Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 2(3) 2004 339–361

Discrete limit theorems for general Dirichlet series. III

A. Laurinčikas[a*], R. Macaitienė[†]

Department of Mathematics and Informatics,
Vilnius University,
Naugarduko 24, LT-03225 Vilnius, Lithuania

Received 24 February 2004; accepted 23 April 2004

Abstract: Here we prove a limit theorem in the sense of the weak convergence of probability measures in the space of meromorphic functions for a general Dirichlet series. The explicit form of the limit measure in this theorem is given.

Keywords: Dirichlet series, probability measure, random element, weak convergence
MSC (2000): 11M41, 30B50, 60B10

[a] Partially supported by Lithuanian Foundation of Studies and Science

1 Introduction

Let $s = \sigma + it$ be a complex variable, and let $\mathbb{R}$ and $\mathbb{C}$ denote the set of real and complex numbers, respectively. The series

$$\sum_{m=1}^{\infty} a_m e^{-\lambda_m s}, \tag{1}$$

where $a_m \in \mathbb{C}$ and $\lambda_m \in \mathbb{R}$, $0 < \lambda_1 < \lambda_2 < \ldots$, $\lim_{m\to\infty} \lambda_m = +\infty$, is called a general Dirichlet series with coefficients a_m and exponents λ_m. If $\lambda_m = \log m$, then we have the ordinary Dirichlet series

$$\sum_{m=1}^{\infty} \frac{a_m}{m^s}.$$

* E-mail: antanas.laurincikas@maf.vu.lt
† E-mail: renata.macaitiene@mif.vu.lt

Suppose that (1) absolutely converges for $\sigma > \sigma_a$ and has a sum $f(s)$. Then the function $f(s)$ is regular in the half-plane $\sigma > \sigma_a$. The value-distribution of the function $f(s)$ is complicated, therefore probabilistic methods were used to begin attacking the problem. It is convenient to state probabilistic results for functions given by Dirichlet series in the form of limit theorems in the sense of the weak convergence of probability measures. This idea belongs to H. Bohr who jointly B. Jessen [2], [3] obtained limit theorems for the Riemann zeta-function. Later many mathematicians (A. Wintner, V. Borchsenius, A. Selberg, A. Ghosh, P.D.T.A. Elliott, E. Stankus, D. Joyner, D. Hejhal, E. M. Nikishin, B. Bagchi, K. Matsumoto, R. Garunkštis, W. Schwarz, J. Steuding, R. Šleževičienė, R. Kačinskaitė, J. Ignatavičiūtė, I. Belov, the authors and others) extended and generalized Bohr–Jessen's results. In general, the probabilistic value-distribution of the ordinary Dirichlet series has been widly studied, while results of such a kind for general Dirichlet series are not numerous.

Denote by $meas\{A\}$ the Lebesgue measure of the set $A \subset \mathbb{R}$, and let $\nu_T^t(...) = \frac{1}{T}\text{meas}\{t \in [0;T] : \; ....\}$, where in place of dots, a condition satisfied by t is written. In [14] the distribution function $\nu_T^t(F(t) < x)$, where $F(t) = \Re f(\sigma+it)$ or $F(t) = \Im f(\sigma+it)$, was considered in connection with the Besicovitch classes.

Let G be a region on the complex plane $\mathbb{C}$. Denote by $H(G)$ the space of analytic functions on G equipped with the topology of uniform convergence on compacta, and let $\mathcal{B}(S)$ stand for the class of Borel sets of the space S. The paper [6] is devoted to weak convergence of the probability measure

$$\nu_T^\tau\big(f(s+i\tau) \in A\big), \quad A \in \mathcal{B}(H(D_a)),$$

where $D_a = \{s \in \mathbb{C} : \sigma > \sigma_a\}$. It was proved that the latter measure converges weakly to some probability measure on $\big(H(D_a), \mathcal{B}(H(D_a))\big)$ as $T \to \infty$, while in the case of a linear independence of the system $\{\lambda_m\}$ over the field of rational numbers the explicit form of the limit measure was given.

The further investigations in this field are related to limit theorems in the space of meromorphic functions. Denote by $M(G)$ the space of meromorphic functions on G equipped with the topology of uniform convergence on compacta. Suppose that the function $f(s)$ is meromorphically continuable to the region $\sigma > \sigma_1$ with some $\sigma_1 < \sigma_a$, all poles in this region being included in a compact set. Moreover, we assume that, for $\sigma > \sigma_1$, the estimates

$$f(\sigma+it) = O(|t|^\alpha), \qquad |t| \geq t_0, \; \alpha > 0, \tag{2}$$

and t_0 is a fixed positive number, and

$$\frac{1}{2T}\int_{-T}^{T} \big|f(\sigma+it)\big|^2 \mathrm{d}t = O(T), \quad T \to \infty, \tag{3}$$

are satisfied. Let $D = \{s \in \mathbb{C} : \sigma > \sigma_1\}$. Then in [7] it was proved that the probability measure

$$\nu_T^\tau\big(f(s+i\tau) \in A\big), \quad A \in \mathcal{B}(M(D)),$$

weakly converges to some probability measure P on $\big(M(D), \mathcal{B}(M(D))\big)$, however, the explicit form of this measure was not indicated.

The first attempt to find the limit measure P in the theorem of [7] was made in [9]. Assuming additionally that

$$\lambda_m \geq c(\log m)^{\delta}, \tag{4}$$

where c and δ are some positive constants, and that the set

$$\{\log 2\} \cup \bigcup_{m=1}^{\infty} \{\lambda_m\} \tag{5}$$

is linearly independent over the field of rational numbers; in [9] the explicit form the limit measure P was obtained. Finally, in [4] the number $\log 2$ from (5) was removed, so the explicit form of the measure P is known if the system of exponents $\{\lambda_m\}$ is linearly independent over the field of rational numbers.

In [8] the weak convergence for $\sigma > \sigma_1$ of the probability measure

$$\nu_T^t\big(f(\sigma + it) \in A\big), \quad A \in \mathcal{B}(\mathbb{C}),$$

was considered as $T \to \infty$.

All mentioned above limit theorems for the function $f(s)$ have a continuous character: in these theorems the studied measures are defined by translations $f(\sigma + it)$ or $f(s + i\tau)$, where t and τ vary continuously in the interval $[0; T]$.

An another discrete statement of the problem is also possible. In this case t or τ takes values in some arithmetical progression.

In [10] we began the investigation of the discrete value-distribution of (1) by probabilistic methods, and we proved for it limit theorems in the sense of the weak convergence of probability measures on the complex plane. Discrete limit theorems for $f(s)$ in the space of analytic functions were obtained in [12].

Let

$$\mu_N(...) = \frac{1}{N+1} \#\{0 \leq m \leq N : ...\},$$

where in place of dots a condition satisfied by m is written.

Let γ denote the unit circle on $\mathbb{C}$, i.e. $\gamma = \{s \in \mathbb{C} : |s| = 1\}$, and let

$$\Omega = \prod_{m=1}^{\infty} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m \in \mathbb{N}$. With the product topology and pointwise multiplication, the infinite-dimensional torus Ω is a compact topological Abelian group (proof can be found in [5]). Therefore, there exists the probability Haar measure m_H on $(\Omega, \mathcal{B}(\Omega))$, and this gives a probability space $(\Omega, \mathcal{B}(\Omega), m_H)$. Let $\omega(m)$ stand for the projection of $\omega \in \Omega$ to the coordinate space γ_m.

Suppose, as above, that the function $f(s)$ is meromorphically continuable to the half-plane $\sigma > \sigma_1$, $\sigma_1 < \sigma_a$, and that all poles in this region belong to a compact set. Moreover, we require that, for $\sigma > \sigma_1$, the estimates (2) and (3) should be is satisfied.

Now, for $\sigma > \sigma_1$, on the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ we define a complex-valued random element $f(\sigma, \omega)$ by

$$f(\sigma, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m \sigma},$$

and denote by Q_f the distribution of the random element $f(\sigma, \omega)$. In this paper we suppose that $h > 0$ is fixed and such that $\exp\{\frac{2\pi}{h}\}$ is a rational number. Then in [10] the following theorem was proved.

Theorem 1.1. Suppose that the function $f(s)$ satisfies conditions (2) and (3), $\{\lambda_m\}$ is a sequence of algebraic numbers linearly independent over the field of rational numbers, and satisfies condition (4). Then the probability measure

$$Q_N(A) = \mu_N(f(s + imh) \in A), \quad A \in \mathcal{B}(\mathbb{C}),$$

converges weakly to Q_f as $N \to \infty$.

