matrix $C_r$ consists of nonlinear function of observational weights $p_i$, $i = 1, 2, \ldots, n$. The SOD seeks to design the observation weights in a manner that the solution is able to fulfill the prescribed precisions by estimating the weight of the observations.

In the SOD problem, the covariance matrix of the observations must be selected in a sense that the covariance matrix of the parameters get as close as possible to the Criterion matrix (Baarda 1973). The criterion matrix is an ideal covariance matrix of the network. In other words, the criterion matrix is an artificial variance-covariance matrix possessing an ideal structure (Kuang 1996 p. 207). The objective function for precision is defined as (Kuang 1994):

$$\| G \| = \| C_x - C^*_e \| = \min$$  \hspace{1cm} (19)

where

$$C^*_e = S C_e S^T$$  \hspace{1cm} (20)

and

$$S = I - D^T \left( D D^T \right)^{-1} D$$  \hspace{1cm} (21)

where, $S$ is the similarity transformation matrix, through which, the datum of the network is provided. In other words, the criterion matrix is forced to have an adaptive datum. Given a set of approximate values for the weights, the matrix $G$ can be approximated by Taylor series of linear form as follows:

$$G = G^0 + \sum_{i=1}^{n} \frac{\partial G}{\partial p_i} \Delta p_i$$  \hspace{1cm} (22)

where

$$G^0 = \sigma_0^2 \left[ \left( A^T P A + D^T D \right)^{-1} - D^T \left( D D^T D^T \right)^{-1} \right]$$  \hspace{1cm} (23)

and

$$\frac{\partial G}{\partial p_i} = \sigma_0^2 \left\{ - \left( A^T P A + D^T D \right)^{-1} \left[ A^T \frac{\partial P}{\partial p_i} A \right] \left( A^T P A + D^T D \right)^{-1} \right\} \bigg|_{p\rightarrow p^0}$$  \hspace{1cm} (24)

$\Delta p_i \quad i = 1, 2, \ldots, n$ is the improvement to the initial weights which should be solved optimally. If all pairs of
observations are assumed independent, the partial derivative of the weight matrix \( P \) with respect to the observational weight \( p_i \) is expressed as

\[
\frac{\partial P}{\partial p_i} = \text{diag} ( E_i ) , \quad E_j = \begin{cases} I_{3 \times 3} & \text{if } j = i \\ 0_{3 \times 3} & \text{if } j \neq i \end{cases}, \quad j = 1,2,\ldots,n
\] (25)

where \( I_{3 \times 3} \) is the \( 3 \times 3 \) identity matrix and \( 0_{3 \times 3} \) is the \( 3 \times 3 \) zero matrix.

After the design stage, the observations that have weights equal to zero or have a low weight compared to the other weights will be removed. Observing baselines with different precision in a GPS network is very difficult or almost impossible; therefore, the best decision should be made as to whether to observe it or not. The final optimal values of observational weights are given by:

\[
p_i = p_i^0 + \Delta p_i \quad i = 1, 2, \ldots, n
\] (26)

In reference to Eqs. (19) and (22), the optimization model with different precision in a GPS network is very difficult the other weights will be removed. Observing baselines weights equal to zero or have a low weight compared to the achievable accuracy of the available instruments.

As seen here, the inequality (35) can be written in a matrix and vector form as:

\[
\begin{align*}
\| G \| & = \| \text{vec}(G) \| = \| A_1 W + U_1 \|_2 = (A_1 W + U_1)^T (A_1 W + U_1) \\
& = W^T A_1^T A_1 W + 2 U_1^T A_1 W + U_1^T U_1
\end{align*}
\] (34)

In this context, the inequality constraint will be expressed as follows:

\[
\begin{align*}
\left( G^0 + \sum_{i=1}^n \frac{\partial G}{\partial p_i} \Delta p_i \right)_{kk} & \leq 0 \quad k = 1, 2, \ldots, u
\end{align*}
\] (28)

and

\[
0 \leq p_i \leq p_{i,\text{max}} \Rightarrow -p_i^0 \leq p_i - p_i^0 \leq \frac{\sigma_i^2}{(\sigma_i^2)_{\text{min}}} \cdot -p_i^0
\] (29)

In the inequality (28) the diagonal elements of the \( G \) matrix must be negative. It means the diagonal elements of \( C_k \) should be less or at least equal to diagonal elements of the \( C_k^0 \) matrix. Indeed, the final precision of the network should be more precise than the objective postulated precision or in the worst case it should be equal to that.

