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Miura opers and critical points of master functions

Evgeny Mukhin[1*†], Alexander Varchenko[2‡]

[1] *Department of Mathematical Sciences, Indiana University Purdue University Indianapolis, 402 North Blackford St., Indianapolis, IN 46202-3216, USA*

[2] *Department of Mathematics, University of North Carolina at Chapel Hill, Chapel Hill, NC 27599-3250, USA*

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Abstract: Critical points of a master function associated to a simple Lie algebra $\mathfrak{g}$ come in families called the populations [11]. We prove that a population is isomorphic to the flag variety of the Langlands dual Lie algebra ${}^t\mathfrak{g}$. The proof is based on the correspondence between critical points and differential operators called the Miura opers.
For a Miura oper D, associated with a critical point of a population, we show that all solutions of the differential equation $DY = 0$ can be written explicitly in terms of critical points composing the population.

Keywords: Bethe Ansatz, Miura opers, flag varieties
MSC (2000): 82B23, 17B67, 14M15

1 Introduction

In [18] rational functions were considered which are products of powers of linear functions. It was discovered that under certain conditions all critical points of the rational functions are non-isolated and form non-trivial varieties. It is not clear yet how general that phenomenon is but the phenomenon certainly holds for products of powers of linear functions appearing in representation theory. Those products are called the master functions.

Let $\mathfrak{h}$ be the Cartan subalgebra of a simple Lie algebra $\mathfrak{g}$; $(\ ,\ )$ the Killing form on $\mathfrak{h}^*$; $\alpha_1, \dots, \alpha_r \in \mathfrak{h}^*$ simple roots; $\Lambda_1, \dots, \Lambda_n \in \mathfrak{h}^*$ dominant integral weights; $l_1, \dots, l_r$ non-

* E-mail: mukhin@math.iupui.edu
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negative integers; $z_1, \dots, z_n$ distinct complex numbers. The master function Φ associated with this data is given by formula (1). It is a rational function of $l_1 + \dots + l_r$ variables $t_1^{(1)}, \dots, t_{l_1}^{(1)}, t_1^{(2)}, \dots, t_{l_r}^{(r)}$, and n variables $z_1, \dots, z_n$. We may think that $l_1 + \dots + l_r + n$ particles are given in $\mathbb{C}$. The particle $t_j^{(i)}$ has weight $-\alpha_i$ and the particle z_s has weight Λ_s. The particles interact pairwise. The interaction of particles x and y with weights v and w, respectively, is given by $(x-y)^{(v,w)}$. Then total interaction is the product of such terms over the set of all pairs. The master function describes the interaction of t-particles and z-particles.

The master function appears in hypergeometric solutions to the KZ equations with values in the tensor product of irreducible highest weight representations $L_{\Lambda_1}, \dots, L_{\Lambda_n}$ with highest weights $\Lambda_1, \dots, \Lambda_n$, respectively. The solutions have the form

$$u(z) \;=\; \int \Phi(\boldsymbol{t};\boldsymbol{z})^{1/\kappa}\; A(\boldsymbol{t};\boldsymbol{z})\; d\boldsymbol{t}\;,$$

where κ is the parameter of the KZ equations and $A(\boldsymbol{t};\boldsymbol{z})$ is some explicitly written rational function with values in the tensor product [19].

The master function also appears in the Bethe ansatz of the Gaudin model with values in the same tensor product [17]. In that case the value of the function $A(\,\cdot\,;\boldsymbol{z})$ at a point $\boldsymbol{t}$ is an eigenvector of the commuting Gaudin Hamiltonians if $\boldsymbol{t}$ is a critical point of the master function.

In this paper we study critical points of the master function on the set where all $\{t_j^{(i)}, z_s\}$ are distinct. In other words we study those positions of distinct particles $\{t_j^{(i)}\}$ in the complement to $\{z_s\}$ which extremize the master function.

Critical points of master functions associated to a simple Lie algebra $\mathfrak{g}$ come in families called populations [18, 11]. In this paper we prove that a population is isomorphic to the flag variety of the Langlands dual Lie algebra ${}^t\mathfrak{g}$. The proof is based on the correspondence between critical points and differential operators called the Miura opers.

To every critical point $\boldsymbol{t}$ one assigns a certain linear differential operator $D_{\boldsymbol{t}}$ with coefficients in ${}^t\mathfrak{g}$, called the Miura oper. The differential operators of that type were considered by V. Drinfeld and V. Sokolov in their study of the KdV type equations [5]. On opers and Miura opers see [1, 6, 7, 8, 11, 4, 14].