Now define an $H(D_a)$-valued random element $f(s, \omega)$ on $(\Omega, \mathcal{B}(\Omega), m_H)$ by

$$f(s, \omega) = \sum_{m=1}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m s}, \quad s \in D_a,$$

and denote by $\widehat{Q}_f$ its distribution. Then in [12] the following theorem was obtained.

Theorem 1.2. Suppose that $\{\lambda_m\}$ is a sequence of algebraic numbers linearly independent over the field of rational numbers. Then the probability measure

$$\widehat{Q}_N(A) = \mu_N(f(s + imh) \in A), \quad A \in \mathcal{B}(H(D_a)),$$

converges weakly to $\widehat{Q}_f$ as $N \to \infty$.

The aim of this paper is to obtain a discrete limit theorem for the function $f(s)$ in the space of meromorphic functions.

Let $\mathbb{C}_\infty = \mathbb{C} \cup \{\infty\}$ be the Riemann sphere, and let $d(s_1, s_2)$ be a metric given by the formulae

$$d(s_1, s_2) = \frac{2|s_1 - s_2|}{\sqrt{1 + |s_1|^2}\sqrt{1 + |s_2|^2}}, \qquad d(s, \infty) = \frac{2}{\sqrt{1 + |s|^2}}, \qquad d(\infty, \infty) = 0,$$

where $s, s_1, s_2 \in \mathbb{C}$. This metric is compatible with the topology of $\mathbb{C}_\infty$. Denote by $M(G)$ the space of meromorphic functions $g\colon\ G \to (\mathbb{C}_\infty, d)$ equipped with the topology of uniform convergence on compacta. In this topology, a sequence $\{g_n : g_n \in M(G)\}$ converges to the function $g \in M(G)$ if

$$d(g_n(s), g(s)) \to 0, \qquad n \to \infty,$$

uniformly on compact subsets of G. Consider the weak convergence of the probability measure

$$P_N(A) = \mu_N\big(f(s+imh) \in A\big), \quad A \in \mathcal{B}(M(D)).$$

On the probability space $(\Omega, \mathcal{B}(\Omega), m_H)$ define an $H(D)$-valued random element $f(s,\omega)$ by the formula

$$f(s,\omega) = \sum_{m=0}^{\infty} a_m \omega(m) \mathrm{e}^{-\lambda_m s}, \qquad s \in D. \tag{6}$$

Denote by P_f the distribution of the random element $f(s,\omega)$. Then we have the following statement.

Theorem 1.3. Suppose that $\{\lambda_m\}$ is a system of algebraic numbers linearly independent over the field of rational numbers and satisfies condition (4), and that the function $f(s)$ satisfies conditions (2) and (3). Then the probability measure P_N converges weakly to P_f as $N \to \infty$.

2 Limit theorems for Dirichlet polynomials

Since all poles of the function $f(s)$ in the region D are included in compact set, the number of these poles is finite. Denote them by $s_1, \ldots, s_r$, and let

$$f_1(s) = \prod_{j=1}^{r} \big(1 - \mathrm{e}^{\lambda_1(s_j - s)}\big).$$

Then, clearly, f_1 is a Dirichlet polynomial, and $f_1(s_j) = 0$ for $j = 1, \ldots, r$. Moreover, let

$$f_2(s) = f_1(s) f(s).$$

Then we have, that the function $f_2(s)$ is regular on D. Denote by $|\mathcal{A}|$ the number of elements of the set $\mathcal{A}$. Then, for $\sigma > \sigma_a$, we have

$$\begin{aligned} f_2(s) &= \prod_{j=1}^{r} \big(1 - \mathrm{e}^{\lambda_1(s_j - s)}\big) \sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s} \\ &= \sum_{\mathcal{A} \subseteq \{1,\ldots,r\}} \sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_1 s_j} (-1)^{|\mathcal{A}|} \mathrm{e}^{-(\lambda_m + |\mathcal{A}|\lambda_1)s} \\ &= \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{mj} \mathrm{e}^{-(\lambda_m + j\lambda_1)s}, \end{aligned}$$

with some coefficients a_{mj} satisfying $a_{mj} = O\big(|a_m|\big)$ for $m \in \mathbb{N}$ and $j = 0, 1, \ldots, r$. Here the first sum runs over all the subsets $\mathcal{A}$ of $\{1, \ldots, r\}$.

Obviously, the definition of $f_2(s)$ and conditions (2) and (3) imply, for $\sigma > \sigma_1$, the estimates

$$f_2(s) = O\big(|t|^{\alpha}\big), \qquad |t| \geq t_0, \ \alpha > 0, \tag{7}$$

and

$$\int_{-T}^{T} \left| f_2(\sigma+it) \right|^2 \mathrm{d}t = O(T), \qquad T \to \infty. \tag{8}$$

Let S_1 and S_2 be two metric spaces. For us the following statement will be useful.

Lemma 2.1. Let P, P_n be probability measures on $(S_1, \mathcal{B}(S_1))$, and $h: S_1 \to S_2$ be a continuous function. Then the weak convergence of P_n to P implies the weak convergence of $P_n h^{-1}$ to Ph^{-1} on $(S_2, \mathcal{B}(S_2))$ as $n \to \infty$.

The lemma is a particular case of Theorem 5.1 from [1].

We begin with a limit theorem for the Dirichlet polynomial

$$p_n(s) = \sum_{j=0}^{r} \sum_{m=1}^{n} a_{mj} \mathrm{e}^{-(\lambda_m + j\lambda_1)s}.$$

Lemma 2.2. Suppose that $\{\lambda_m\}$ is a sequence of algebraic numbers linearly independent over the field of rational numbers. Then there exists a probability measure P_{p_n} on $(H(D), \mathcal{B}(H(D)))$ such that the probability measure

$$P_{N,p_n}(A) = \mu_N\big(p_n(s+imh) \in A\big), \qquad A \in \mathcal{B}(H(D)),$$

converges weakly to P_{p_n} as $N \to \infty$.

Proof. Let

$$\Omega_n = \prod_{m=1}^{n} \gamma_m,$$

where $\gamma_m = \gamma$ for all $m = 1, \dots, n$. Define the function $u: \Omega_n \to H(D)$ by the formula

$$u(x_1, \dots, x_n) = \sum_{j=0}^{r} \sum_{m=1}^{n} a_{mj} \mathrm{e}^{-(\lambda_m + j\lambda_1)s} x_1^{-j} x_m^{-1}, \qquad (x_1, \dots, x_n) \in \Omega_n.$$

Then we have that u is a continuous function, and

$$p_n(s+imh) = u(\mathrm{e}^{i\lambda_1 mh}, \dots, \mathrm{e}^{i\lambda_n mh}). \tag{9}$$

Consider the probability measure P'_N on $(\Omega_n, \mathcal{B}(\Omega_n))$ defined by

$$P'_N(A) = \mu_N\big((\mathrm{e}^{i\lambda_1 mh}, \dots, \mathrm{e}^{i\lambda_n mh}) \in A\big).$$

The Fourier transform $g_N(k_1, \dots, k_n)$, $k_1, \dots, k_n \in \mathbb{Z}$, of the measure P'_N is

$$g_N(k_1, \dots, k_n) = \int_{\Omega} x_1^{ik_1} \dots x_n^{ik_n} \mathrm{d}P'_N = \frac{1}{N+1} \sum_{m=0}^{N} \mathrm{e}^{imh \sum_{l=1}^{n} k_l \lambda_l}.$$

Since $\{\lambda_m\}$ is a system of algebraic numbers linearly independent over the field of rational numbers and h has the above properties, we find [11] that

$$g_N(k_1,\dots,k_n)=\begin{cases}1 & if(k_1,\dots,k_n)=(0,\dots,0),\\ \frac{1}{N+1}\frac{1-\exp\{i(N+1)h\sum\limits_{l=1}^{n}k_l\lambda_l\}}{1-\exp\{ih\sum\limits_{l=1}^{n}k_l\lambda_l\}} & if(k_1,\dots,k_n)\neq(0,\dots,0).\end{cases}$$

Consequently,

$$\lim_{N\to\infty}g_N(k_1,\dots,k_n)=\begin{cases}1 & if(k_1,\dots,k_n)=(0,\dots,0),\\ 0 & if(k_1,\dots,k_n)\neq(0,\dots,0).\end{cases}$$

By Theorem 1.3.19 of [5] this implies that the measure P'_N converges weakly to the Haar measure m_{nH} on $(\Omega_n,\mathcal{B}(\Omega_n))$ as $N\to\infty$. Hence, the continuity of the function u, Eq. 9, and Lemma 2.1 yield that the probability measure P_{N,p_n} converges weakly to $P_{p_n}=m_{nH}u^{-1}$ as $N\to\infty$.

