Journal of Theoretical and Applied Mechanics, Sofia, 2014, vol. 44, No. 3, pp. 65–90

DOI: 10.2478/jtam-2014-0017

RESPONSE TO CONCENTRATED MOVING MASSES OF ELASTICALLY SUPPORTED RECTANGULAR PLATES RESTING ON WINKLER ELASTIC FOUNDATION*

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[Received 17 July 2013. Accepted 02 June 2014]

Abstract. The dynamic response to moving concentrated masses of elastically supported rectangular plates resting on Winkler elastic foundation is investigated in this work. This problem, involving non-classical boundary conditions, is solved and illustrated with two common examples often encountered in engineering practice. Analysis of the closed form solutions shows that, for the same natural frequency (i) the response amplitude for the moving mass problem is greater than that one of the moving force problem for fixed Rotatory inertia correction factor R_0 and foundation modulus F_0, (ii) The critical speed for the moving mass problem is smaller than that for the moving force problem and so resonance is reached earlier in the former. The numerical results in plotted curves show that, for the elastically supported plate, as the value of R_0 increases, the response amplitudes of the plate decrease and that, for fixed value of R_0, the displacements of the plate decrease as F_0 increases. The results also show that for fixed R_0 and F_0, the transverse deflections of the plates under the actions of moving masses are higher than those when only the force effects of the moving load are considered. Hence, the moving force solution is not a save approximation to the moving mass problem. Also, as the mass ratio Γ approaches zero, the response amplitude of the moving mass problem approaches that one of the moving force problem of the elastically supported rectangular plate resting on constant Winkler elastic foundation.
Key words: Natural frequency, moving force, moving mass, resonance, critical speed, mass ratio.

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1. Introduction

The problem of analyzing the dynamic response of the elastic structures under the action of moving masses continues to motivate a variety of investigations [1–6]. Structural members are commonly modelled as a beam or as a plate in most analytical studies in Engineering and Mathematical Physics.

The effects of moving loads on solid bodies are dual [1]. On the one hand is the gravitational effect of the moving load while on the other hand is the inertia effect of the mass of the load on the vibrating solid bodies. The governing differential equation of motion becomes complex and cumbersome when the inertia effect of the moving load is considered and no longer has constant coefficients. In particular, the coefficients become variable and singular. The problem is termed moving force problem if the inertia effect of the moving load is neglected, and when it is retained, it is termed moving mass problem.

The problem of determining the dynamic response of structures (beams or plates) under the action of moving concentrated masses has been almost exclusively reserved for structures having the normal ideal boundary conditions. Such ideal boundary conditions include among others, Clamped edge, Free edge, Simply supported edge and Sliding edge boundary conditions. For practical applications in many cases, it is more realistic to consider non-classical boundary conditions, because the ideal boundary conditions can seldom be realized. A common example is the elastically supported end conditions. As a problem of this kind, Wilson [7] studied the response of a cantilever plate strip restrained elastically against rotation and subjected to a moving normal line load. In a later development, Saito et al [8] presented a theoretical analysis of the steady state response of a plate strip constrained elastically along its edges against rotation and translation under the action of a moving transverse line load. The first five speeds of the applied load for which a resonance effect occurs in the system are plotted as functions of the edge constraint parameters.

Several researchers have performed tremendous efforts in the study of dynamics of structures under moving loads [9–16]. Recently, Oni and Awodola [17] considered the dynamic behaviour under moving concentrated masses of elastically supported finite Bernoulli-Euler beam on Winkler foundation. The technique was based on the generalized finite integral transform method.

More recently, efforts have been made in the study of two-dimensional (plate) structures under moving loads [18–21]. Awodola and Oni [19] investigated the dynamic response to moving masses of rectangular plates with general boundary conditions and resting on variable Winkler foundation, they considered general classical boundary conditions only in their work. In all these investigations, extension of the theory to cover two-dimensional (plate) problem has not been affected, when the plate possesses non-classical boundary conditions. Therefore, this study concerns the response to moving concentrated masses of rectangular plate with non-classical boundary conditions (elastically supported) and resting on Winkler elastic foundation.

2. Governing equation

The structure under consideration is assumed to be carrying an arbitrary number (say N) of concentrated masses M_i moving with constant velocities c_i, $i = 1, 2, 3, \ldots, N$ along a straight line parallel to the x-axis (no difficulty arises by assuming that masses travel in an arbitrary path) issuing from point $y = s$ on the y-axis. The equation governing the dynamic transverse displacement $W(x, y, t)$ of the rectangular plate when it is resting on a Winkler foundation and traversed by several moving concentrated masses is the fourth order partial differential equation given by [19,20]:

$$\begin{aligned}(1)\quad \nabla^4 W(x,y,t) &+ \mu\frac{\partial^2 W(x,y,t)}{\partial t^2} \\ &= \mu R_0\left[\frac{\partial^4}{\partial t^2\partial x^2} + \frac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t) - F_0 W(x,y,t) \\ &\quad + \sum_{i=1}^{N}[M_i g\delta(x - c_i t)\delta(y - s) \\ &\quad - M_i\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x - c_i t)\delta(y - s)]\end{aligned}$$

where:

$$D = \frac{Eh^2}{12(1 - v)}, \tag{2}$$

is the bending rigidity of the plate, ∇^2 is the two-dimensional Laplacian operator, $W(x, y, t)$ is the transverse displacement, h is the plate's thickness, E is the Young's Modulus, v is the Poisson's ratio ($v < 1$), μ is the mass per unit area of the plate, R_0 is the Rotatory inertia correction factor, F_0 is the foundation's stiffness, x and y are respectively the spatial coordinates in x and y directions and t is the time coordinate, g is the acceleration due to gravity and $\delta(\cdot)$ is the Dirac–Delta function.

The initial conditions, without any loss of generality, is taken as:

$$W(x,y,t) = 0 = \frac{\partial W(x,y,t)}{\partial t}. \tag{3}$$

3. Analytical approximate solution

In the first instance, we consider rectangular plate elastically supported at edges $y = 0$, $y = L_Y$ with simple support at edges $x = 0$, $x = L_X$.