Different critical points correspond to different Miura opers. The Miura opers corresponding to critical points of a given population form an equivalence class with respect to suitable gauge equivalence. We show that the equivalence class of Miura opers is isomorphic to the flag variety of ${}^t\mathfrak{g}$.

In [11, 4] we considered Miura opers for Lie algebras of types A_r, B_r, C_r, G_2 and using the opers proved that a population of critical points of types A_r, B_r, C_r, G_2 is isomorphic to the corresponding flag variety. The proof, suggested in the present paper, is more direct and works for any simple Lie algebra.

If $D_{\boldsymbol{t}}$ is the Miura oper corresponding to a critical point $\boldsymbol{t}$, then the set of solutions of the differential equation $D_{\boldsymbol{t}} Y \;=\; 0$ with values in a suitable space is an important characteristics of the critical point. We used that characteristics for sl_{r+1} Miura opers in [11] to give a bound from above for the number of populations of critical points of the corresponding sl_{r+1} master function. That statement in [11] was in some sense opposite to the Bethe ansatz conjectures, see [11].

It turns out that for any simple Lie algebra $\mathfrak{g}$ and any critical point $\boldsymbol{t}$ of a $\mathfrak{g}$ master function all solutions of the differential equation $D_{\boldsymbol{t}} Y = 0$ can be written explicitly in terms of critical points composing the population originated at $\boldsymbol{t}$. Thus the population of critical points "solves" the Miura differential equation $D_{\boldsymbol{t}} Y = 0$. This is the second main result of the paper.

When this paper was being written preprint [8] by E. Frenkel appeared. The preprint is devoted to the same fact that the variety of gauge equivalent ${}^t\mathfrak{g}$ Miura opers is isomorphic to the flag variety of ${}^t\mathfrak{g}$. One of the main claims of [8] is Corollary 3.3. In our opinion the proofs leading to Corollary 3.3 in [8] are sometimes incomplete. Moreover in our paper [16] we construct a counterexample to the statement of Corollary 3.3.

The idea of this paper was originated in discussions with E. Frenkel in the spring of 2002. As a result of those discussions two papers appeared: this one (see its preprint version in [15]) and [8].

We thank E. Frenkel for stimulating meetings which originated this paper. We thank P. Belkale and S. Kumar for numerous useful discussions.

The paper is organized as follows. In Section 2 we introduce populations of critical points. In Section 3 we discuss elementary properties of Miura opers corresponding to critical points. In Section 4 we prove that the variety of gauge equivalent Miura opers is isomorphic to the flag variety, see Theorem 4.3. We discuss the relations between the Bruhat cell decomposition of the flag variety and populations of critical points in Section 5. The main result there is Corollary 5.4 describing the structure of connected components of the critical set of master functions. In Section 6 we give explicit formulas for solutions of the differential equation $D_{\boldsymbol{t}} Y = 0$, see Theorems 6.5, 6.7, 6.8.

2 Master functions and critical points, [11]

2.1 Kac-Moody algebras

Let $A = (a_{i,j})_{i,j=1}^r$ be a generalized Cartan matrix, $a_{i,i} = 2$, $a_{i,j} = 0$ if and only $a_{j,i} = 0$, $a_{i,j} \in \mathbb{Z}_{\le 0}$ if $i \ne j$. We assume that A is symmetrizable, i.e. there exists a diagonal matrix $D = \mathrm{diag}\{d_1, \dots, d_r\}$ with positive integers d_i such that $B = DA$ is symmetric.

Let $\mathfrak{g} = \mathfrak{g}(A)$ be the corresponding complex Kac-Moody Lie algebra (see [10], §1.2), $\mathfrak{h} \subset \mathfrak{g}$ the Cartan subalgebra. The associated scalar product is non-degenerate on $\mathfrak{h}^*$ and $\dim \mathfrak{h} = r + 2d$, where d is the dimension of the kernel of the Cartan matrix A.

Let $\alpha_i \in \mathfrak{h}^*$, $\alpha_i^\vee \in \mathfrak{h}$, $i = 1, \dots, r$, be the sets of simple roots, coroots, respectively. We have

$$(\alpha_i, \alpha_j) = d_i \, a_{i,j},$$
$$\langle \lambda, \alpha_i^\vee \rangle = 2(\lambda, \alpha_i)/(\alpha_i, \alpha_i), \qquad \lambda \in \mathfrak{h}^*.$$

In particular, $\langle\alpha_j,\alpha_i^\vee\rangle = a_{i,j}$.