Now let $g(m)$, $m\in\mathbb{N}$, be complex numbers such that $|g(m)|=1$ for all $m\in\mathbb{N}$. Define

$$p_n(s,g)=\sum_{j=0}^{r}\sum_{m=1}^{n}a_{mj}g(m)\mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Lemma 2.3. The probability measure

$$\tilde{P}_{N,p_n}(A)=\mu_T\big(p_n(s+imh,g)\in A\big),\qquad A\in\mathcal{B}(H(D)),$$

also converges weakly to the measure $m_{nH}u^{-1}$ as $N\to\infty$.

Proof. Let $\theta_m=\arg g(m)$, $m=0,1,...,n$, and let the function $u_1:\Omega_n\to\Omega_n$ be given by the formula

$$u_1(x_1,\dots,x_n)=(x_1\mathrm{e}^{-i\theta_1},\dots,x_n\mathrm{e}^{-i\theta_n}),\qquad (x_1,\dots,x_n)\in\Omega_n.$$

Then we have that

$$p_n(s+imh,g)=u\big(u_1(e^{i\lambda_1 mh},...,e^{i\lambda_n mh})\big).$$

Therefore, by the proof of Lemma 2.2, the measure $\widetilde{P}_{T,p_n}$ converges weakly to the measure $m_{nH}(uu_1)^{-1}$ as $N\to\infty$. Since the Haar measure is invariant with respect to translations by points from Ω_n, P_{N,p_n} converges weakly to $(m_{nH}u_1^{-1})u^{-1}=m_{nH}u^{-1}$ as $N\to\infty$.

3 Approximation in the mean

In this section we approximate the function $f_2(s)$, defined at the beginning of Section 2, by an absolutely convergent Dirichlet series in the mean. Let $\sigma_2=\sigma_a-\sigma_1$. For $\sigma\in[-\sigma_2,\sigma_2]$ define

$$l_n(s)=\frac{s}{\sigma_2}\Gamma\Big(\frac{s}{\sigma_2}\Big)\mathrm{e}^{(\lambda_n+j\lambda_1)s},$$

where, as usual, $\Gamma(s)$ is the gamma-function. Obviously, $\sigma_2 > 0$. For $\sigma > \sigma_1$ consider the function

$$g_n(s) = \frac{1}{2\pi i}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} f_2(s+z) l_n(z) \frac{\mathrm{d}z}{z}\,.$$

Lemma 3.1. The function $g_n(s)$ has the expansion

$$g_n(s) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{mj} \exp\big\{ - \mathrm{e}^{-(\lambda_m - \lambda_n)\sigma_2}\big\} \mathrm{e}^{-(\lambda_m + j\lambda_1)s}, \tag{10}$$

the series being absolutely convergent for $\sigma > \sigma_1$.

Proof. Since $\sigma_2 = \sigma_a - \sigma_1$, it follows that $\sigma + \sigma_2 > \sigma_a$ for $\sigma > \sigma_1$. Therefore, for $\Re z = \sigma_2$,

$$f_2(s+z) = \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{mj}\mathrm{e}^{-(\lambda_m + j\lambda_1)(s+z)}.$$

We put

$$k_n(m) = \frac{1}{2\pi i}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} l_n(s) \mathrm{e}^{-(\lambda_m + j\lambda_1)s}\frac{\mathrm{d}s}{s}\,,$$

and consider the series

$$\sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{mj} k_n(m) \mathrm{e}^{-(\lambda_m + j\lambda_1)(s+z)}. \tag{11}$$

Since

$$k_n(m) = O\bigg(\mathrm{e}^{-(\lambda_m + j\lambda_1)\sigma_2}\int_{-\infty}^{\infty} \big|l_n(\sigma_2 + it)\big|\,\mathrm{d}t\bigg) = O\Big(\mathrm{e}^{-(\lambda_m + j\lambda_1)\sigma_2}\Big),$$

the series (11) converges absolutely for $\sigma > \sigma_a - \sigma_2$, i. e., for $\sigma > \sigma_1$. Hence we may interchange summation and integration in the definition of $g_n(s)$. This gives

$$\begin{aligned} g_n(s) &= \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{mj}\mathrm{e}^{-(\lambda_m + j\lambda_1)s}\frac{1}{2\pi i}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} l_n(z)\mathrm{e}^{-(\lambda_m + j\lambda_1)z}\frac{\mathrm{d}z}{z} \\ &= \sum_{j=0}^{r}\sum_{m=1}^{\infty} a_{mj} k_n(m)\mathrm{e}^{-(\lambda_m + j\lambda_1)s}. \end{aligned} \tag{12}$$

Using the equality

$$\frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} \Gamma(s) b^{-s}\mathrm{d}s = \mathrm{e}^{-b}, \quad c > 0,\ b > 0,$$

we find that

$$\frac{1}{2\pi i}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \frac{s}{\sigma_2}\Gamma\Big(\frac{s}{\sigma_2}\Big)\mathrm{e}^{-(\lambda_m - \lambda_n)s}\frac{\mathrm{d}s}{s}$$

$$= \frac{1}{2\pi i}\int_{\sigma_2 - i\infty}^{\sigma_2 + i\infty} \Gamma\Big(\frac{s}{\sigma_2}\Big)\mathrm{e}^{-(\lambda_m - \lambda_n)\left(-\frac{s}{\sigma_2}\right)\sigma_2}\mathrm{d}\frac{s}{\sigma_2} = \exp\big\{ - \mathrm{e}^{(\lambda_m - \lambda_n)\sigma_2}\big\}.$$

Hence and from Eq. 12 the lemma follows.

Lemma 3.2. Let T_0 and $T \geq \delta > 0$ be real numbers, $\mathcal{T}$ be a finite set in the interval $[T_0 + \frac{\delta}{2}, T_0 + T - \frac{\delta}{2}]$. Moreover, let

$$N_\delta(x) = \sum_{\substack{t \in \mathcal{T} \\ |t-x|<\delta}} 1,$$

and let $S(x)$ be a complex-valued continuous function on $[T_0, T + T_0]$ having a continuous derivative on $(T_0, T + T_0)$. Then

$$\sum_{t \in \mathcal{T}} N_\delta^{-1}(t)|S(t)|^2 \leq \frac{1}{\delta} \int\limits_{T_0}^{T_0+T} |S(x)|^2 \mathrm{d}x + \Bigg(\int\limits_{T_0}^{T_0+T} |S(x)|^2 \mathrm{d}x \Bigg)^{\frac{1}{2}} \Bigg(\int\limits_{T_0}^{T_0+T} |S'(x)|^2 \mathrm{d}x \Bigg)^{\frac{1}{2}}.$$

Proof. This is Lemma 1.4 of [13].

Lemma 3.3. Let $T \to \infty$. Then, for $\sigma > \sigma_1$,

$$\int\limits_0^T |f_2'(\sigma + it)|^2 = O(T).$$

Proof. By the Cauchy formula

$$f_2'(s) = \frac{1}{2\pi i} \int\limits_{|z-s|=\delta} \frac{f_2(z)}{(z-s)^2} \mathrm{d}z,$$

where the circle $|z - s| = \delta$ lies in the half-plane $\sigma > \sigma_1$. Then for some $\sigma' > \sigma_1$ and bounded τ by Eq. 8

$$\int\limits_0^T |f_2'(\sigma + it)|^2 \mathrm{d}t = \int\limits_0^T \Bigg| \frac{1}{2\pi i} \int\limits_{|z-s|=\delta} \frac{f_2(z)}{(z-s)} \mathrm{d}z \Bigg|^2 \mathrm{d}t = O\Bigg(\int\limits_0^T \big| f_2(\sigma' + it + i\tau) \big|^2 \mathrm{d}t \Bigg) = O(T).$$

Lemma 3.4. Let K be a compact subset of D. Then

$$\lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \sup_{s \in K} \big| f_2(s + imh) - g_n(s + imh) \big| = 0.$$

Proof. We change the contour in the integral for $g_n(s)$. The integrand has a simple pole at the point $z = 0$. Suppose $\sigma \in [\sigma_1 + \eta, \sigma_4]$, $\eta > 0$, when $s \in K$, and put $\sigma_3 = \sigma_1 + \frac{\eta}{2}$.