If $\frac{\partial^2 W(x,y,t)}{\partial x^2} \equiv W_{xx}(x,y,t)$, $\frac{\partial^2 \Psi_{ni}(x,t)}{\partial x^2} \equiv \Psi_{ni,xx}(x,t)$ and so on, the boundary conditions can be written as [17]:

$$W(0,y,t) = 0, \qquad W(L_X,y,t) = 0; \tag{4}$$

$$W_{yy}(x,0,t) - k_1 W_y(x,0,t) = 0, \qquad W_{yy}(x,L_Y,t) - k_1 W_y(x,L_Y,t) = 0; \tag{5}$$

$$W_{xx}(0,y,t) = 0, \qquad W_{xx}(L_X,y,t) = 0; \tag{6}$$

$$W_{yyy}(x,0,t) + k_2 W(x,0,t) = 0, \qquad W_{yyy}(x,L_Y,t) + k_2 W(x,L_Y,t) = 0. \tag{7}$$

Secondly, we consider an elastic rectangular plate resting on a variable Pasternak elastic foundation and having elastic supports at all its edges, the boundary conditions are given by [17]:

$$W_{xx}(0,y,t) - k_1 W_x(0,y,t) = 0, \qquad W_{xx}(L_X,y,t) - k_1 W_x(L_X,y,t) = 0; \tag{8}$$

$$W_{yy}(x,0,t) - k_1 W_y(x,0,t) = 0, \qquad W_{yy}(x,L_Y,t) - k_1 W_y(x,L_Y,t) = 0; \tag{9}$$

$$W_{xxx}(0,y,t) + k_2 W(0,y,t) = 0, \qquad W_{xxx}(L_X,y,t) + k_2 W(L_X,y,t) = 0; \tag{10}$$

$$W_{yyy}(x,0,t) + k_2 W(x,0,t) = 0, \qquad W_{yyy}(x,L_Y,t) + k_2 W(x,L_Y,t) = 0, \tag{11}$$

where: k_1 and k_2 are the stiffness against rotation and the stiffness against translation respectively. Evidently, an exact closed form solution of the above fourth order partial differential equation (1) does not exist. Consequently, an approximate solution is sought. Thus, the technique based on separation of variable described in [9] is employed. This versatile technique requires that the solution of equation (1) takes the form:

$$W(x,y,t) = \sum_{n=1}^{\infty} \phi_n(x,y) T_n(t), \tag{12}$$

Where: ϕ_n are the known eigen functions of the plate with the same boundary conditions and have the form of [12]:

$$\nabla^4\phi_n - \omega_n^4\phi_n = 0, \tag{13}$$

where:

$$\omega_n^4 = \frac{\Omega_n^2\mu}{D}, \tag{14}$$

Ω_n, $n = 1, 2, 3, \ldots$, are the natural frequencies of the dynamical system and $T_n(t)$ are amplitude functions, which have to be calculated. In order to solve the equation (1), it is rewritten as:

$$\begin{aligned}(15)\quad & \frac{D}{\mu}\,\nabla^4 W(x,y,t) + \frac{\partial^2 W(x,y,t)}{\partial t^2} \\ & = R_0\left[\frac{\partial^4}{\partial t^2\partial x^2} + \frac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t) - \frac{F_0}{\mu}W(x,y,t) \\ & + \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x-c_i t)\delta(y-s)\right. \\ & \left. -\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x-c_i t)\delta(y-s)\right]\end{aligned}$$

Rewriting the right hand side of equation (15) in the form of a series, we have:

$$\begin{aligned}(16)\quad & R_0\left[\frac{\partial^4}{\partial t^2\partial x^2} + \frac{\partial^4}{\partial t^2\partial y^2}\right]W(x,y,t) - \frac{F_0}{\mu}W(x,y,t) \\ & [12pt] + \sum_{i=1}^{N}\left[\frac{M_i g}{\mu}\delta(x-c_i t)\delta(y-s)\right. \\ & \left. -\frac{M_i}{\mu}\left(\frac{\partial^2}{\partial t^2} + 2c_i\frac{\partial^2}{\partial t\partial x} + c_i^2\frac{\partial^2}{\partial x^2}\right)W(x,y,t)\delta(x-c_i t)\delta(y-s)\right] \\ & = \sum_{n=1}^{\infty}\phi_n(x,y)B_n(t)\end{aligned}$$

When equation (12) is used in equation (16), integrating on area A of the plate and considering the orthogonality of $\phi_n(x,y)$, we have:

$$(17)\quad \sum_{n=1}^{\infty}\int_A \Big\{ R_0\left[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t)+\phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)\right]$$

$$-\frac{F_0}{\mu}\phi_n(x,y)\phi_p(x,y)T_n(t)+\sum_{i=1}^{N}\Big[\frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_i t)\delta(y-s)$$

$$-\frac{M_i}{\mu}\left(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t)+2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t)\right.$$

$$\left.+c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t)\ \right)\delta(x-c_i t)\delta(y-s)\Big]\Big\}\, dA$$

$$=\sum_{n=1}^{\infty}\int_A \phi_n(x,y)\phi_p(x,y)B_n(t)\, dA$$

Considering the orthogonality of $\phi_n(x,y)$, we have that:

$$(18)\quad B_n(t)$$

$$=\frac{1}{P^*}\sum_{n=1}^{\infty}\int_A \Big\{ R_0[\phi_{n,xx}(x,y)\phi_p(x,y)T_{n,tt}(t)+\phi_{n,yy}(x,y)\phi_p(x,y)T_{n,tt}(t)]$$

$$-\frac{F_0}{\mu}\phi_n(x,y)\phi_p(x,y)T_n(t)+\sum_{i=1}^{N}\Big[\frac{M_i g}{\mu}\phi_p(x,y)\delta(x-c_i t)\delta(y-s)$$

$$-\frac{M_i}{\mu}(\phi_n(x,y)\phi_p(x,y)T_{n,tt}(t)+2c_i\phi_{n,x}(x,y)\phi_p(x,y)T_{n,t}(t)$$

$$+c_i^2\phi_{n,xx}(x,y)\phi_p(x,y)T_n(t))\delta(x-c_i t)\delta(y-s)\Big]\Big\}\, dA$$

where: $P^* = \int_A \phi_p^2 dA$.

Using (18) and taking into account (13) and (14), equation (15) can be written as:

$$(19) \quad \phi_n(x,y)\left[\frac{D\omega_n^4}{\mu}T_n(t)+T_{n,tt}(t)\right]$$
$$=\frac{\phi_n(x,y)}{P^*}\sum_{q=1}^{\infty}\int_A\Bigg\{R_0[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t)+\phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)]$$
$$-\frac{F_0}{\mu}\phi_q(x,y)\phi_p(x,y)T_q(t)+\sum_{i=1}^{N}\left[\frac{M_ig}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)\right.$$
$$-\frac{M_i}{\mu}\left(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)+2c_i\phi_{q,x}(x,y)\phi_p(x,y)T_{q,t}(t)\right.$$
$$\left.\left.+c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t)\ \right)\delta(x-c_it)\delta(y-s)\right]\Bigg\}\,dA.$$

Equation (19) implies that:

$$(20) \quad T_{n,tt}(t)+\frac{D\omega_n^4}{\mu}T_n(t)$$
$$=\frac{1}{P^*}\sum_{q=1}^{\infty}\int_A\Bigg\{R_0[\phi_{q,xx}(x,y)\phi_p(x,y)T_{q,tt}(t)+,\phi_{q,yy}(x,y)\phi_p(x,y)T_{q,tt}(t)]$$
$$-\frac{F_0}{\mu}\phi_q(x,y)\phi_p(x,y)T_q(t)+\sum_{i=1}^{N}\left[\frac{M_ig}{\mu}\phi_p(x,y)\delta(x-c_it)\delta(y-s)\right.$$
$$-\frac{M_i}{\mu}(\phi_q(x,y)\phi_p(x,y)T_{q,tt}(t)+2c_i\phi_{q,x}(x,y)\phi_p(x,y)T_{q,t}(t)$$
$$\left.+c_i^2\phi_{q,xx}(x,y)\phi_p(x,y)T_q(t))\delta(x-c_it)\delta(y-s)\right]\Bigg\}\,dA.$$

Equation (29) is a set of coupled second order ordinary differential equations.