The inequality (29) is a constraint for the weight improvement \( \Delta p_i \) \( i = 1, 2, \ldots, n \). The weights of observations should be non-negative and bounded by maximum achievable accuracy of the available instruments. \( \sigma_i^2 \) and \( (\sigma_i^2)_{\text{min}} \) \( i = 1, \ldots, n \) are the a priori variance factor and the minimum variance which can be achieved for each observable \( l_i \) \( i = 1, \ldots, n \), respectively.

Equation (27) presents a general mathematical programming problem with two linear inequality constraints (28) and (30). Depending on the selection of the matrix norm in Eq. (27), one has to apply a certain mathematical programming problem technique to obtain the solution.

Studies show that \( L_2 \) norm has more advantages than the other norm spaces (especially in the fitting of network variance covariance to criterion matrices) (Kuang 1993a). If the vec operator is applied to the linear function of \( G \) matrix, Eq. (22) can be written as follows:

\[
\text{vec} \left( G^0 + \sum_{i=1}^n \frac{\partial G}{\partial p_i} \Delta p_i \right) = \text{vec} \left( G^0 \right) + \sum_{i=1}^n \text{vec} \left( \frac{\partial G}{\partial p_i} \right) \Delta p_i
\]

\[
= U_1 + A_1 W
\] (30)

where

\[
U_1 = \text{vec} \left( G^0 \right)
\] (31)

and

\[
A_1 = \left[ \begin{array}{ccc} \text{vec} \left( \frac{\partial G}{\partial p_1} \right) & \text{vec} \left( \frac{\partial G}{\partial p_2} \right) & \ldots & \text{vec} \left( \frac{\partial G}{\partial p_n} \right) \end{array} \right]
\] (32)

and

\[
W = (\Delta p_1 \quad \Delta p_2 \ldots \quad \Delta p_n)
\] (33)

where the vec operator produces a vector by stacking the columns of the matrix in a single column vector. Eventually, if one applies the \( L_2 \)-norm to Eq. (27) by considering Eq. (30), the objective function for optimization can be restated in the matrix and vector form as:

\[
\| G \| = \| \text{vec}(G) \| = \| A_1 W + U_1 \|_2 = (A_1 W + U_1)^T (A_1 W + U_1) = W^T A_1^T A_1 W + 2 U_1^T A_1 W + U_1^T U_1
\] (34)

In this context, the inequality constraint will be expressed as follows:

\[
\left( G^0 + \sum_{i=1}^n \frac{\partial G}{\partial p_i} \Delta p_i \right)_{kk} \leq 0 \Rightarrow \text{diag} \left( G^0 \right) + \sum_{i=1}^n \text{diag} \left( \frac{\partial G}{\partial p_i} \right) \Delta p_i \leq 0
\] (35)

\[
A_{11} W + U_{11} \leq 0
\] (36)

\[
U_{11} = \text{diag} \left( G^0 \right)
\] (37)

\[
A_{11} = (I \Theta I)^T A_1
\] (38)

where \( \Theta \) stands for the Khatri-Rao product (Kuang 1992). As seen here, the inequality (35) can be written in a matrix form as Eq. (36).

Equation (34) presents a least squares problem with linear inequality constraints (29) and (36). This problem can be solved through quadratic programming. It has a unique solution if the matrix \( A_1^T A_1 \) is positive definite and all the constraints are consistent. The \( G \) is a nonlinear matrix with respect to weights which is linearized by applying the Taylor-series expansion. To compensate for the linearization error of a nonlinear function, the optimization problem should be solved iteratively.
To illustrate the advantages of the analytical method in optimization and design of GPS networks, a real network with 8 stations is established and used by the authors. A total of 28 baselines of the network were measured independently with a pair of dual frequency receivers. Moreover, a computer program is written by the authors in order to implement the analytical method in optimization and design of local GPS network using the Matlab software.

3 Numerical results

A standard output from the baseline processing adjustment is the error ellipse which graphically portrays the region of positioning uncertainty associated with the adjusted coordinates at a given statistical confidence level. The largest dimension of the error ellipse is called the semi-major axis. Its length indicates the maximum expected position error at a selected confidence level, usually 95% of statistical confidence. The semi-major axis of the point ellipse will be compared with the positioning accuracy requirements established for the objective of the project.