Then, by the residue theorem, we find

$$g_n(s) = \frac{1}{2\pi i} \int\limits_{\sigma_3-\sigma-i\infty}^{\sigma_3-\sigma+i\infty} f_2(s+z) l_n(z) \frac{\mathrm{d}z}{z} + f_2(s). \tag{13}$$

Now let L be a simple closed contour lying in D and enclosing the set K, and let δ stand for the distance of L from the set K. Then by the Cauchy formula, for $s \in K$, we have

$$f_2(s+imh) - g_n(s+imh) = \frac{1}{2\pi i} \int\limits_L \frac{f_2(z+imh) - g_n(z+imh)}{z-s} \mathrm{d}z.$$

Therefore,

$$\sup_{s\in K} |f_2(s+imh) - g_n(s+imh)| \le \frac{1}{2\pi\delta} \int\limits_L |f_2(z+imh) - g_n(z+imh)||\mathrm{d}z|.$$

Hence, for sufficiently large N, we obtain

$$\frac{1}{N+1} \sum_{m=0}^{N} \sup_{s\in K} |f_2(s+imh) - g_n(s+imh)| =$$

$$\begin{aligned}
&= O\biggl(\frac{1}{N} \sum_{m=0}^{N} \frac{1}{2\pi\delta} \int\limits_L |f_2(z+imh) - g_n(z+imh)||\mathrm{d}z|\biggr) \\
&= O\biggl(\frac{1}{N} \int\limits_L |\mathrm{d}z| \sum_{m=0}^{2N} |f_2(\Re z+imh) - g_n(\Re z+imh)|\biggr) + O\biggl(\frac{1}{N\delta}\biggr) \\
&= O\biggl(\frac{1}{N\delta}\biggr) + O\biggl(\frac{1}{N} \sup_{\substack{\sigma \\ s\in L}} \sum_{m=0}^{2N} |f_2(\sigma+imh) - g_n(\sigma+imh)|\biggr).
\end{aligned} \tag{14}$$

Remembering Eq. 13, we have

$$f_2(s+imh) - g_n(s+imh) = O\biggl(\int\limits_{-\infty}^{\infty} |f_2(\sigma_3+imh+i\tau)||l_n(\sigma_3-\sigma+i\tau)|\mathrm{d}\tau\biggr).$$

Hence, taking $u = [\frac{|\tau|}{h}] + 1$, where $[x]$ denotes the integer part of x, we obtain that

$$\frac{1}{N} \sum_{m=0}^{2N} |f_2(\sigma+imh) - g_n(\sigma+imh)| = O\biggl(\int\limits_{-\infty}^{\infty} |l_n(\sigma_3-\sigma+i\tau)| \frac{1}{N} \sum_{m=-u}^{2N+u} |f_2(\sigma_3+imh)|\mathrm{d}\tau\biggr).$$

By Lemmas 3.2 and 3.3, and estimate (8)

$$\begin{aligned}
\sum_{m=-u}^{2N+u} |f_2(\sigma+imh)|^2 &\le \frac{1}{h} \int\limits_{-uh}^{(2N+u)h} |f_2(\sigma+it)|^2 \mathrm{d}t \\
&+ \biggl(\int\limits_{-uh}^{(2N+u)h} |f_2(\sigma+it)|^2 \mathrm{d}t \int\limits_{-uh}^{(2N+u)h} |f_2'(\sigma+it)|^2 \mathrm{d}t\biggr)^{\frac{1}{2}} = O(2N+2u).
\end{aligned}$$

Therefore the Cauchy - Schwarz inequality yields

$$\frac{1}{N+1}\sup_{\substack{\sigma\\ s\in L}}\sum_{m=0}^{2N}|f_2(\sigma+imh)-g_n(\sigma+imh)|=$$
$$=O\Bigg(\sup_{\substack{\sigma\\ s\in L}}\int_{-\infty}^{\infty}|l_n(\sigma_3-\sigma+i\tau)|\mathrm{d}\tau\Bigg(\frac{1}{N+1}\sum_{m=-u}^{2N+u}|f_2(\sigma_3+imh)|^2\Bigg)^{\frac{1}{2}}\Bigg)=$$
$$=O\Bigg(\sup_{\substack{\sigma\\ s\in L}}\int_{-\infty}^{\infty}|l_n(\sigma_3-\sigma+i\tau)|\frac{2N+2u}{N+1}\mathrm{d}\tau\Bigg)=$$
$$=O\Bigg(\sup_{\substack{\sigma\\ s\in L}}\int_{-\infty}^{\infty}|l_n(\sigma_3-\sigma+i\tau)|(1+|\tau|)\mathrm{d}\tau\Bigg). \tag{15}$$

We can choose the number δ so that the inequality $\sigma_3-\sigma\le-\frac{\eta}{4},\ s\in L$, should be satisfied. In this case by definition of $l_n(s)$, we have that

$$\lim_{n\to\infty}\sup_{\sigma\le-\frac{\eta}{4}}\int_{-\infty}^{\infty}|l_n(\sigma+it)|(1+|t|)\mathrm{d}t=0.$$

The above along with estimates (14) and (15) prove the lemma.

Let $a_h=\{\mathrm{e}^{-i\lambda_m h}:m\in\mathbb{N}\}$, $h>0$. Then a_h is a one-parameter group. We define the one-parameter family φ_h of transformations on Ω by $\varphi_h(\omega)=a_h\omega$ for $\omega\in\Omega$. Then φ_h is a measurable measure preserving transformation on the probability space $(\Omega,\mathcal{B}(\Omega),m_H)$.

Now we recall some basic facts of ergodic theory, see for example [15]. Let $\mathcal{G}$ be a compact topological Abelian group with Haar measure $m_{\mathcal{G}}$. Let $\{\mathcal{G}_h\}$ be an one-parameter group of measurable transformations on $\mathcal{G}$. A set $A\in\mathcal{B}(\mathcal{G})$ is called an invariant set with respect to the group $\{\mathcal{G}_h\}$ if, for each h, the sets A and $A_h=\mathcal{G}_h(A)$ differ from one another by a set of zero $m_{\mathcal{G}}$-measure, i.e. $m_{\mathcal{G}}(A\Delta A_h)=0$, where $A\Delta A_h$ denotes the symmetric difference of sets A and A_h. A one-parameter group $\{\mathcal{G}_h\}$ is called ergodic if its σ-field of invariant sets consists only of sets having $m_{\mathcal{G}}$-measure equal to 0 or 1. For these and other facts of ergodic theory, see for example, [15].

Lemma 3.5. The one-parameter group φ_h is ergodic.

Proof. This is Lemma 7 from [10].

Let, for $s\in D$,

$$f_2(s,\omega)=\sum_{j=0}^{r}\sum_{m=1}^{\infty}a_m\omega^j(1)\omega(m)\mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Then we have that $f_2(s,\omega)$ is a product of two $H(D)$-valued random elements,

$$f_1(s,\omega)=\prod_{j=1}^{r}\left(1-\omega(1)\mathrm{e}^{\lambda_1(s_j-s)}\right)$$

and $f(s,\omega)$, and therefore it is an $H(D)$-valued random element.

Denote by $E\xi$ the mean of the random element ξ.

Lemma 3.6. Let T be a measurable measure preserving ergodic transformation on probability space $(\widehat{\Omega},\mathcal{F},m)$. Then for every $g \in L_1(\widehat{\Omega},\mathcal{F},m)$

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1} g(T^k\omega) = Eg$$

for almost all $\omega \in \widehat{\Omega}$.

The lemma is the well-known Birkhoff-Khinchine theorem, see for example [16].

Lemma 3.7. Let $\sigma > \sigma_1$ and $N \to \infty$. Then

$$\sum_{m=0}^{N}\left|f_2(\sigma+imh,\omega)\right|^2 = BN$$

for almost all $\omega \in \Omega$.