Expressing the Dirac–Delta function in the Fourier cosine series, as:

$$(21) \quad \begin{aligned}\delta(x-c_it)&=\frac{1}{L_X}\left[1+2\sum_{j=1}^{\infty}\cos\frac{j\pi c_it}{L_X}\cos\frac{j\pi x}{L_X}\right],\\ \delta(y-s)&=\frac{1}{L_Y}\left[1+2\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\cos\frac{k\pi y}{L_Y}\right],\end{aligned}$$

equation (20) then becomes:

$$
\begin{aligned}
(22)\quad & \frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\Bigg\{ R_0 P_1^* \frac{d^2T_q(t)}{dt^2} - \frac{F_0}{\mu} P_2^* T_q(t) \\
& - \sum_{i=1}^{N} \frac{M_i}{L_X L_Y \mu}\Bigg[2\Bigg(\frac{P_2^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y} P_3^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X} P_3^{***}(j) \\
& \quad +2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y} P_3^{****}(j,k)\Bigg) \frac{d^2T_q(t)}{dt^2} \\
& \quad + 4c_i \Bigg(\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y} P_4^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X} P_4^{***}(j) \\
& \quad +2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y} P_4^{****}(j,k)\Bigg) \frac{dT_q(t)}{dt} \\
& \quad + 2c_i^2 \Bigg(\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y} P_5^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi c_i t}{L_X} P_5^{***}(j) \\
& + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi c_i t}{L_X}\cos\frac{k\pi s}{L_Y} P_5^{****}(j,k)\Bigg) T_q(t) \Bigg]\Bigg\} = \sum_{i=1}^{N}\frac{M_i g}{P^*\mu}\phi_p(c_i t, s),
\end{aligned}
$$

where:

$$
\begin{aligned}
& \alpha_n^2 = \frac{D\omega_n^4}{\mu}, \quad P_1^* = \int_0^{L_X}\int_0^{L_Y} \left[\phi_{n,xx}(x,y) + \phi_{n,yy}(x,y)\right]\phi_p(x,y)\,dy\,dx, \\
& P_2^* = \int_0^{L_X}\int_0^{L_Y} \phi_n(x,y)\,\phi_p(x,y)\,dy\,dx, \\
& P_3^{**}(k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{k\pi y}{L_Y}\,\phi_n(x,y)\,\phi_p(x,y)\,dy\,dx, \\
& P_3^{***}(j) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\phi_n(x,y)\,\phi_p(x,y)\,dy\,dx, \\
& P_3^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\,\phi_n(x,y)\,\phi_p(x,y)\,dy\,dx, \\
& P_4^* = \int_0^{L_X}\int_0^{L_Y} \phi_{n,x}(x,y)\,\phi_p(x,y)\,dy\,dx,
\end{aligned}
$$

$$P_4^{**}(k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{k\pi y}{L_Y}\,\phi_{n,x}(x,y)\,\phi_p(x,y)\,dy\,dx \tag{23}$$

$$P_4^{***}(j) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\phi_{n,x}(x,y)\,\phi_p(x,y)\,dy\,dx,$$

$$P_4^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\,\phi_{n,x}(x,y)\,\phi_p(x,y)\,dy\,dx,$$

$$P_5^{*} = \int_0^{L_X}\int_0^{L_Y} \phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx,$$

$$P_5^{**}(k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{k\pi y}{L_Y}\,\phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx,$$

$$P_5^{***}(j) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx$$

$$P_5^{****}(j,k) = \int_0^{L_X}\int_0^{L_Y} \cos\frac{j\pi x}{L_X}\cos\frac{k\pi y}{L_Y}\,\phi_{n,xx}(x,y)\,\phi_p(x,y)\,dy\,dx.$$

Equation (22) is the transformed equation governing the problem of the rectangular plate on a variable Winkler elastic foundation. In what follows, ϕ_n(x, y) are assumed to be the products of the beam functions ψ_{ni}(x) and ψ_{nj}(y) [22]. That is:

$$\phi_n(x,y) = \psi_{ni}(x)\psi_{nj}(y). \tag{24}$$

These beam functions can be defined, respectively as:

$$\psi_{ni}(x) = \sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}, \tag{25}$$

$$\psi_{nj}(y) = \sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}. \tag{26}$$

where: A_{ni}, A_{nj}, B_{ni}, B_{nj}, C_{ni} and C_{nj} are constants determined by the boundary conditions. Ω_{ni} and Ω_{nj} are called the mode frequencies.

In order to solve equation (22), we shall consider only one mass M travelling with uniform velocity c along the line $y = s$. The solution for any arbitrary number of moving masses can be obtained by superposition of the individual solution, since the governing differential equation is linear. Thus, for the single mass M_1 equation (22) reduces to:

$$\frac{d^2T_n(t)}{dt^2} + \alpha_n^2 T_n(t) - \frac{1}{P^*}\sum_{q=1}^{\infty}\left\{R_0 P_1^*\frac{d^2T_q(t)}{dt^2} - \frac{F_0}{\mu}P_2^* T_q(t)\right. \tag{27}$$

$$-\Gamma\left[2\left(\frac{P_2^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)\right.\right.$$

$$\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2}$$

$$+4c\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)\right.$$

$$\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt}$$

$$+2c^2\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)\right.$$

$$\left.\left.\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\right]\right\}=\frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s)$$

where:

$$\Gamma=\frac{M}{L_XL_Y\mu}. \tag{28}$$

Equation (27) is the fundamental equation of our problem, when the rectangular plate has arbitrary end support conditions. In what follows, we shall discuss two special cases of equation (27) namely; the moving force and the moving mass problems.

Case I: Moving force problem

An approximate model of the differential equation describing the response of a rectangular plate resting on a variable Winkler elastic foundation and traversed by a moving force would be obtained by setting $\Gamma = 0$ in equation (27).

Thus, setting $\Gamma = 0$ in equation (27), we have:

$$\frac{d^2T_n(t)}{dt^2}+\alpha_n^2T_n(t)-\frac{P_1^*R_0}{P^*}\sum_{q=1}^{\infty}\frac{d^2T_q(t)}{dt^2}+\frac{P_2^*F_0}{\mu P^*}\sum_{q=1}^{\infty}T_q(t)$$
$$=\frac{Mg}{P^*\mu}\Psi_{pi}(ct)\Psi_{pj}(s). \tag{29}$$

Evidently, an exact analytical solution to this equation is not possible. Consequently, the approximate analytical solution technique, which is a modification of the asymptotic method of Struble extensively discussed in [17, 23] shall be used to be employed. By this technique, after some rigorous mathematical procedures, without much difficulty however, the homogeneous part of equation (29) after some simplifications and rearrangements becomes:

$$\frac{d^2T_n(t)}{dt^2} + \gamma_{sf}^2 T_n(t) = 0. \tag{30}$$

where:

$$\gamma_{sf} = \gamma_s \left[1 + \frac{\varepsilon_0 P_1^*}{2}\right], \tag{31}$$

is the modified frequency corresponding to the frequency of the free system due to the presence of the rotatory inertia. It is observed that when $\varepsilon_0 = 0$, we recover the frequency of the moving force problem when the rotatory inertia effect is neglected.