A real GPS network is established for the research purposes by the authors around the campus of the Meybod University, Yazd province, Iran. This network is used to evaluate this proposed approach. The datum of the network is provided by a minimum number of constraints, so only S1 is considered as a fixed station. The initial network configuration consists of all 28 possible baselines, which are observed by a pair of dual frequency receivers without any repeated baselines.

In due course, first, the network with 28 independent baselines is processed by available commercial software. The configuration of the network stations, all baselines and their 95% error ellipses for this scheme are shown in Fig. 2 and the magnitude of the semi-major and minor axes of absolute error ellipse are tabulated in the first column (I column) of Table 2.

Second, the optimization method described above is implemented for the network. The optimization algorithm found 10 low weight baselines that do not have any significant contribution to the improvement of the network precision. Since the weights of the 10 non-significant baselines are negligible in comparison to the weights of the other baselines, these low weight baselines can be easily eliminated among the observational plan of the network without any undesirable influence on the precision of the network.

The optimized baselines, which are selected through the computer program (18 baselines out of a total of 28 baselines), are processed by a commercial software for GPS baseline processing. The size of the semi-major and semi-minor axes of the absolute ellipse errors are tabulated in the second column (II column) of Table 2. The network stations and the 18 optimized baselines are shown in Fig. 3. The lists of these selected baselines are tabulated in second column of Table 1. The corrections of initial weights presented in Eq. (38) are illustrated as a bar-chart in Fig. 4. The baselines with negative or near zero correction are illustrated in red and could be removed from the observing plan due to their low weight in comparison to the significant amount of weights with blue color.

Third and last, as an additional effort, 18 baselines are randomly selected among the whole 28 baselines without any priori planning. The network with these random baselines is processed and the precision results of the network (i.e. semi-major and minor axes of the network’s points) are sorted in third column (III column) of Table 2. These baselines are listed in the third column of Table 1. The distribution of these 18 random baselines throughout the network is shown in Fig. (5).

Numerically, with comparison of the three columns of Table 2, it can be deduced that deleting 10 baselines has no significant influence on the size of semi-major axis of 95% error ellipses. Omitting the 10 baselines of the network without any planning and idea for exact optimization destroys the station’s precision of the network. A dramatic growth in the size of semi-major axes of stations is seen. For instance the S3 station has approximately 80% growth in the magnitude of its semi-major axis. After optimization process, almost 36% of all baselines are deleted from the original scheme (28 baselines); while only about 10% increase is shown in the average size of the most error ellipses. The situation is very different in the network with randomly selected baselines. The influence and importance of optimization of monitoring networks are demonstrated through this comparison.

4 Discussion and conclusion

To achieve high precision satellite positioning (up to sub-centimeter or better) all of the possible baselines (or most of them) of the GPS campaign network should be observed for a long time. Moreover, accuracies sufficient for GPS structural deformation surveys are only achieved by applying precise carrier beat phase and relative positioning techniques.
Table 1. List of 18 optimized and randomly selected baselines among the 28.

<table>
<thead>
<tr>
<th>Baseline Number</th>
<th>Optimized Selected Baselines</th>
<th>Accidentally Selected Baselines</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>From</td>
<td>To</td>
</tr>
<tr>
<td>1</td>
<td>S1</td>
<td>S2</td>
</tr>
<tr>
<td>2</td>
<td>S1</td>
<td>S3</td>
</tr>
<tr>
<td>3</td>
<td>S2</td>
<td>S3</td>
</tr>
<tr>
<td>4</td>
<td>S1</td>
<td>S4</td>
</tr>
<tr>
<td>5</td>
<td>S1</td>
<td>S6</td>
</tr>
<tr>
<td>6</td>
<td>S1</td>
<td>S7</td>
</tr>
<tr>
<td>7</td>
<td>S2</td>
<td>S5</td>
</tr>
<tr>
<td>8</td>
<td>S2</td>
<td>S6</td>
</tr>
<tr>
<td>9</td>
<td>S2</td>
<td>S7</td>
</tr>
<tr>
<td>10</td>
<td>S2</td>
<td>S8</td>
</tr>
<tr>
<td>11</td>
<td>S3</td>
<td>S5</td>
</tr>
<tr>
<td>12</td>
<td>S3</td>
<td>S6</td>
</tr>
<tr>
<td>13</td>
<td>S3</td>
<td>S7</td>
</tr>
<tr>
<td>14</td>
<td>S3</td>
<td>S8</td>
</tr>
<tr>
<td>15</td>
<td>S4</td>
<td>S8</td>
</tr>
<tr>
<td>16</td>
<td>S5</td>
<td>S7</td>
</tr>
<tr>
<td>17</td>
<td>S5</td>
<td>S8</td>
</tr>
<tr>
<td>18</td>
<td>S6</td>
<td>S8</td>
</tr>
</tbody>
</table>