Let $\mathcal{P} = \{\lambda \in \mathfrak{h}^* \,|\, \langle\lambda,\alpha_i^\vee\rangle \in \mathbb{Z}\}$ and $\mathcal{P}^+ = \{\lambda \in \mathfrak{h}^* \,|\, \langle\lambda,\alpha_i^\vee\rangle \in \mathbb{Z}_{\geq 0}\}$ be the sets of integral and dominant integral weights.

Fix $\rho \in \mathfrak{h}^*$ such that $\langle\rho,\alpha_i^\vee\rangle = 1$, $i = 1,\dots,r$. We have $(\rho,\alpha_i) = (\alpha_i,\alpha_i)/2$.

The Weyl group $\mathcal{W} \in \mathrm{End}(\mathfrak{h}^*)$ is generated by reflections s_i, $i = 1,\dots,r$,

$$s_i(\lambda) = \lambda - \langle\lambda,\alpha_i^\vee\rangle\alpha_i, \qquad \lambda \in \mathfrak{h}^*.$$

We use the notation

$$w\cdot\lambda = w(\lambda+\rho)-\rho, \qquad w\in\mathcal{W},\ \lambda\in\mathfrak{h}^*,$$

for the shifted action of the Weyl group.

The Kac-Moody algebra $\mathfrak{g}(A)$ is generated by $\mathfrak{h}$, $e_1,\dots,e_r, f_1,\dots,f_r$ with defining relations

$$\begin{aligned}
[e_i,f_j] &= \delta_{i,j}\,\alpha_i^\vee, \qquad i,j=1,\dots r,\\
[h,h'] &= 0, \qquad h,h'\in\mathfrak{h},\\
[h,e_i] &= \langle\alpha_i,h\rangle\,e_i, \qquad h\in\mathfrak{h},\ i=1,\dots r,\\
[h,f_i] &= -\langle\alpha_i,h\rangle\,f_i, \qquad h\in\mathfrak{h},\ i=1,\dots r,
\end{aligned}$$

and the Serre's relations

$$(\mathrm{ad}\,e_i)^{1-a_{i,j}}\,e_j = 0, \qquad (\mathrm{ad}\,f_i)^{1-a_{i,j}}\,f_j = 0,$$

for all $i \neq j$. The generators $\mathfrak{h}$, $e_1,\dots,e_r, f_1,\dots,f_r$ are called the Chevalley generators.

Denote $\mathfrak{n}_+$ (resp. $\mathfrak{n}_-$) the subalgebra generated by $e_1,\dots,e_r$ (resp. $f_1,\dots,f_r$). Then $\mathfrak{g} = \mathfrak{n}_+\oplus\mathfrak{h}\oplus\mathfrak{n}_-$. Set $\mathfrak{b}_\pm = \mathfrak{h}\oplus\mathfrak{n}_\pm$.

Let $\mathfrak{g} = \oplus_j\mathfrak{g}^j$ be the canonical grading of $\mathfrak{g}$. Here we have $e_i\in\mathfrak{g}^1$, $f_i\in\mathfrak{g}^{-1}$, $\mathfrak{n}_+ = \oplus_{j>0}\mathfrak{g}^j$, $\mathfrak{h} = \mathfrak{g}^0$, $\mathfrak{n}_- = \oplus_{j<0}\mathfrak{g}^j$.

For a vector space X we denote $M(X)$ the space of X-valued rational functions on $\mathbb{C}$.

We denote $\bar{M}(\mathfrak{n}_+)$ the completion of the space $M(\mathfrak{n}_+)$ with respect to the canonical grading. An element of $\bar{M}(\mathfrak{n}_+)$ is a formal sum $\sum_{j>0}u_j$, where $u_j : \mathbb{C}\to\mathfrak{g}^j$ are rational functions.

The Kac-Moody algebra ${}^t\mathfrak{g} = \mathfrak{g}({}^tA)$ corresponding to the transposed Cartan matrix ${}^tA$ is called *Langlands dual* to $\mathfrak{g}$. Let ${}^t\alpha_i\in{}^t\mathfrak{h}^*$, ${}^t\alpha_i^\vee\in{}^t\mathfrak{h}$, $i=1,\dots,r$, be the sets of simple roots, coroots of ${}^t\mathfrak{g}$, respectively. Then

$$\langle{}^t\alpha_i,{}^t\alpha_j^\vee\rangle = \langle\alpha_j,\alpha_i^\vee\rangle = a_{i,j}$$

for all i,j.