Proof. Let, for $m \in \mathbb{N}$,

$$f_{mj}(\sigma,\omega) = a_{mj}\omega^j(1)\omega(m)\mathrm{e}^{-(\lambda_m+j\lambda_1)\sigma},$$

and let, for a fixed j,

$$f_j(\sigma,\omega) = \sum_{m=1}^{\infty} f_{mj}(\sigma,\omega).$$

Then we have

$$\begin{aligned} E(f_{mj},\overline{f_{kj}}) &= a_{mj}\overline{a_{kj}}\mathrm{e}^{-(\lambda_m+j\lambda_1)\sigma}\mathrm{e}^{-(\lambda_k+j\lambda_1)\sigma}\int\limits_{\Omega}\omega^j(1)\omega^{-j}(1)\omega(m)\overline{\omega(k)}m_H(\mathrm{d}\omega) \\ &= \begin{cases} |a_{mj}|^2\mathrm{e}^{-2(\lambda_m+j\lambda_1)\sigma}, & m=k, \\ 0, & m\neq k, \end{cases} \end{aligned}$$

since

$$\int\limits_{\Omega}\omega(m)\overline{\omega(k)}m_H(\mathrm{d}\omega) = \begin{cases} 1, & m=k, \\ 0, & m\neq k. \end{cases}$$

Thus we have that the random variables $f_{mj}(\sigma,\omega)$ are orthogonal. In [9] it was proved that, for $\sigma > \sigma_1$,

$$\sum_{m=1}^{\infty}|a_m|^2\mathrm{e}^{-2\lambda_m\sigma} < \infty.$$

Consequently,

$$E|f_j(\sigma,\omega)|^2 = \sum_{m=1}^{\infty}E|f_{m,j}(\sigma,\omega)|^2 = O\bigg(\sum_{m=1}^{\infty}|a_m|^2\mathrm{e}^{-2(\lambda_m+j\lambda_1)\sigma}\bigg) < \infty, \quad j=0,..,r.$$

Hence,

$$E\big|f_2(\sigma,\omega)\big|^2 = \sum_{j=0}^{r} E|f_j(\sigma,\omega)|^2 < \infty. \tag{16}$$

Clearly,

$$|f_2(\sigma,\varphi_h^m(\omega))|^2 = \big|f_2(\sigma,a_{mh}\omega)\big|^2 = \big|f_2(\sigma+imh,\omega)\big|^2. \tag{17}$$

Therefore, in view of (16) and Lemmas 3.5 and 3.6, we find

$$\begin{aligned}\lim_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\big|f_2(\sigma+imh,\omega)\big|^2 &= \lim_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\big|f_2(\sigma,\varphi_h^m(\omega))\big|^2\\ &= E\big|f_2(\sigma,\omega)\big|^2 < \infty\end{aligned}$$

for almost all $\omega\in\Omega$. Hence the lemma follows.

Let, for $\omega\in\Omega$,

$$g_n(s,\omega) = \sum_{j=1}^{r}\sum_{m=1}^{\infty} a_{mj}\omega^j(1)\omega(m)\exp\{-\mathrm{e}^{-(\lambda_m-\lambda_n)\sigma_2}\}\mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Clearly, the last series converges absolutely for $\sigma>\sigma_1$.

Lemma 3.8. Let K be a compact subset of D. Then

$$\lim_{n\to\infty}\limsup_{N\to\infty}\frac{1}{N+1}\sum_{m=0}^{N}\sup_{s\in K}|f_2(s+imh,\omega)-g_n(s+imh,\omega)| = 0$$

for almost all $\omega\in\Omega$.

Proof. In virtue of Lemma 3.7, the proof is similar to that of Lemma 3.4.

4 Limit theorems for $g_n(s)$

In this section we consider the weak convergence of two probability measures on $(H(D),\mathcal{B}(H(D)))$, namely,

$$P_{N,n}(A) = \mu_N\big(g_n(s+imh)\in A\big),$$

and

$$\widehat{P}_{N,n}(A) = \mu_N\big(g_n(s+imh,\omega)\in A\big)$$

as $N\to\infty$. To investigate the weak convergence of these measures we need a metric on $H(D)$ which induces its topology. It is known, (see, for example, Lemma 1.7.1 of [5]), that there exists a sequence $\{K_n\}$ of compact subsets of D such that $D=\cup_{n=1}^{\infty}K_n$, $K_n\subset K_{n+1}$, and if K is a compact subset of D, then $K\subseteq K_n$ for some n. Then

$$\varrho(f,g) = \sum_{n=1}^{\infty}\frac{\varrho_n(f,g)}{1+\varrho_n(f,g)}, \qquad f,g\in H(D),$$

where

$$\varrho_n(f,g) = \sup_{s \in K_n} |f(s) - g(s)|$$

is a metric on $H(D)$ which induces its topology.

Lemma 4.1. There exists a probability measure P_n on $(H(D), \mathcal{B}(H(D)))$ such that both the measures $P_{N,n}$ and $\widehat{P}_{N,n}$ converge weakly to P_n as $N \to \infty$.

Proof. Define, for a positive integer M,

$$\begin{aligned} g_{n,M}(s) \quad &= \textstyle\sum_{j=0}^{r} \sum_{m=1}^{M} a_{mj} \exp\big\{-\mathrm{e}^{(\lambda_m-\lambda_n)\sigma_2}\big\} \mathrm{e}^{-(\lambda_m+j\lambda_1)s}, \\ g_{n,M}(s,\omega) &= \textstyle\sum_{m=1}^{M} a_{mj}\omega^j(1)\omega(m) \exp\big\{-\mathrm{e}^{(\lambda_m-\lambda_n)\sigma_2}\big\} \mathrm{e}^{-(\lambda_m+j\lambda_1)s}, \end{aligned}$$

and let

$$\begin{aligned} P_{N,n,M}(A) \quad &= \mu_N\big(g_{n,M}(s+imh) \in A\big), \qquad A \in \mathcal{B}(H(D)), \\ \widehat{P}_{N,n,M}(A) &= \mu_N\big(g_{n,M}(s+imh,\omega) \in A\big), \qquad A \in \mathcal{B}(H(D)). \end{aligned}$$

By Lemmas 2.2 and 2.3, both the measures $P_{N,n,M}$ and $\widehat{P}_{N,n,M}$ converge weakly to the same measure $P_{n,M}$ as $N \to \infty$. Similarly as in Theorem 5.5.2 of [5], we obtain that the family of probability measures $\{P_{n,M}\}$ is tight for fixed n. Hence, by the Prokhorov theorem, (see for example, Theorem 1.1.12 of [5]), it is relatively compact.

By the definition of $g_n(s)$ and $g_{n,M}(s)$, we have

$$\lim_{M\to\infty} g_{n,M}(s) = g_n(s),$$

and since the series for $g_n(s)$ converges absolutely for $\sigma > \sigma_1$, the convergence is uniform on compact subsets of D. Hence, for every $\varepsilon > 0$,

$$\begin{aligned} &\lim\nolimits_{n\to\infty} \limsup\nolimits_{N\to\infty} \mu_N\big(\varrho(g_{n,M}(s+imh), g_n(s+imh)) \geq \varepsilon\big) \\ &\leq \lim\nolimits_{M\to\infty} \limsup\nolimits_{N\to\infty} \tfrac{1}{(N+1)\epsilon} \sum_{m=0}^{N} \varrho\big(g_{n,M}(s+imh), g_n(s+imh)\big) = 0. \end{aligned} \tag{18}$$

Let θ_N be a random variable on a certain probability space $(\widehat{\Omega}, \mathcal{B}(\widehat{\Omega}), \mathbb{P})$ with distribution

$$\mathbb{P}(\theta_N = mh) = \frac{1}{N+1}, \qquad m = 0, 1, ..., N.$$

We put

$$X_{N,n,M}(s) = g_{n,M}(s + i\theta_N).$$

Denote by $\xrightarrow{\mathcal{D}}$ the convergence in distribution. Then by the above remark

$$X_{N,n,M}(s) \xrightarrow[N\to\infty]{\mathcal{D}} X_{n,M},$$

where $X_{n,M}$ is an $H(D)$-valued random element with the distribution $P_{n,M}$. Now let

$$X_{N,n}(s) = g_n(s + i\theta_N).$$

Then, in view of Eq. 18, for every $\varepsilon > 0$

$$\lim_{N\to\infty} \limsup_{N\to\infty} \mathbb{P}\big(\varrho(X_{N,n,M}(s), X_{N,n}(s)) \geq \varepsilon\big) = 0. \tag{19}$$

Since the family $\{P_{n,M}\}$ is relatively compact, there exists a subsequence P_{n,M_1} which converges weakly to P_n, say, as $M_1 \to \infty$. Then

$$X_{n,M_1} \xrightarrow[M_1\to\infty]{\mathcal{D}} P_n. \tag{20}$$

The space $H(D)$ is separable. Therefore, by relations (18)–(20) and Theorem 1.2.4 of [5],

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} P_n. \tag{21}$$

This means that there is a probability measure P_n such that $P_{N,n}$ converges weakly to P_n as $N \to \infty$. On the other hand, relation (21) shows that the measure P_n is independent of the choice of the subsequence P_{n,M_1}. This and the relative compactness of $\{P_{n,M}\}$ imply the weak convergence of $P_{n,M}$ to P_n as $M \to \infty$, and also the relation

$$X_{n,M} \xrightarrow[M\to\infty]{\mathcal{D}} P_n. \tag{22}$$

Now repeating the same arguments for the random elements

$$\begin{aligned} \widehat{X}_{N,n,M}(s,\omega) &= g_{n,M}(s+i\theta,\omega), \\ \widehat{X}_{N,n}(s,\omega) &= g_n(s+i\theta,\omega), \end{aligned}$$

and taking into account relation (22), we obtain that the measure $\widehat{P}_{N,n}$ also converges weakly to P_n as $N \to \infty$. The lemma is proved.