Table 2. Magnitude of the semi major and minor axes of the 95% of error ellipses in three states. (First column with all 28 baselines, Second column with 18 optimized baselines and Third column with 18 random baselines).

<table>
<thead>
<tr>
<th>POINT NAME</th>
<th>(I) WITH ALL BASELINES</th>
<th>(II) WITH OPTIMIZED BASELINES</th>
<th>(III) WITH RANDOM BASELINES</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>A(mm)</td>
<td>B(mm)</td>
<td>A(mm)</td>
</tr>
<tr>
<td>S1</td>
<td>0.0</td>
<td>0.0</td>
<td>0.0</td>
</tr>
<tr>
<td>S2</td>
<td>3.5</td>
<td>3.0</td>
<td>3.6</td>
</tr>
<tr>
<td>S3</td>
<td>3.3</td>
<td>2.9</td>
<td>3.4</td>
</tr>
<tr>
<td>S4</td>
<td>2.9</td>
<td>2.6</td>
<td>3.2</td>
</tr>
<tr>
<td>S5</td>
<td>3.2</td>
<td>2.8</td>
<td>3.2</td>
</tr>
<tr>
<td>S6</td>
<td>3.1</td>
<td>2.7</td>
<td>3.7</td>
</tr>
<tr>
<td>S7</td>
<td>3.2</td>
<td>2.7</td>
<td>3.7</td>
</tr>
<tr>
<td>S8</td>
<td>3.3</td>
<td>2.8</td>
<td>3.6</td>
</tr>
</tbody>
</table>

Through the optimization process, the observation of a minimum number of baselines instead of all of the network baselines is achieved in a manner where the postulated precision of the network is fulfilled. This optimization procedure is particularly suited for optimal design of GPS networks for engineering purposes.

The optimal baseline configuration method suggested removing the baselines that have the lowest weight corrections in the optimization process. Therefore, this method assures the optimal solution for GPS local networks, through the analytical method where quadratic programing is adopted.

This article revealed that to reach the optimized precision, there is no need to observe all of the possible baselines and this finding confirmed the importance of optimization and design of GPS networks. The network assessed in this study shows a 36% decrease in the number of baselines which caused a maximum of 10% increase in the size of the semi-major axes of stations. It can be seen that a significant decrease in the number of optimized baselines, rather than using all baselines, does not result in a significant increase in the magnitude of semi-major and minor axes of the ellipses error. The results indicate the validity and ability of the analytical method in the optimization and design of observational plan of local GPS networks;
Optimal observational planning of local GPS networks: assessing an analytical method

Moreover, it is not essential to observe all of the independent baselines of any network to achieve the postulated criteria.

As a concluding remark it could be claimed that: if surveyors are reluctant to use the optimization process and eliminate some baselines without using proper and exact design in order to decrease their costs, it may destroy the precision of the network (see Table 2).

Campaign networks are demanded for high precision, but data collection for these networks should be repeated in required intervals; therefore, optimization of such networks is a vital part of decreasing the cost of the monitoring. It is recommended not to ignore the analytical optimization of GPS campaign networks. In addition it is recommended to compare the results of the analytical method with the artificial intelligent (AI) techniques in the optimization and design of real GPS network for future research.

Acknowledgement: The authors would like to thank Dr. Amiri Simkooei and Dr. Eshagh and both anonymous reviewers of the Journal of Geodetic Science for their constructive comments on this manuscript.
References


Kuang, S. h. and A. Chrzansowski, 1992, Multi-objective optimization design of geodetic networks. Manuscripta Geodaetica 17, 233-244.