2.2 The definition of master functions and critical points

We fix a Kac-Moody algebra $\mathfrak{g} = \mathfrak{g}(A)$, a non-negative integer n, a collection of dominant integral weights $\boldsymbol{\Lambda} = (\Lambda_1,\dots,\Lambda_n)$, $\Lambda_i\in\mathcal{P}^+$, and points $\boldsymbol{z} = \{z_1,\dots,z_n\}\subset\mathbb{C}$. We assume that $z_i \neq z_j$ if $i \neq j$. We often do not stress the dependence of our objects on these parameters.

In addition we choose a collection of non-negative integers $\boldsymbol{l} = (l_1, \ldots, l_r) \in \mathbb{Z}^r_{\geq 0}$. The choice of $\boldsymbol{l}$ is equivalent to the choice of the weight

$$\Lambda_\infty = \sum_{i=1}^{n} \Lambda_i - \sum_{j=1}^{r} l_j \alpha_j \in \mathcal{P}.$$

The weight Λ_∞ will be called *the weight at infinity*.

The *master function* $\Phi(\boldsymbol{t}; \Lambda_\infty)$ is defined by

$$\Phi(\boldsymbol{t}; \Lambda_\infty) = \Phi(\boldsymbol{t}; \boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty) = \prod_{1 \leq s < u \leq n} (z_s - z_u)^{(\Lambda_s, \Lambda_u)} \times \tag{1}$$

$$\prod_{i=1}^{r} \prod_{j=1}^{l_i} \prod_{s=1}^{n} (t_j^{(i)} - z_s)^{-(\Lambda_s, \alpha_i)} \prod_{i=1}^{r} \prod_{1 \leq j < s \leq l_i} (t_j^{(i)} - t_s^{(i)})^{(\alpha_i, \alpha_i)} \prod_{1 \leq i < j \leq r} \prod_{s=1}^{l_i} \prod_{k=1}^{l_j} (t_s^{(i)} - t_k^{(j)})^{(\alpha_i, \alpha_j)},$$

see [19]. The function Φ is a function of variables $\boldsymbol{t} = (t_j^{(i)})$, where $i = 1, \ldots, r$, and $j = 1, \ldots, l_i$, of variables $z = (z_1, \ldots, z_n)$, weights $\boldsymbol{\Lambda}$, and integers $\boldsymbol{l}$. The main variables are $\boldsymbol{t}$, the other variables will be considered as parameters.

The function Φ is symmetric with respect to permutations of variables with the same upper index.

A point $\boldsymbol{t}$ with complex coordinates is called a *critical point* if the following system of algebraic equations is satisfied

$$-\sum_{s=1}^{n} \frac{(\Lambda_s, \alpha_i)}{t_j^{(i)} - z_s} + \sum_{s,\ s \neq i} \sum_{k=1}^{l_s} \frac{(\alpha_s, \alpha_i)}{t_j^{(i)} - t_k^{(s)}} + \sum_{s,\ s \neq j} \frac{(\alpha_i, \alpha_i)}{t_j^{(i)} - t_s^{(i)}} = 0, \tag{2}$$

where $i = 1, \ldots, r$ and $j = 1, \ldots, l_i$. In other words, the point $\boldsymbol{t}$ is a critical point if

$$\left(\Phi^{-1} \frac{\partial \Phi}{\partial t_j^{(i)}} \right)(\boldsymbol{t}) = 0, \qquad i = 1, \ldots, r, \quad j = 1, \ldots, l_i.$$

Note that the product of symmetric groups $S_{\boldsymbol{l}} = S_{l_1} \times \cdots \times S_{l_r}$ acts on the critical set of the master function permuting the coordinates with the same upper index. All orbits have the same cardinality $l_1! \cdots l_r!$. We do not make distinction between critical points in the same orbit.

2.3 Polynomials representing critical points

Let $\boldsymbol{t}$ be a critical point of the master function $\Phi = \Phi(\boldsymbol{t}; \Lambda_\infty)$. Introduce an r-tuple of polynomials $\boldsymbol{y} = (y_1(x), \ldots, y_r(x))$,

$$y_i(x) = \prod_{j=1}^{l_i} (x - t_j^{(i)}). \tag{3}$$

Each polynomial is considered up to multiplication by a non-zero number. The tuple defines a point in the direct product $\boldsymbol{P}(\mathbb{C}[x])^r$ of r copies of the projective space associated with the vector space of polynomials in x. We say that the tuple $\boldsymbol{y}$ *represents the critical point*.

