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On the topographic bias and density distribution in modelling the geoid and orthometric heights

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Abstract: It is well known that the success in precise determinations of the gravimetric geoid height (N) and the orthometric height (H) rely on the knowledge of the topographic mass distribution. We show that the residual topographic bias due to an imprecise information on the topographic density is practically the same for N and H, but with opposite signs. This result is demonstrated both for the Helmert orthometric height and for a more precise orthometric height derived by analytical continuation of the external geopotential to the geoid.
This result leads to the conclusion that precise gravimetric geoid heights cannot be validated by GNSS-levelling geoid heights in mountainous regions for the errors caused by the incorrect modelling of the topographic mass distribution, because this uncertainty is hidden in the difference between the two geoid estimators.

Keywords: analytical continuation, geoid height, orthometric height, topographic bias, topographic density

1 Introduction

The concept of topographic bias was introduced by Sjöberg (2007) as the error caused by using analytical continuation of the external gravity field in gravimetric geoid determination, primarily in applying the KTH method of Least Squares Modification of Stokes formula with Additive corrections (LSMSA) with the subtraction of the topographic bias as one of several corrections (Sjöberg 2003, 2007; Sjöberg and Bagherbandi 2017, Chaps. 5 and 6). However, as will be shown here, the topographic bias is also a problem in the commonly practised Remove-Compute-Restore (RCR) technique for geoid determination (e.g. Forsberg 1993; Ellmann and Vanicek 2007), as the lack in removal and restoration of the contributions of the topography leads to a residual topographic bias.

Orthometric and geoid heights are affected by the same topographic mass distribution, and here we will study the relation between the corrections for topographic density distribution (TDD) in these heights. It is important for the study, as illustrated in the example below, that at each observation/computational site on or above the Earth's surface the topographic mass provides exactly the same contribution to the geoid and orthometric heights but with opposite signs.

Example 1: Let the points Q and P be located at sea (geoid) level and in free air vertically above the geoid, respectively. If the volume between P and Q is filled with topographic mass, the geoid height will increase, but as the geodetic height of P does not change, the orthometric height of P will decrease. The height changes depend on the topography and its density distribution.

2 The topographic bias in gravimetric geoid determination

The error caused by the harmonic analytical continuation of the disturbing potential T down to sea level (denoted T^*), where the true potential T_g at the geoid (denoted by subscript g) is not harmonic inside the topographic masses, was defined by Sjöberg (2007) as the topographic bias:

$$N_{bias} = \frac{T^* - T_g}{\gamma_0}, \tag{1}$$

where γ_0 is normal gravity at the reference ellipsoid. He also presented a simple formula for estimating the bias for a constant TDD:

$$N_{bias} = \frac{2\pi\mu}{\gamma_0}\left(H^2 + \frac{2H^3}{3R}\right), \tag{2}$$

where H is the orthometric height and $\mu = G\times\rho$ is the gravitational constant times mean TDD between the geoid and topographic surface along the vertical at the site of computation, and this formula can be refined for a variable vertical density distribution (Sjöberg 2007). Equation (2) was

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also supported by several studies (Sjöberg 2009a-c). At this point it should be noted, as already stated in Sjöberg (2007) and emphasized in Sjöberg (2017), that determining T^* by a series of spherical harmonics includes not only the topographic bias but also other errors, as the external type series does not converge inside the topographic masses.

The subtraction of Eq. (2) is an important correction that reaches about 9 m for Mt. Everest with an elevation of 8864 m when assuming a topographic density of 2.67 g/cm^3 in the LSMSA method of geoid determination.

However, Eq. (2) also provides an estimate for correcting or at least estimating the uncertainty due to the residual topographic density in RCR applications of geoid determination as can be shown as follows. If the standard density used when removing and restoring the effects of the topography on the geoid is μ_0 and the correct local density is $\mu_0 + d\mu$, then the residual topographic bias to correct for in the RCR method is

$$dN_{bias} = \frac{2\pi d\mu}{\gamma_0}\left(H^2 + \frac{2H^3}{3R}\right). \quad (3)$$

If one assumes that $d\mu$ is 10 % of μ_0(cf. Martinec 1998, p. 82) with the "standard" density 2.67 g/cm^3, Eq. (3) yields residual biases of 1.1, 4.5 and 89.5 cm for H being 1, 2 and 8.864 km, respectively.

If the surface Bouguer anomaly is used for downward continuation to the sphere prior to Stokes integration, there will be another bias in the RCR technique (Sjöberg 2014 and 2015), but not in the LSMSA method. (However, this bias is not essential for this study.)