5 Limit theorems for $f_2(s)$

In this section, we will consider the weak convergence of probability measures

$$\begin{aligned} P_{N,f_2}(A) &= \mu_N\big(f_2(s+imh) \in A\big), \qquad A \in \mathcal{B}(H(D)), \\ \widehat{P}_{N,f_2}(A) &= \mu_N\big(f_2(s+imh,\omega) \in A\big), \qquad A \in \mathcal{B}(H(D)). \end{aligned}$$

Lemma 5.1. There exists a probability measure P on $(H(D), \mathcal{B}(H(D)))$ such that both the measures P_{N,f_2} and $\widehat{P}_{N,f_2}$ converge weakly to P as $N \to \infty$.

Proof. The way of the proof is the same as in Lemma 4.1. By this lemma, the measures $P_{N,n}$ and $\widehat{P}_{N,n}$ converge weakly to the same measure P_n as $N \to \infty$. Taking

$$X_{N,n}(s) = g_n(s+i\theta_N),$$

we have that

$$X_{N,n} \xrightarrow[N\to\infty]{\mathcal{D}} X_n, \tag{23}$$

where X_n is an $H(D)$-valued random element with the distribution P_n. Also, the family $\{P_n\}$ is relatively compact. Applying Lemma 3.4, we find that, for every $\varepsilon > 0$,

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mu_N\big(\varrho(f_2(s+imh), g_n(s+imh)) \geq \varepsilon\big)$$
$$\leq \lim_{n\to\infty} \limsup_{N\to\infty} \frac{1}{(N+1)\varepsilon} \sum_{m=0}^{N} \varrho(f_2(s+imh), g_n(s+imh)) = 0.$$

Now we set

$$Y_N(s) = f_2(s+i\theta_N).$$

Then we can write

$$\lim_{n\to\infty} \limsup_{N\to\infty} \mathbb{P}\big(\varrho(X_{N,n}(s), Y_N(s)) \geq \varepsilon\big) = 0. \tag{24}$$

From the relative compactness of $\{P_n\}$ there exists a subsequence $\{P_{n_1}\}$ of $\{P_n\}$ which converges weakly to P as $n_1 \to \infty$, i.e.,

$$X_{n_1} \xrightarrow[n_1\to\infty]{\mathcal{D}} P. \tag{25}$$

Using (23)–(25) and applying Theorem 1.2.4 of [5] again, we obtain that

$$Y_N \xrightarrow[n\to\infty]{\mathcal{D}} P.$$

This means that the measure P_{N,f_2} converges weakly to P as $N \to \infty$. The relative compactness of $\{P_n\}$ and relation (25) show that

$$X_n \xrightarrow[n\to\infty]{\mathcal{D}} P. \tag{26}$$

Repeating the above arguments, and using (26) and Lemma 3.8, we obtain that the measure $\widehat{P}_{N,f_2}$ also converges weakly to P as $N \to \infty$.

The aim of the next lemma is to identify the limit measure in Lemma 5.1.

As in Section 3, for $s \in D$, we define

$$f_2(s,\omega) = \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{mj}\omega^j(1)\omega(m)\mathrm{e}^{-(\lambda_m+j\lambda_1)s}.$$

Let P_{f_2} be the distribution of the random element $f_2(s,\omega)$.

Lemma 5.2. The measure P in Lemma 5.1 coincides with P_{f_2}.

Proof. Let $A \in \mathcal{B}(H(D))$ be a continuity set of P. Then by Lemma 5.1

$$\lim_{N\to\infty} \mu_N\big(f_2(s+imh,\omega) \in A\big) = P(A) \tag{27}$$

for almost all $\omega \in \Omega$. Now we fix the set A and define the random variable θ on $(\Omega, \mathcal{B}(\Omega))$ by the formula

$$\theta(\omega) = \begin{cases} 1 & if\, f_2(s,\omega) \in A, \\ 0 & if\, f_2(s,\omega) \notin A. \end{cases}$$

Then

$$E(\theta) = \int_\Omega \theta \mathrm{d}m_H = m_H\big(\omega \in \Omega : \ f_2(s,\omega) \in A\big) = P_{f_2}(A) < \infty. \tag{28}$$

By Lemmas 3.5 and 3.6 it follows that

$$\lim_{N\to\infty} \frac{1}{N+1} \sum_{m=0}^{N} \theta(\varphi_h^m(\omega)) = E(\theta) \tag{29}$$

for almost all $\omega \in \Omega$. However, the definitions of θ and φ_h give

$$\frac{1}{N+1} \sum_{m=0}^{N} \theta(\varphi_h^m(\omega)) = \mu_N\big(f_2(s+imh,\omega) \in A\big). \tag{30}$$

From Eqs. (28)–(30) we obtain

$$\lim_{N\to\infty} \mu_N\big(f_2(s+imh,\omega) \in A\big) = P_{f_2}(A)$$

for almost all $\omega \in \Omega$. From this and Eq. (27), we have that $P(A) = P_{f_2}(A)$ for any continuity set A of P. Since all continuity sets constitute a determining class, hence $P(A) = P_{f_2}(A)$ for all $A \in \mathcal{B}(H(D))$, and the lemma is proved.

6 A limit theorem for $f_1(s)$

First we observe that

$$f_1(s) = \prod_{j=1}^{r} \big(1 - \mathrm{e}^{\lambda_1(s_j - s)}\big) = \sum_{m=0}^{r} b_m \mathrm{e}^{-\lambda_1 ms}$$

is a Dirichlet polynomial with some coefficients b_m and exponents $m\lambda_1$, and define the $H(D)$-valued random element $f_1(s,\omega)$ by

$$f_1(s,\omega) = \prod_{j=1}^{r} \big(1 - \omega(1)\mathrm{e}^{\lambda_1(s_j - s)}\big) = \sum_{m=0}^{r} b_m \omega^m(1) \mathrm{e}^{-\lambda_1 ms}.$$

We define the probability measure

$$P_{N,f_1}(A) = \mu_N\big(f_1(s+imh) \in A\big), \quad A \in \mathcal{B}(H(D)).$$

Lemma 6.1. The probability measure P_{N,f_1} converges weakly to the distribution P_{f_1} of the random element $f_1(s,\omega)$ as $N \to \infty$.

Proof. The lemma is obtained in the same way as Lemmas 2.2 and 5.2.

7 A two-dimensional limit theorem

In Sections 5 and 6 we proved discrete limit theorems for the functions $f_1(s)$ and $f_2(s)$ in the space $H(D)$. Now we will prove a joint discrete limit theorem for these functions.

Let $H^2(D) = H(D) \times H(D)$. Denote by P_{f_1,f_2} the distribution of the $H^2(D)$-valued random element

$$F(s,\omega) = \big(f_1(s,\omega), f_2(s,\omega)\big), \quad \omega \in \Omega, s \in D,$$

and let

$$P_{N,f_1,f_2}(A) = \mu_N\big((f_1(s+imh), f_2(s+imh)) \in A\big), \qquad A \in \mathcal{B}(H^2(D)).$$

Lemma 7.1. The probability measure P_{N,f_1,f_2} converges weakly to P_{f_1,f_2} as $N \to \infty$.

For the proof of this lemma we need the following results.

Lemma 7.2. The family of probability measures $\{P_{N,f_1,f_2}\}$ is relatively compact.