It is convenient to think that the tuple $(1,\dots,1)$ of constant polynomials represents in $\boldsymbol{P}(\mathbb{C}[x])^r$ the critical point of the master function with no variables. This corresponds to the case when $\boldsymbol{l} = (0,\dots,0)$ and $\Lambda_\infty = \sum_{s=1}^n \Lambda_s$.

Introduce polynomials

$$T_i(x) = \prod_{s=1}^n (x - z_s)^{\langle \Lambda_s, \alpha_i^\vee \rangle}, \qquad i = 1,\dots,r. \tag{4}$$

We say that a given tuple $\boldsymbol{y}$ is *generic with respect to weights* $\Lambda_1,\dots,\Lambda_n$ *and points* $z_1,\dots,z_n$ if

- each polynomial $y_i(x)$ has no multiple roots;
- all roots of $y_i(x)$ are different from roots of the polynomial T_i;
- any two polynomials $y_i(x)$, $y_j(x)$ have no common roots if $i \neq j$ and $a_{i,j} \neq 0$.

A tuple is generic if it represents a critical point.

Let $W(f,g) = f'g - fg'$ be the Wronskian of functions f,g of x.

A tuple $\boldsymbol{y}$ is called *fertile*, if for every $i = 1,\dots,r$ there exists a polynomial $\tilde{y}_i$ satisfying the equation

$$W(y_i, \tilde{y}_i) \;=\; T_i \prod_{j,\; j\neq i} y_j^{-\langle \alpha_j, \alpha_i^\vee \rangle} \;=\; T_i \prod_{j,\; j\neq i} y_j^{-a_{i,j}} \,. \tag{5}$$

The polynomial $\tilde{y}_i$ considered up to multiplication by a non-zero number has the form

$$\tilde{y}_i(x) \;=\; c_1\, y_i(x) \int T_i(x) \prod_{j=1}^r y_j(x)^{-a_{i,j}}\, dx \;+\; c_2\, y_i(x)\,, \tag{6}$$

where c_1, c_2 are complex numbers, $c_1 \neq 0$.

If $\boldsymbol{y}$ is fertile and $i \in \{1,\dots,r\}$, then the tuple

$$\boldsymbol{y}^{[i]} \;=\; (y_1,\dots,\tilde{y}_i,\dots,y_r) \qquad \in \qquad \boldsymbol{P}(\mathbb{C}[x])^r \tag{7}$$

is called *the immediate descendant* of $\boldsymbol{y}$ in the i-th direction.

Theorem 2.1 ([11]).

(i) A generic tuple $\boldsymbol{y} = (y_1,\dots,y_r)$, with $\deg y_j = l_j$, represents a critical point of the master function $\Phi(\boldsymbol{t};\Lambda_\infty)$, with $\Lambda_\infty = \sum_{s=1}^n \Lambda_s - \sum_{j=1}^r l_j\, \alpha_j$, if and only if it is fertile.

(ii) If $\boldsymbol{y}$ represents a critical point, $i \in \{1,\dots,r\}$, and $\boldsymbol{y}^{[i]}$ is an immediate descendant of $\boldsymbol{y}$, then $\boldsymbol{y}^{[i]}$ is fertile.

Let $\boldsymbol{y}$ represent a critical point of $\Phi(\boldsymbol{t};\Lambda_\infty)$. Let $i \in \{1,\dots,r\}$ and let $\boldsymbol{y}^{[i]}$ be an immediate descendant of $\boldsymbol{y}$ in the i-th direction. Denote $\tilde{l}_i = \deg \tilde{y}_i$ and $\Lambda_\infty^{[i]} = \sum_{s=1}^n \Lambda_s -$

$\tilde{l}_i\alpha_i - \sum_{j=1,\ j\neq i}^{r} l_j\alpha_j$. Assume that $\boldsymbol{y}^{[i]}$ is generic, then $\boldsymbol{y}^{[i]}$ represents a critical point of the master function $\Phi(\boldsymbol{t};\Lambda_\infty^{[i]})$. If $\tilde{l}_i \neq l_i$, then

$$\Lambda_\infty^{[i]} \ = \ s_i \cdot \Lambda_\infty \ ,$$

where $s_i\cdot$ is the shifted action of the i-th reflection of the Weyl group.