In order to solve the non-homogenous equation (29), the differential operator which acts on $T_n(t)$ is replaced by the equivalent free system operator, defined by the modified frequency γ_{sf}. Thus:

$$\frac{d^2T_n(t)}{dt^2} + \gamma_{sf}^2 T_n(t) = K_0 \Psi_{pi}(ct)\Psi_{pj}(s), \tag{32}$$

where:

$$K_0 = \frac{Mg}{P^*\mu}. \tag{33}$$

Therefore, the moving force problem is reduced to the non-homogeneous ordinary differential equation, given as:

$$\begin{aligned}\frac{d^2T_n(t)}{dt^2} &+ \gamma_{sf}^2 T_n(t)\\ &= K_0 \Psi_{pj}(s)\left[\sin\alpha_{pi}t + A_{pi}\cos\alpha_{pi}t + B_{pi}\sinh\alpha_{pi}t + C_{pi}\cosh\alpha_{pi}t\right],\end{aligned} \tag{34}$$

where: $\alpha_{pi} = \dfrac{\Omega_{pi}c}{L_X}$.

One obtains expression for $T_n(t)$ when equation (34) is solved in conjunction with the initial conditions. Thus, in view of equation (12), one obtains:

$$(35)\quad W(x,y,t)=\sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty}\frac{K_0\Psi_{pj}(s)}{\gamma_{sf}[\gamma_{sf}^4-\alpha_{pi}^4]}\Big\{[\gamma_{sf}^2-\alpha_{pi}^2]$$

$$\times[C_{pi}\gamma_{sf}(\cosh\alpha_{pi}t-\cos\gamma_{sf}t)+B_{pi}(\gamma_{sf}\sinh\alpha_{pi}t-\alpha_{pi}\sin\gamma_{sf}t)]$$

$$+[\gamma_{sf}^2+\alpha_{pi}^2][A_{pi}\gamma_{sf}(\cos\alpha_{pi}t-\cos\gamma_{sf}t)\ -(\alpha_{pi}\sin\gamma_{sf}t-\gamma_{sf}\sin\alpha_{pi}t)]\Big\}$$

$$\times\left[\sin\frac{\Omega_{ni}x}{L_X}+A_{ni}\cos\frac{\Omega_{ni}x}{L_X}+B_{ni}\sinh\frac{\Omega_{ni}x}{L_X}+C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}\right]$$

$$\times\left[\sin\frac{\Omega_{nj}y}{L_Y}+A_{nj}\cos\frac{\Omega_{nj}y}{L_Y}+B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y}+C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}\right].$$

Equation (35) represents the transverse displacement response to a moving force of a rectangular plate resting on Winkler elastic foundation.

Case II: Moving mass problem

If the mass of the moving load is commensurable with that of the structure, the inertia effect of the moving mass is not negligible. Thus $\Gamma\neq 0$ and one is required to solve the entire equation (27) when no term of the coupled differential equation is neglected. This is termed the moving mass problem.

Thus, equation (27) can be rewritten in the form

$$(36)\quad \left[1+\frac{2\varepsilon}{P^*}\left(\frac{P_2^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)\right.\right.$$

$$\left.\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\right]\frac{d^2T_n(t)}{dt^2}$$

$$+\frac{4\varepsilon c}{P^*}\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)\right.$$

$$\left.+\ 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_n(t)}{dt}$$

$$+\left[\gamma_{sf}^2+\frac{2\varepsilon c^2}{P^*}\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)\right.\right.$$

$$\left.\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)\right]T_n(t)$$

$$+\frac{\varepsilon}{P^*}\sum_{\substack{q=1\\q\neq n}}\left[2\left(\frac{P_2^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_3^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j)\right.\right.$$

$$\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right)\frac{d^2T_q(t)}{dt^2}$$

$$+4c\left(\frac{P_4^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_4^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j)\right.$$

$$\left.+\,2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right)\frac{dT_q(t)}{dt}$$

$$+2c^2\left(\frac{P_5^*}{2}+\sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}P_5^{**}(k)+\sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j)\right.$$

$$\left.\left.+2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right)T_q(t)\right]$$

$$=\frac{\varepsilon g L_X L_Y}{P^*}\Psi_{pi}(ct)\Psi_{pj}(s)$$

where: $\varepsilon=\dfrac{M}{L_X L_Y\mu}$.

We rearrange equation (36) to take the form:

$$\frac{d^2T_n(t)}{dt^2} + \frac{\mu_0 R_2(t)}{1+\mu_0 R_1(t)}\frac{dT_n(t)}{dt} + \frac{\gamma_{sf}^2 + \mu_0 R_3(t)}{1+\mu_0 R_1(t)}T_n(t) + \frac{\mu_0}{1+\mu_0 R_1(t)}\sum_{\substack{q=1\\ q\neq n}}^{\infty}\left[R_1(t)\,\frac{d^2T_q(t)}{dt^2} + R_2(t)\frac{dT_q(t)}{dt} + R_3(t)T_q(t)\right] = \frac{\mu_0 g L_X L_Y}{[1+\mu_0 R_1(t)]P^*}\Psi_{pi}(ct)\Psi_{pj}(s) \tag{37}$$

where ε has been written as a function of the mass ratio μ_0,

$$\begin{aligned} R_1(t) &= \frac{2}{P^*}\left[\frac{P_2^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\,P_3^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_3^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_3^{****}(j,k)\right]\\ R_2(t) &= \frac{2c}{P^*}\left[\frac{P_4^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\,P_4^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_4^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_4^{****}(j,k)\right]\\ R_3(t) &= \frac{2c^2}{P^*}\left[\frac{P_5^*}{2} + \sum_{k=1}^{\infty}\cos\frac{k\pi s}{L_Y}\,P_5^{**}(k) + \sum_{j=1}^{\infty}\cos\frac{j\pi ct}{L_X}P_5^{***}(j) + 2\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}\cos\frac{j\pi ct}{L_X}\cos\frac{k\pi s}{L_Y}P_5^{****}(j,k)\right] \end{aligned} \tag{38}$$

Considering the homogeneous part of the equation (37) and going through similar arguments and analysis as in [17, 23], the modified frequency corresponding to the frequency of the free system due to the presence of the moving mass is obtained as:

$$\beta_{sf} = \gamma_{sf}\left[1 - \frac{\mu_0}{2}\left(R_1 - \frac{R_3}{\gamma_{sf}^2}\right)\right], \tag{39}$$

retaining terms to $o(\mu_0)$ only.