3 The topographic bias in orthometric heights

Consider the geometric relation between the geodetic height (h), the orthometric height (H) and the geoid height (N) given e.g. in Heiskanen and Moritz (1967, p. 325) and Sjöberg and Bagherbandi (2017, p. 113):

$$h = H + N. \quad (4)$$

Let us assume that the density is constant but unknown. As the determination of h by satellite positioning does not depend on the topographic density distribution μ as do H and N when calculated with gravimetric data, one obtains by differentiating Eq. (4) w.r.t. μ:

$$0 = dH(\mu) + dN(\mu) \text{ or } dH(\mu) = -dN(\mu), \quad (5)$$

implying that any small change in μ results in identical small changes dH and dN in orthometric height and geoid height but with opposite signs. Below we will demonstrate this result for Helmert orthometric heights in Sect. 3.1 as well as for precise orthometric heights in Sect. 3.2.

The orthometric height is given by (e.g. Heiskanen and Moritz 1967, Sect. 4.4 ; Sjöberg and Bagherbandi 2017, Sect.3.5.2)

$$H = \frac{C}{\bar{g}}, \quad (6a)$$

where

$$C = -\frac{1}{H}\int_0^H g dh = W_0 - W \quad (6b)$$

is the geopotential number obtained by precise levelling, and $\bar{g}$ is the mean gravity value along the plumb-line between the geoid and the computation point on the Earth's surface. Finally, W_0 and W are the Earth's gravity potentials at the geoid and computation point, respectively. The TDD is essential in estimating the mean gravity $\bar{g}$, while C is obtained by precise levelling.

3.1 Helmert orthometric heights

In Helmert orthometric heights $\left(H^O\right)$ the mean gravity is defined by

$$\bar{g}^O = g_P + \frac{1}{2}F - 2\pi\mu_0 H^O, \quad (7)$$

where F is the free-air correction. Hence, if the standard density μ_0 needs a correction $d\mu$, the mean gravity in Eq. (7) need a correction $dg^O = -2\pi H^O d\mu$, which implies that $H^O = C/\bar{g}^O$ is in error by

$$dH^O = \frac{C}{\bar{g}^O} - \frac{C}{\bar{g}^O + dg^O} = H^O \frac{dg^O}{\bar{g}^O + dg^O}. \quad (8)$$

Using the approximation $\bar{g}^O + dg^O \approx \gamma_0$, one obtains the approximate error of the Helmert orthometric height due to the imprecise topographic density as

$$dH \approx dH^O \approx H^O \frac{dg^O}{\gamma_0} = -\frac{2\pi\left(H^O\right)^2 d\mu}{\gamma_0} \approx -dN_{bias}. \quad (9)$$

Hence, the error is approximately the same as the residual topographic bias in the geoid height but with opposite sign.

3.2 Precise orthometric heights

A precise orthometric height requires a precise approximation of mean gravity ($\bar{g}$) as given by Eqs. (6a,b). Here we will first rewrite the gravity potential on the geoid as

$$W_0 = T_g + U_g = T^\star - \gamma_0 N_{bias} + U_g = W^\star - \gamma_0 N_{bias}, \quad (10)$$

where U is normal gravity. As $W^\star$, the geopotential analytically continued down to the geoid, can be determined by a Taylor expansion of W at the surface point P:

$$\begin{aligned} W^\star &= W_P + \sum_{k=1}^{\infty} \frac{(-H)^k}{k!} \left(\frac{\partial^k W}{\partial h^k} \right)_P \\ &= W_P - \sum_{k=1}^{\infty} \frac{(-H)^k}{k!} \left(\frac{\partial^{k-1} g}{\partial h^{k-1}} \right)_P, \end{aligned} \quad (11)$$

one obtains the mean gravity from Eqs. (6a, b), (10) and (11) as

$$\bar{g} = \frac{W_0 - W_P}{H} = \frac{W^\star - W_P - \gamma_0 N_{bias}}{H} = \bar{g}^O + d\bar{g}, \quad (12a)$$

where $\bar{g}^O$ was defined in Eq. (7) as the mean gravity used in the Helmert orthometric height, and

$$d\bar{g} = -\frac{1}{2}\frac{\partial \delta g}{\partial h} - \gamma_0 \frac{dN_{bias}}{H} + \sum_{k=3}^{\infty} \frac{(-H)^{k-1}}{k!} \left(\frac{\partial^{k-1} g}{\partial h^{k-1}} \right)_P, \quad (12b)$$

where dN_{bias} was given in Eq. (3). Here δg is the gravity disturbance.
As a result Eq. (6a) becomes

$$H = \frac{C}{\bar{g}^O + d\bar{g}} = H^O - H^O \frac{d\bar{g}}{\bar{g}}, \quad (13)$$

or, if one uses the approximations $H \approx H^O$ in Eq. (12b) and $\bar{g} \approx \gamma_0$ in Eq. (13), one finally obtains

$$H \approx H^O - \frac{H^O}{2\gamma_0}\frac{\partial \delta g}{\partial h} - \frac{1}{\gamma_0} \sum_{k=3}^{\infty} \frac{\left(-H^O\right)^k}{k!} \left(\frac{\partial^{k-1} g}{\partial h^{k-1}} \right) + dN_{bias}. \quad (14)$$