Proof. In Lemmas 5.1, 5.2 and 6.1 we proved that the probability measures P_{N,f_1} and P_{N,f_2} converge weakly to the measures P_{f_1} and P_{f_2}, respectively, as $N \to \infty$. Therefore, the family of probability measures $\{P_{N,f_j}\}$, $j = 1, 2$, is relatively compact. The space $H(D)$ is a complete separable space. Hence, by the Prokhorov theorem the family $\{P_{N,f_j}\}$, $j = 1, 2$ is tight. Thus we have that for every $\epsilon > 0$ there exists a compact subset $K_j \subset D$ such that

$$P_{N,f_j}(H(D)\backslash K_j) < \frac{\epsilon}{2}, \quad j = 1, 2. \tag{31}$$

Let θ_N be a random variable defined in the proof of Lemma 4.1, and

$$\begin{aligned} f_{1,N}(s) &= f_1(s+i\theta_N), \\ f_{2,N}(s) &= f_2(s+i\theta_N), \\ F_N(s) &= \big(f_{1,N}(s), f_{2,N}(s)\big). \end{aligned}$$

Then inequality (31) and the definition of P_{N,f_j} yield

$$\mathbb{P}\big(f_{j,N} \in H(D)\backslash K_j\big) < \frac{\epsilon}{2}, \quad j = 1, 2. \tag{32}$$

Let $K = K_1 \times K_2$. Then K is a compact subset of $H^2(D)$. In virtue of (32)

$$\begin{aligned} P_{N,f_1,f_2}(H^2(D)\backslash K) &= \mathbb{P}\big(F_N \in H^2(D)\backslash K\big) \\ &= \mathbb{P}\Big(\bigcup_{j=1}^{2}(f_{j,N}(s) \in H(D)\backslash K_j)\Big) \\ &\le \sum_{j=1}^{2} \mathbb{P}\big(f_{j,N}(s) \in H(D)\backslash K_j\big) < \epsilon. \end{aligned} \tag{33}$$

Relation (33) shows that the family $\{P_{N,f_1,f_2}\}$ is tight. By the Prokhorov theorem it is relatively compact. The lemma is proved.

Let $s_1, ..., s_l$ be arbitrary points on D, and put $\sigma' = \min_{1\le k\le l} \Re s_k$. Then $\widehat{\sigma} = \sigma_1 - \sigma' < 0$, and we set

$$\widehat{D} = \{s \in \mathbb{C} : \sigma > \widehat{\sigma}\}.$$

Moreover, let $u_{jk},\ j = 1,2,\ 1 \le k \le l$, be arbitrary complex numbers, and define a function $v :\ H^2(D) \to H(\widehat{D})$ by the formula

$$v(f_1, f_2) = \sum_{j=1}^{2} \sum_{k=1}^{l} u_{jk} f_j(s_k + s), \quad s \in \widehat{D},\ f_j \in H(D),\ j = 1,2. \tag{34}$$

Let

$$W_v(s) = v\big(f_1(s), f_2(s)\big).$$

Lemma 7.3. The relation

$$W_v(s + i\theta_N) \underset{N\to\infty}{\xrightarrow{\mathcal{D}}} v\big(F(s,\omega)\big)$$

holds.

Proof. By the the definition of the function v,

$$W_v(s) = \sum_{j=1}^{2} \sum_{k=1}^{l} u_{jk} f_j(s_k + s)$$

for $\sigma > \sigma_1$. In the region $\sigma > \sigma_a$ the $f_1(s)$ and $f_2(s)$ are presented by absolutely convergent Dirichlet series

$$f_1(s) = \prod_{j=1}^{r} \big(1 - \mathrm{e}^{\lambda_1(s_j - s)}\big) = \sum_{j=0}^{r} b_j \mathrm{e}^{-\lambda_1 j s}$$

and

$$\begin{aligned} f_2(s) &= \prod_{j=1}^{r} \big(1 - \mathrm{e}^{\lambda_1(s_j - s)}\big) \sum_{m=1}^{\infty} a_m \mathrm{e}^{-\lambda_m s} \\ &= \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{m,j} \mathrm{e}^{-(\lambda_m + j\lambda_1)s}. \end{aligned}$$

This shows that for $s \in \widehat{D}$

$$\begin{aligned} W_v(s) &= \sum_{k=1}^{l} u_{1k} f_1(s_k + s) + \sum_{k=1}^{l} u_{2k} f_2(s_k + s) \\ &= \sum_{k=1}^{l} u_{1k} \sum_{j=0}^{r} b_j \mathrm{e}^{-\lambda_1 j (s_k + s)} + \sum_{k=1}^{l} u_{2k} \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{mj} \mathrm{e}^{-(\lambda_m + j\lambda_1)(s_k + s)} \\ &= \sum_{j=0}^{r} \widetilde{b}_j \mathrm{e}^{-\lambda_1 j s} + \sum_{k=1}^{l} u_{2k} \sum_{j=0}^{r} \sum_{m=1}^{\infty} \widetilde{a}_{mjk} \mathrm{e}^{-(\lambda_m + j\lambda_1)s} = Z_1(s) + Z_2(s), \end{aligned}$$

where

$$\widetilde{b}_j = \sum_{k=1}^{l} u_{1k}\widehat{b}_j e^{-\lambda_1 s_k}, \quad \widehat{b}_j = \begin{cases} b_j, & j \le r, \\ 0, & \text{j>r}, \end{cases}$$

and

$$\widetilde{a}_{mjk} = \widehat{a}_{mj} e^{-(\lambda_m + j\lambda_1)s_k}, \quad \widehat{a}_{mj} = \begin{cases} a_{mj}, & j \le r, \\ 0, & j > r. \end{cases}$$

$Z_1(s)$ is a Dirichlet polynomial, and $Z_2(s)$ is a linear combination of Dirichlet series satisfying conditions (7) and (8). Since the function $f_2(s)$ is regular in the half-plane $\sigma > \sigma_1$, the function $Z_2(s)$ is regular in $\widehat{D}$.

Since the set $\{\lambda_m\}$ is a system of algebraic numbers linearly independent over the field of rational numbers, and the exponents have a property of type (4), we have, that the probability measure

$$\mu_N\big(W_v(s+imh) \in A\big) = \mu_N\big((Z_1(s+imh) + Z_2(s+imh)) \in A\big), \quad A \in \mathcal{B}(H(\widehat{D})), \tag{35}$$

converges weakly to the distribution of the $H(\widehat{D})$-valued random element

$$W_v(s,\omega) = \sum_{j=0}^{r} \widetilde{b}_j \omega(1) e^{-\lambda_1 js} + \sum_{k=1}^{l} u_{2k} \sum_{j=0}^{r} \sum_{m=1}^{\infty} \widetilde{a}_{m,j,k} \omega^j(1)\omega(m) e^{-(\lambda_m + j\lambda_1)s} \tag{36}$$

as $N \to \infty$. The proof of this is obtained in a similar way like, for example, for the function $f_2(s)$. First we prove a limit theorem for the measure

$$\mu_N\big((Z_1(s) + \sum_{k=1}^{l} u_{2k} \sum_{j=0}^{r} \sum_{m=1}^{M} \widetilde{a}_{mjk} e^{-(\lambda_m + j\lambda_1)s}) \in A\big), \quad A \in \mathcal{B}(H(\widehat{D})).$$

After this, applying an approximation of the function $W_v(s)$ in the mean, it remains only to use some elements of the ergodic theory to obtain the explicit form of the limit measure, and it turns out that this limit measure coincides with the distribution of the random element defined by Eq. (36).

However, by the definition of v

$$\begin{aligned} W_v(s,\omega) &= \sum_{k=1}^{l} u_{1k} \sum_{j=0}^{r} b_j \omega^j(1) e^{-\lambda_1 j(s_k+s)} + \sum_{k=1}^{l} u_{2k} \sum_{j=0}^{r} \sum_{m=1}^{\infty} a_{m,j,k}\omega^j(1)\omega(m) e^{-(\lambda_m + j\lambda_1)(s_k+s)} \\ &= \sum_{j=0}^{2} \sum_{k=1}^{r} u_{jk} f_j(s_k + s, \omega) = v\big(f_1(s,\omega), f_2(s,\omega)\big). \end{aligned}$$

Therefore, measure (35) converges weakly to the distribution of the random element $v\big(f_1(s,\omega), f_2(s,\omega)\big)$ as $N \to \infty$.