Thus, to solve the non-homogeneous equation (37), the differential operator which acts on $T_n(t)$ and $T_q(t)$ is replaced by the equivalent free system operator defined by the modified frequency β_{sf}. Therefore, taking into account equations (25) and (26), we have:

$$\frac{d^2T_n(t)}{dt^2} + \beta_{sf}^2 T_n(t) = G_0\Psi_{pj}(s)\sin\alpha_{pi}t + A_{pi}\cos\alpha_{pi}t + B_{pi}\sinh\alpha_{pi}t + C_{pi}\cosh\alpha_{pi}t, \tag{40}$$

where:

$$G_0 = \frac{\mu_0 g L_X L_Y}{P^*}. \tag{41}$$

It is noticed, that equation (40) is analogous to equation (34) with β_{sf} and G_0 replacing γ_{sf} and K_0, respectively. Therefore, when equation (40) is solved in conjunction with the initial conditions, one obtains expression for $T_n(t)$ and in view of equation (20), one obtains:

$$W(x,y,t) = \sum_{ni=1}^{\infty}\sum_{nj=1}^{\infty}\frac{G_0\Psi_{pj}(s)}{\beta_{sf}[\beta_{sf}^4 - \alpha_{pi}^4]}\Big\{[\beta_{sf}^2 - \alpha_{pi}^2]$$
$$\times [C_{pi}\beta_{sf}(\cosh\alpha_{pi}t - \cos\beta_{sf}t) + B_{pi}(\beta_{sf}\sinh\alpha_{pi}t - \alpha_{pi}\sin\beta_{sf}t)]$$
$$+ [\beta_{sf}^2 + \alpha_{pi}^2][A_{pi}\beta_{sf}(\cos\alpha_{pi}t - \cos\beta_{sf}t) - (\alpha_{pi}\sin\beta_{sf}t - \beta_{sf}\sin\alpha_{pi}t)]\Big\}$$
$$\times \left[\sin\frac{\Omega_{ni}x}{L_X} + A_{ni}\cos\frac{\Omega_{ni}x}{L_X} + B_{ni}\sinh\frac{\Omega_{ni}x}{L_X} + C_{ni}\cosh\frac{\Omega_{ni}x}{L_X}\right]$$
$$\times \left[\sin\frac{\Omega_{nj}y}{L_Y} + A_{nj}\cos\frac{\Omega_{nj}y}{L_Y} + B_{nj}\sinh\frac{\Omega_{nj}y}{L_Y} + C_{nj}\cosh\frac{\Omega_{nj}y}{L_Y}\right] \tag{42}$$

Equation (42) is the transverse displacement response to a moving mass of a rectangular plate resting on Winkler elastic foundation. The constants A_{ni}, A_{pi}, A_{nj}, A_{pj}, B_{ni}, B_{pi}, B_{nj}, B_{pj}, C_{ni}, C_{pi}, C_{nj} and C_{pj} are to be determined from the choice of the end support condition.

4. Analysis of the solution

At this juncture, we examine the conditions under which the deflections of an elastic plate may grow without bound. Equation (35) clearly shows that the rectangular plate on a variable Winkler elastic foundation and traversed by a moving force reaches a state of resonance, whenever:

$$\gamma_{sf} = \frac{\Omega_{pi} c}{L_X}, \tag{43}$$

while equation (42) shows that the same plate under the action of a moving mass experiences resonance effect, whenever:

$$\beta_{sf} = \frac{\Omega_{pi} c}{L_X}, \tag{44}$$

where:

$$\beta_{sf} = \gamma_{sf} \left[1 - \frac{\mu_0}{2} \left(R_1 - \frac{R_3}{\gamma_{sf}^2} \right) \right]. \tag{45}$$

Equations (44) and (45) imply, that:

$$\beta_{sf} = \gamma_{sf} \left[1 - \frac{\mu_0}{2} \left(R_1 - \frac{R_3}{\gamma_{sf}^2} \right) \right] = \frac{\Omega_{pi} c}{L_X}. \tag{46}$$

Consequently, from equations (43) and (46), for the same natural frequency, the system critical speed (and the natural frequency) of a rectangular plate traversed by a moving mass is smaller than that of the same system traversed by a moving force. Thus, for the same plate natural frequency, the resonance is reached earlier when we consider the moving mass system than when we consider the moving force system.

5. Illustrative examples

a. Simple-Elastic Rectangular plate

The plate is taken to be simply supported at $x = 0$ and $x = L_X$ and at the edges $y = 0$ and $y = L_Y$, it is taken to be elastically supported. Using the conditions (4–7) in equations (25) and (26), the following values of the constants and the frequency equation are obtained for the elastic edges.

$$C_{nj} = \left(\left[\frac{\Omega_{nj}}{L_Y} - k_1 r_2\right]\sin\Omega_{nj} + \left[k_1 + \frac{r_2\Omega_{nj}}{L_Y}\right]\cos\Omega_{nj} - \frac{r_1\Omega_{nj}}{L_Y}\sinh\Omega_{nj} + k_1 r_1\cosh\Omega_{nj}\right) \Bigg/ \left(k_1 r_1 \sin\Omega_{nj} - \frac{r_1\Omega_{nj}}{L_Y}\cos\Omega_{nj} + \left[\frac{r_3\Omega_{nj}}{L_Y} - k_1\right]\sinh\Omega_{nj} + \left[\frac{\Omega_{nj}}{L_Y} - k_1 r_3\right]\cosh\Omega_{nj}\right)$$
$$= \left(-\left[\frac{r_2\Omega_{nj}^3}{L_Y^3} + k_2\right]\sin\Omega_{nj} + \left[\frac{\Omega_{nj}^3}{L_Y^3} - k_2 r_2\right]\cos\Omega_{nj} - k_2 r_1\sinh\Omega_{nj} - \frac{r_1\Omega_{nj}^3}{L_Y^3}\cosh\Omega_{nj}\right) \Bigg/ \left(\frac{r_1\Omega_{nj}^3}{L_Y^3}\sin\Omega_{nj} + k_2 r_1\cos\Omega_{nj} + \left[\frac{\Omega_{nj}^3}{L_Y^3} + k_2 r_3\right]\sinh\Omega_{nj} + \left[\frac{r_3\Omega_{nj}^3}{L_Y^3} + k_2\right]\cosh\Omega_{nj},\right) \quad (47)$$

$A_{nj} = r_1 C_{nj} + r_2$ and $B_{nj} = r_3 C_{nj} + r_1$, where:

$$r_1 = \frac{\frac{\Omega_{nj}^4}{L_Y^4} + k_1 k_2}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2}; \quad r_2 = \frac{-\frac{2k_1\Omega_{nj}^3}{L_Y^3}}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2} \quad \text{and} \quad r_3 = \frac{\frac{-2k_2\Omega_{nj}}{L_Y}}{\frac{\Omega_{nj}^4}{L_Y^4} - k_1 k_2}. \quad (48)$$

Equation (47) when simplified yields:

$$\tan\Omega_{nj} = \tanh\Omega_{nj}, \quad (49)$$

which is termed the frequency equation for the elastic edge, such that:

$$\Omega_1 = 3.927, \quad \Omega_2 = 7.069, \quad \Omega_3 = 10.210, \ldots \quad (50)$$