Alternatively one obtains directly by differentiating Eq. (6a) and noting Eqs. (12a) and (12b):

$$dH(\mu) = \frac{C}{\bar{g}^2}\frac{\gamma_0 dN_{bias}}{H} = \frac{\gamma_0}{\bar{g}} dN_{bias} \approx dN_{bias}. \quad (15)$$

Again we notice that the change in orthometric height equals the change in geoid height with opposite sign.

4 Consequences

In this section we will look at some implications caused by the fact that the residual topographic bias dN_{bias} impairs both the gravimetrically determined geoid height and the estimated orthometric height.
First we consider the geoid height determined by GNSS-levelling, theoretically given by

$$N = h - H, \quad (16)$$

but in practice typically estimated by satellite/GNSS positioning with the geodetic and orthometric heights provided according to Helmert's method:

$$N_{GNSS} = h - H^O, \quad (17)$$

leading to the following error due to the residual topographic bias (see Eq. 9):

$$dN_{GNSS} = -dH^O \approx dN_{bias}, \quad (18)$$

i.e. the same residual error as in the geoid height estimated by gravimetry. This result reveals that GNSS-levelling is useless for validating the topographic density used in a gravimetric model applied in mountainous regions.
A similar problem occurs in studying the performance of the residual (e) from geodetic heights (h) determined by the GNSS, orthometric heights (H) and gravimetric geoid heights (N) as suggested by Foroughi et al. (2017):

$$e = h - H - N. \quad (19)$$

Again, as the orthometric and geoid height estimates are both in error by the residual topographic bias but with opposite signs, and the observation h is practically immune to topographic density errors, any resulting significant error due to the lack of information about the TDD vanishes in Eq. (19). Hence, as this formula propagates all errors from the observations as well as lack of theoretical rigour in geodetic, orthometric and geoid heights, it cannot be used for studying the sensitivity to the choice of topographic density in H or N.

4.1 A numerical example

In 2009 numerical comparisons of several software products for quasigeoid determination were carried out by Ågren et al. (2009) using the data available from the Auvergne geoid computation test area in France (Duquenne 2007). Recently Foroughi et al. (2017) used the data from the same test area for orthometric and geoid height computations in a comparison of the classical height system (geoid plus orthometric height) versus the modern height system (quasigeoid and normal height) by comparing the residuals in Eq. (19) for the classical system and the corresponding residual formula

$$E = h - H^N - \zeta \quad (20)$$

for the modern system. They computed the mean and its standard deviation (STD) by Eq. (19) with the results 18.7 ± 2.9 cm and 13.2 ± 3.3 cm for all 558 points of the test area and the 75 points used in the 2009 test, respectively. However, only the second result with 75 points is of interest here for comparison with those data already used by Ågren et al. (2009) for quasigeoid determination.

The LSMSA technique for quasigeoid determination, used by the KTH team (Ågren et al. 2009), performed at least as good as any of the other methods in the comparison in 2009, and their result was used by Foroughi et al. (2017) for comparison. With these data the mean off-set and STD when applying Eq. (20) became 12.5 ± 3.34 cm. Hence, it follows that there is no significant difference in the mean off-sets when using Eqs. (19) and (20), and the STDs are practically the same. This result supports the theoretical discussion above.

5 Concluding remarks

We have shown, in accord with our postulate in the introduction, that a sufficiently accurate choice of a topographic density model in gravimetric geoid determination cannot be validated by GNSS-levelling, nor by comparing geodetic heights from satellite positioning with those from orthometric plus gravimetric geoid height estimates. The reason for this failure is that the residual topographic biases in the gravimetric and geometric geoid estimates are practically the same, and the residual error due to topographic density in the estimated orthometric height is also practically the same but with opposite sign. As a result, we conclude that today there is no simple geodetic way of validating estimated geoid and orthometric heights to the 1 cm level for orthometric heights in high mountains, as the error due to incorrect TDD increases with the square of the elevation. This conclusion is obvious, if one realizes that the problems of computing the geoid and orthometric heights are actually gravimetric inverse problems, which cannot be solved by geodetic observables alone. Such problems are, of course, avoided by defining the height system by normal heights with the quasigeoid as the vertical reference surface, as this system is independent of the topographic density distribution, as once suggested by M.S. Molodensky.

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