Proof of Lemmas 7.1. By Lemma 7.2 it follows that there exists a sequence $N_1 \to \infty$ such that the measure P_{N_1,f_1,f_2} converges weakly to some probability measure P on $\big(H^2(D), \mathcal{B}(H^2(D))$ as $N_1 \to \infty$. Let

$$F_1(s) = \big(f_{11}(s), f_{12}(s)\big)$$

be an $H^2(D)$-valued random element with distribution P. Then by the choice of N_1 we have that

$$F_{N_1} \underset{N_1\to\infty}{\xrightarrow{\mathcal{D}}} F_1. \tag{37}$$

The function v is continuous. Hence, by Lemma 2.1, we have that

$$v(F_{N_1}) \underset{N_1\to\infty}{\xrightarrow{\mathcal{D}}} v(F_1).$$

By the definition of $W_v(s)$

$$W_v(s+i\theta_{N_1}) \underset{N_1\to\infty}{\xrightarrow{\mathcal{D}}} v(F_1). \tag{38}$$

Denoting $F(s) = \big(f_1(s), f_2(s)\big)$, by Lemma 7.3 we have

$$W_v(s+i\theta_{N_1}) \underset{N_1\to\infty}{\xrightarrow{\mathcal{D}}} v(F).$$

From this and relation (38)

$$v(F) \overset{\mathcal{D}}{=} v(F_1). \tag{39}$$

Now let $v_1 : \ H(\widehat{D}) \to \mathbb{C}$ be defined by the formula

$$v_1(f) = f(0), \quad f \in H(\widehat{D}).$$

Then the function v_1 is measurable, and (39) gives

$$v_1(v(F)) \overset{\mathcal{D}}{=} v_1(v(F_1)),$$

or

$$v(F)(0) \overset{\mathcal{D}}{=} v(F_1)(0).$$

By the definition of v we find

$$\sum_{j=1}^{2}\sum_{k=1}^{l} u_{jk} f_j(s_k,\omega) \overset{\mathcal{D}}{=} \sum_{j=1}^{2}\sum_{k=1}^{l} u_{jk} f_{1j}(s_k) \tag{40}$$

with arbitrary complex numbers u_{jk}. The hyperplanes in the space $\mathbb{R}^{4n}$ generates a determining class [1]. Then, the hyperplanes also form a determining class in $\mathbb{C}^{2n}$. Hence, and from (40), we see that $\mathbb{C}^{2n}$-valued random elements $f_j(s_k,\omega)$ and $f_{1j}(s_k)$, $j=1,2$, $k=1,...,l$, have the same distribution.

Denote by K a compact subset of D, and let $\varphi_1, \varphi_2 \in H(D)$. Moreover, let for every $\epsilon > 0$,

$$G = \{(g_1, g_2) \in H^2(D) : \ \sup_{s\in K} |g_j(s) - \varphi_j(s)| \le \epsilon, j = 1,2\}.$$

We choose a sequence $\{s_k\}$ to be dense in K. Moreover, let

$$G_l = \{(g_1, g_2) \in H^2(D) : \ |g_j(s_k) - \varphi_j(s_k)| \le \epsilon, j = 1,2, \ k = 1, ..., l\}.$$

Then the above mentioned properties of the random elements $f_j(s_k,\omega)$ and $f_{1j}(s_k,\omega)$ show that

$$m_H\big(\omega\in\Omega:\ F(s,\omega)\in G_l\big)=\mathbb{P}\big(F_1(s)\in G_l\big). \tag{41}$$

Since the sequence $\{s_k\}$ is dense in K, we have $G_1\supset G_2\supset\ldots$ and $G_l\to G$ as $l\to\infty$. If $l\to\infty$ in Eq. (41), we find

$$m_H\big(\omega\in\Omega:\ F(s,\omega)\in G\big)=\mathbb{P}\big(F_1(s)\in G\big). \tag{42}$$

The space $H^2(D)$ is separable. Thus finite intersections of the spheres form a determining class [1]. Hence, Eq. (42) implies

$$F=F_1.$$

This and relation (37) yield

$$F_{N_1}\ \xrightarrow[N_1\to\infty]{\mathcal{D}}\ F. \tag{43}$$

Consequently, the measure P_{N_1,f_1,f_2} converges weakly to the distribution of random element

$$F=\big(f_1(s,\omega),f_2(s,\omega)\big)$$

as $N_1\to\infty$. Since by Lemma 7.2 the family $\{P_{N_1,f_1,f_2}\}$ is relatively compact and the random element F in (43) is independent of the choice of N_1, by Theorem 1.1.9 of [5] and Lemma 7.2 we obtain the assertion of the lemma.

8 Proof of the Theorem

Define the function u: $H^2(D)\to M(D)$ by the formula

$$u(g_1,g_2)=\frac{g_2}{g_1},\qquad g_1,g_2\in H(D).$$

The metric d satisfies

$$d(g_1,g_2)=d\Big(\frac{1}{g_1},\frac{1}{g_2}\Big).$$

Therefore, the function u is continuous. Thus, by Lemmas 2.1 and 7.1 we deduce that the probability measure

$$\begin{aligned}P_N\ &=\mu_N\big(f(s+imh)\in A\big)=P_{N,f_1,f_2}u^{-1}\\ &=\mu_N\Bigg(\tfrac{f_2(s+imh)}{f_1(s+imh)}\in A\Bigg),\quad A\in\mathcal{B}(M(D))\end{aligned}$$

converges weakly to the distribution of the random element $\frac{f_2(s,\omega)}{f_1(s,\omega)}$. However,

$$\begin{aligned}\frac{f_2(s,\omega)}{f_1(s,\omega)}&=\frac{\sum_{j=0}^{r}\sum_{m=1}^{\infty}a_{mj}\omega^j(1)\omega(m)\mathrm{e}^{-(\lambda_m+j\lambda_1)s}}{\prod_{j=1}^{r}\big(1-\omega(1)\mathrm{e}^{\lambda_1(s_j-s)}\big)}\\ &=\frac{\prod_{j=1}^{r}\big(1-\omega(1)\mathrm{e}^{\lambda_1(s_j-s)}\big)\sum_{m=1}^{\infty}a_m\omega(m)\mathrm{e}^{-\lambda_m s}}{\prod_{j=1}^{r}\big(1-\omega(1)\mathrm{e}^{\lambda_1(s_j-s)}\big)}\\ &=f(s,\omega).\end{aligned}$$

Therefore, the limit measure is $m_H(\omega\in\Omega:\ f(s,\omega)\in A)$, $A\in\mathcal{B}(M(D))$. The theorem is proved.

References

[1] P. Billingsley: *Convergence of Probability Measures*, Wiley, New York, 1968.

[2] H. Bohr and B. Jessen: "Über die Werverteilung der Riemannschen Zeta function". Erste Mitteilung, *Acta Math.*, Vol. 54, (1930), pp. 1–35.

[3] H. Bohr and B. Jessen: "Über die Werverteilung der Riemannschen Zeta function". Zweite Mitteilung, *Acta Math.*, Vol. 54, (1932), pp. 1–55.

[4] J. Genys and A. Laurinčikas: "Value distribution of general Dirichlet series IV", *Lith.Math.J.,* Vol. 43, No. 3, (2003), pp. 342–358; *Lith.Math.J.*, Vol.43, No. 3, (2003), pp. 281–294 (in Russian).

[5] A. Laurinčikas: *Limit Theorems for the Riemann Zeta-Function*, Kluwer, Dordrecht, 1996.

[6] A. Laurinčikas: "Value distribution of general Dirichlet series", In: B. Grigelionis et al. (Eds.): *Probab.Theory and Math. Statistics; Proceedings of the seventh Vilnius*, TEV, Vilnius, (1999), pp. 405–414.

[7] A. Laurinčikas: "Value distribution of general Dirichlet series. II", *Lith.Math.J.*, Vol. 41, No. 4, (2001), pp. 351–360.

[8] A. Laurinčikas: "Limit theorems for general Dirichlet series", *Theory Stoch.Proc.*, Vol. 8, No. 24, (2002), pp. 256–268.

[9] A. Laurinčikas, W. Schwarz and J. Steuding: "Value distribution of general Dirichelet series. III", In: A. Dubickas et al. (Eds.): *Analytic and Probab. Methods in Number Theory, Proc. The Third Palanga Conf.*, TEV, Vilnius, (2002), pp. 137–156.

[10] A. Laurinčikas and R. Macaitienė: "Discrete limit theorems for general Dirichlet series. I", *Chebyshevski sbornik*, Vol. 4, No. 3, (2003), pp. 156–170.

[11] R. Macaitienė: "Discrete limit theorems for general Dirichlet polynomials", *Lith. Math. J.*, Vol. 42(spec. issue), (2002), pp. 705–709.

[12] R. Macaitienė: "Discrete limit theorems for general Dirichlet series. II", *Lith. Math. J.* , (to appear).

[13] H.L. Montgomery: *Topics in multiplicative number theory*, Springer, Berlin, 1971.

[14] E.M. Nikishin: "Dirichlet series with independent exponents and certain of their applications", *Matem.sb*, Vol. 96, No. 1, (1975), pp. 3–40 (in Russian).

[15] Y.G. Sinaĭ: *Introduction to Ergodic Theory*, Princeton Univ. Press, 1976.

[16] A.A. Tempelman: *Ergodic Theorems on Groups* , Mokslas, Vilnius, 1986.