For the simple edges, it can be shown that:

$$A_{ni} = 0, \quad B_{ni} = 0, \quad C_{ni} = 0, \quad \text{and} \quad \Omega_{ni} = n_i\pi, \quad (51)$$

Similarly:

$$A_{pi}=0,\quad B_{pi}=0,\quad C_{pi}=0,\quad \text{and}\quad \Omega_{pi}=p_i\pi. \tag{52}$$

Using (47), (48), (49), (51) and (52) in equations (35) and (42), one obtains the displacement response, respectively to a moving force and a moving mass of a simple-elastic rectangular plate resting on a constant Winkler elastic foundation.

b. Elastic support at all edges

Using the conditions (8-11) in equations (25) and (26), one obtains:

$$\begin{aligned} C_{ni}=&\left(\left[\frac{\Omega_{ni}}{L_X}-k_1r_2(i)\right]\sin\Omega_{ni}+\left[k_1+\frac{r_2(i)\Omega_{ni}}{L_X}\right]\cos\Omega_{ni}-\frac{r_1(i)\Omega_{ni}}{L_X}\sinh\Omega_{ni}\right.\\ &\left.+k_1r_1(i)\cosh\Omega_{ni}\right)\Bigg/\left(k_1r_1(i)\sin\Omega_{ni}-\frac{r_1(i)\Omega_{ni}}{L_X}\cos\Omega_{ni}\right.\\ &\left.+\left[\frac{r_3(i)\Omega_{ni}}{L_X}-k_1\right]\sinh\Omega_{ni}+\left[\frac{\Omega_{ni}}{L_X}-k_1r_3(i)\right]\cosh\Omega_{ni}\right)\\ =&\left(-\left[\frac{r_2(i)\Omega_{ni}^3}{L_X^3}+k_2\right]\sin\Omega_{ni}+\left[\frac{\Omega_{ni}^3}{L_X^3}-k_2r_2(i)\right]\cos\Omega_{ni}-k_2r_1(i)\sinh\Omega_{ni}\right.\\ &\left.-\frac{r_1(i)\Omega_{ni}^3}{L_X^3}\cosh\Omega_{ni}\right)\Bigg/\left(\frac{r_1(i)\Omega_{ni}^3}{L_X^3}\sin\Omega_{ni}+k_2r_1(i)\cos\Omega_{ni}\right.\\ &\left(+\left[\frac{\Omega_{ni}^3}{L_X^3}+k_2r_3(i)\right]\sinh\Omega_{ni}+\left[\frac{r_3(i)\Omega_{ni}^3}{L_X^3}+k_2\right]\cosh\Omega_{ni}\right) \end{aligned} \tag{53}$$

$A_{ni}=r_1(i)C_{ni}+r_2(i)$ and $B_{ni}=r_3(i)C_{ni}+r_1(i)$, where:

$$r_1(i)=\frac{\frac{\Omega_{ni}^4}{L_X^4}+k_1k_2}{\frac{\Omega_{ni}^4}{L_X^4}-k_1k_2};\quad r_2(i)=\frac{-\frac{2k_1\Omega_{ni}^3}{L_X^3}}{\frac{\Omega_{ni}^4}{L_X^4}-k_1k_2}\quad\text{and}\quad r_3(i)=\frac{\frac{-2k_2\Omega_{ni}}{L_X}}{\frac{\Omega_{ni}^4}{L_X^4}-k_1k_2}. \tag{54}$$

and

$$(55)\quad C_{nj}=\left(\left[\frac{\Omega_{nj}}{L_Y}-k_1r_2(j)\right]\sin\Omega_{nj}+\left[k_1+\frac{r_2(j)\Omega_{nj}}{L_Y}\right]\cos\Omega_{nj}-\frac{r_1(j)\Omega_{nj}}{L_Y}\sinh\Omega_{nj}\right.$$

$$\left.+k_1r_1(j)\cosh\Omega_{nj}\right)\Bigg/\left(k_1r_1(j)\sin\Omega_{nj}-\frac{r_1(j)\Omega_{nj}}{L_Y}\cos\Omega_{nj}\right.$$

$$\left.+\left[\frac{r_3(j)\Omega_{nj}}{L_Y}-k_1\right]\sinh\Omega_{nj}+\left[\frac{\Omega_{nj}}{L_Y}-k_1r_3(j)\right]\cosh\Omega_{nj}\right)$$

$$=\left(-\left[\frac{r_2(j)\Omega^3_{nj}}{L^3_Y}+k_2\right]\sin\Omega_{nj}+\left[\frac{\Omega^3_{nj}}{L^3_Y}-k_2r_2(j)\right]\cos\Omega_{nj}-k_2r_1(j)\sinh\Omega_{nj}\right.$$

$$\left.-\frac{r_1(j)\Omega^3_{nj}}{L^3_Y}\cosh\Omega_{nj}\right)\Bigg/\left(\frac{r_1(j)\Omega^3_{nj}}{L^3_Y}\sin\Omega_{nj}+k_2r_1(j)\cos\Omega_{nj}\right.$$

$$\left.+\left[\frac{\Omega^3_{nj}}{L^3_Y}+k_2r_3(j)\right]\sinh\Omega_{nj}+\left[\frac{r_3(j)\Omega^3_{nj}}{L^3_Y}+k_2\right]\cosh\Omega_{nj}\right)$$

where: $A_{nj}=r_1(j)C_{nj}+r_2(j)$, $B_{nj}=r_3(j)C_{nj}+r_1(j)$,

$$(56)\quad r_1(j)=\frac{\frac{\Omega^4_{nj}}{L^4_Y}+k_1k_2}{\frac{\Omega^4_{nj}}{L^4_Y}-k_1k_2};\quad r_2(j)=\frac{-\frac{2k_1\Omega^3_{nj}}{L^3_Y}}{\frac{\Omega^4_{nj}}{L^4_Y}-k_1k_2}\quad\text{and}\quad r_3(j)=\frac{\frac{-2k_2\Omega_{nj}}{L_Y}}{\frac{\Omega^4_{nj}}{L^4_Y}-k_1k_2}.$$

Equations (53) and (54) when simplified yield:

$$(57)\qquad \tan\Omega_{ni}=\tanh\Omega_{ni}\quad\text{and}\quad\tan\Omega_{nj}=\tanh\Omega_{nj}$$

Using (53), (54), (55), (56) and (57) in equations (35) and (42), one obtains the transverse-displacement response, respectively to a moving force and a moving mass of an elastically supported rectangular plate resting on a constant Winkler elastic foundation.

6. Numerical calculations and discussion of results

A rectangular plate of length $L_Y = 0.914$ m and breadth $L_X = 0.457$ m is considered to carry out the calculations of practical interests in dynamics of structures and Engineering design for the illustrative examples. The velocity of the mass is assumed to be 0.8123 m/s. The values for E, S and Γ are chosen to be 3.109×10^9 kg/m^2, 0.4 m and 0.2, respectively. the deflections of the plate for the illustrative examples are calculated and plotted against time t. The deflections of the plate for the illustrative examples are calculated and plotted

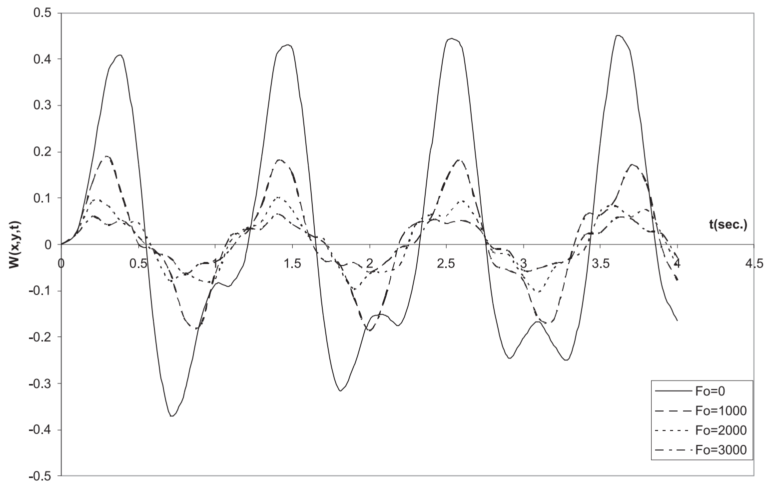

Fig. 1. Deflection of moving mass for simple-elastic rectangular plate on Winkler foundation for various values of foundation modulus F_0

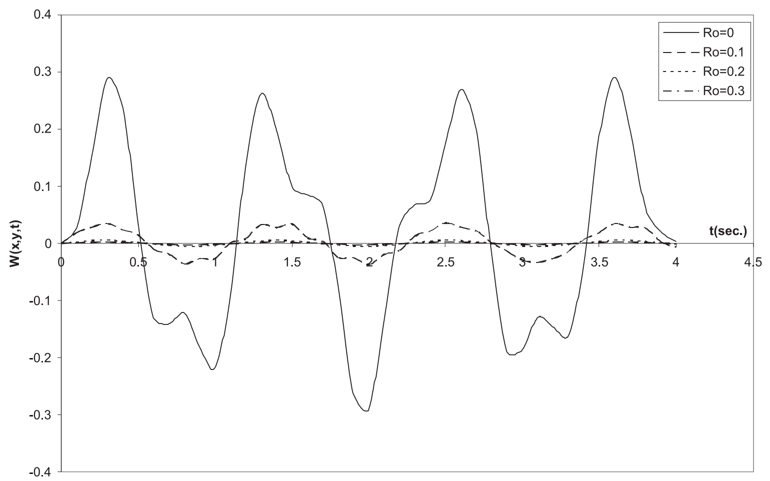

Fig. 2. Deflection profile of moving mass for simple-elastic plate on Winkler foundation for various values of rotatory inertia correction factor R_0

against time t for various values of the foundation moduli F_0 and the rotatory inertia correction factor R_0. Figures 1 and 2 display the effects of foundation modulus F_0 and rotatory inertial correction factor R_0 on the transverse deflection of moving mass for simple-elastic rectangular plate, respectively. The

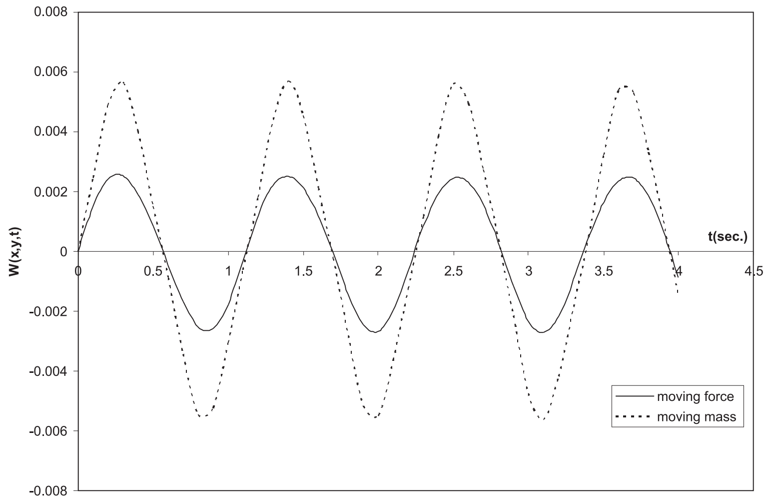

Fig. 3. Comparison of the deflections of moving force and moving mass cases for simple-elastic rectangular plate on Winkler foundation

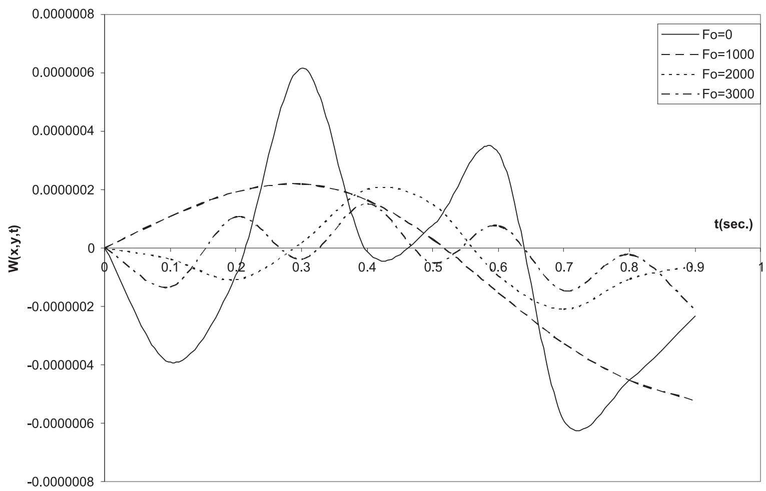

Fig. 4. Deflection of moving mass for elastic-elastic rectangular plate on Winkler foundation for various values of foundation modulus F_0

graphs show that the response amplitudes decrease as the values of F_0 and R_0 increase.

Figure 3 compares the displacement curves of the moving force and moving mass for simple-elastic rectangular plate for fixed F_0 and R_0, it is shown that the response amplitude of moving mass is greater than that one

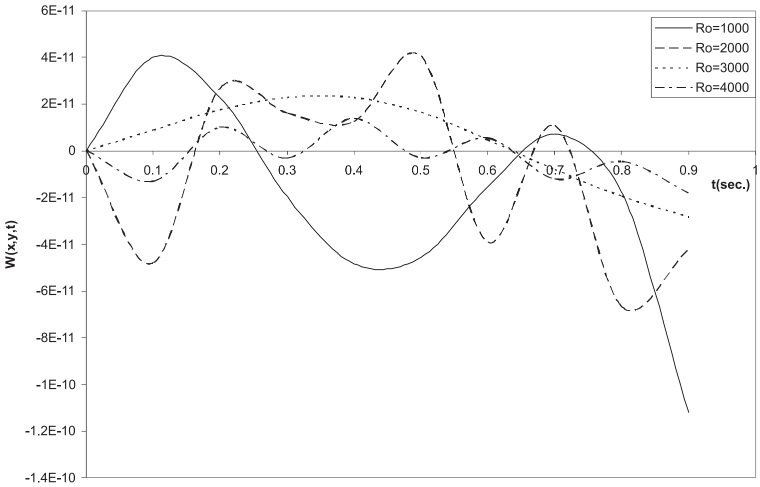

Fig. 5. Deflection of moving force for elastic-elastic plate on Winkler foundation for various values of rotatory inertial correction factor R_0

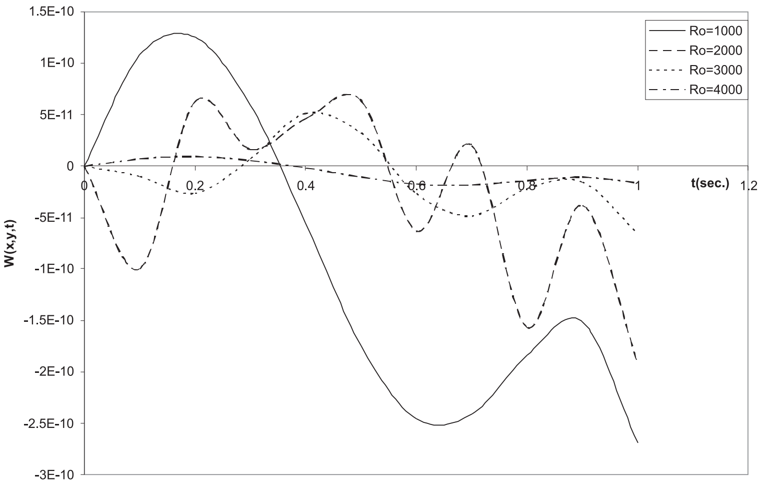

Fig. 6. Deflection of moving mass for elastic-elastic plate on Winkler foundation for various values of rotatory inertial correction factor R_0

of moving force problem. This result holds for other choices of non-classical boundary conditions.

The effect of F_0 on the transverse deflection of moving mass for elasti-

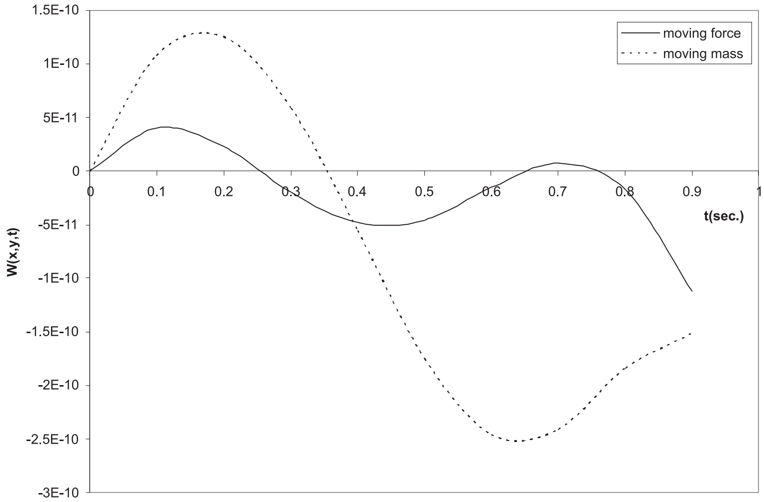

Fig. 7. Comparison of the deflections of moving force and moving mass cases for elastic-elastic rectangular plate on Winkler foundation

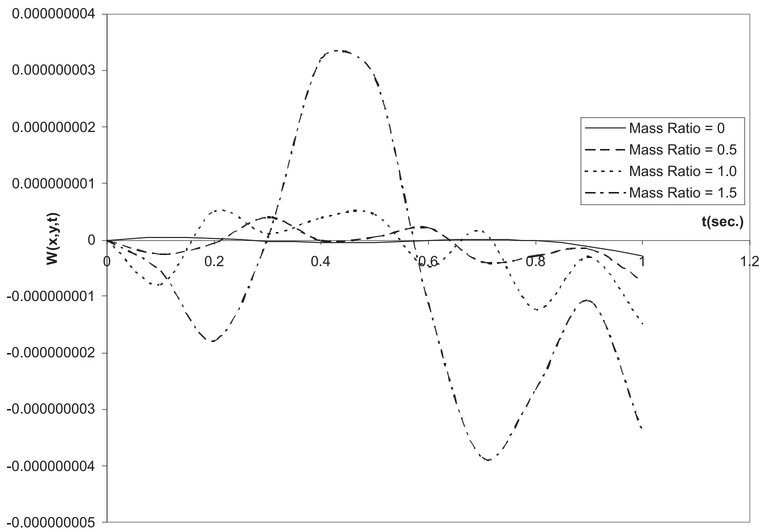

Fig. 8. Deflection of moving mass for elastic-elastic plate on Winkler foundation for various values of the Mass Ratio

cally supported (elastic-elastic) rectangular plate is presented in Fig. 4. The graph shows that the response amplitudes decrease as the value of F_0 increases.

Figures 5 and 6 display the effects of rotatory inertial correction factor R_0 on the transverse deflection of moving force and moving mass for the elastically supported plate, respectively. The graphs show that the response amplitudes decrease as the value of R_0 increases.

For the purpose of comparison, Fig. 7 shows that the response amplitude of moving mass is greater than that of moving force problem for the elastic-elastic rectangular plate resting on a constant Winkler elastic foundation. It is shown in Fig. 8 that as the mass ratio Γ reduces the response amplitude also reduces and approaches the moving force solution as Γ approaches zero.

7. Conclusion

The dynamic response to moving masses of elastically supported rectangular plates resting on constant Winkler elastic foundation is considered in this work. The fourth order partial differential equation governing the system is a non-homogenous equation with variable and singular coefficients. The method based on Separation of variables is used to transform the governing equation to a set of coupled second order ordinary differential equations. The modified Struble's technique and the method of integral transformations are employed to obtain the closed form solution of the transformed equation for both cases of moving force and moving mass problems.

From the analyses of the solutions, for the same natural frequency, the critical speed (and the natural frequency) for the system of an elastically supported rectangular plate traversed by a moving mass is smaller than that of the same system traversed by a moving force. Thus, for the same natural frequency of the plate, the resonance is reached earlier when we consider the moving mass system than when we consider the moving force system. The analyses shows, that the moving force solution is not an upper bound for the accurate solution of the moving mass problem and that as the rotatory inertia correction factor increases, the response amplitudes of the plates decrease for both cases of moving force and moving mass problem. The displacements of the rectangular plates resting on Winkler elastic foundations decrease as the foundation modulus increases when the rotatory inertia correction factor is fixed.

Furthermore, the response amplitude for the moving mass problem is greater than that of the moving force problem for fixed values of rotatory inertia correction factor and foundation modulus, this implies that resonance is reached earlier in moving mass problem than in moving force problem, it is therefore unsafe to rely on the moving force solutions. Also, as the mass ratio Γ approaches zero, the response amplitude of the moving mass problem approaches that of the moving force problem of the rectangular plate resting on constant Winkler elastic foundation. Finally, the results in this work agree with what obtain in literature [17–21]. Hence, the method employed in this work is accurate and the solutions are convergent.

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