2.4 Simple reproduction procedure

Let $\boldsymbol{y}$ represent a critical point of $\Phi(\boldsymbol{t},\Lambda_\infty)$. The tuples $\boldsymbol{y}^{[i]} = (y_1,\dots,\tilde{y}_i,\dots,y_r) \in \boldsymbol{P}(\mathbb{C}[x])^r$, where $\tilde{y}_i$ is given by (6) and c_1, c_2 are arbitrary numbers, not both equal to zero, form a one-parameter family. The parameter space of the family is identified with the projective line $\boldsymbol{P}^1$ with projective coordinates $(c_1 : c_2)$. We have a map

$$Y_{\boldsymbol{y},i} \ : \ \boldsymbol{P}^1 \ \to \boldsymbol{P}(\mathbb{C}[x])^r \ ,$$

which sends a point $c = (c_1 : c_2)$ to the corresponding tuple $\boldsymbol{y}^{[i]}$. Almost all tuples $\boldsymbol{y}^{[i]}$ are generic. The exceptions form a finite set in $\boldsymbol{P}^1$.

Thus, starting with a tuple $\boldsymbol{y}$, representing a critical point of the master function $\Phi(\boldsymbol{t};\Lambda_\infty)$, and an index $i \in \{1,\dots,r\}$, we construct a family $Y_{\boldsymbol{y},i} : \boldsymbol{P}^1 \to \boldsymbol{P}(\mathbb{C}[x])^r$ of fertile tuples. For almost all $c \in \boldsymbol{P}^1$ (with finitely many exceptions only), the tuple $Y_{\boldsymbol{y},i}(c)$ represents a critical point of the master function associated with integral dominant weights $\Lambda_1,\dots,\Lambda_n$, points $z_1,\dots,z_n$, and a suitable weight at infinity.

We call this construction *the simple reproduction procedure in the i-th direction.*

2.5 General reproduction procedure

Assume that a tuple $\boldsymbol{y} \in \boldsymbol{P}(\mathbb{C}[x])^r$ represents a critical point of the master function $\Phi(\boldsymbol{t};\Lambda_\infty)$.

Let $\boldsymbol{i} = [i_1, i_2, \dots, i_k],\ i_j \in \{1,\dots,r\}$, be a sequence of natural numbers. We define a k-parameter family of fertile tuples

$$Y_{\boldsymbol{y},\boldsymbol{i}} \ : \ (\boldsymbol{P}^1)^k \ \to \boldsymbol{P}(\mathbb{C}[x])^r$$

by induction on k, starting at $\boldsymbol{y}$ and successively applying the simple reproduction procedure in directions $i_1,\dots,i_k$. The image of this map is denoted $P_{\boldsymbol{y},\boldsymbol{i}}$.

For a given $\boldsymbol{i} = [i_1,\dots,i_k]$, almost all tuples $Y_{\boldsymbol{y},\boldsymbol{i}}(\boldsymbol{c})$ represent critical points of master functions associated to weights $\Lambda_1,\dots,\Lambda_n$, points $z_1,\dots,z_n$, and suitable weights at infinity. Exceptional values of $c \in (\boldsymbol{P}^1)^k$ are contained in a proper algebraic subset.

It is easy to see that if $\boldsymbol{i}' = [i'_1, i'_2, \dots, i'_{k'}],\ i_j \in \{1,\dots,r\}$, is a sequence of natural numbers, and the sequence $\boldsymbol{i}'$ is contained in the sequence $\boldsymbol{i}$ as an ordered subset, then $P_{\boldsymbol{y},\boldsymbol{i}'}$ is a subset of $P_{\boldsymbol{y},\boldsymbol{i}}$.

The union

$$P_{\boldsymbol{y}} \ = \ \cup_{\boldsymbol{i}} \ P_{\boldsymbol{y},\boldsymbol{i}} \ \subset \boldsymbol{P}(\mathbb{C}[x])^r \ ,$$

where the summation is over all of sequences $\boldsymbol{i}$, is called *the population of critical points associated* with the Kac-Moody algebra $\mathfrak{g}$, integral dominant weights $\Lambda_1, \dots, \Lambda_n$, points $z_1, \dots, z_n$, and *originated* at y.

If two populations intersect, then they coincide.

If the Weyl group is finite, then all tuples of a population consist of polynomials of bounded degree. Thus, if the Weyl group of $\mathfrak{g}$ is finite, then a population is a projective irreducible variety.

Every population P has a tuple $\boldsymbol{y} = (y_1, \dots, y_r)$, $\deg y_i = l_i$, such that the weight $\Lambda_\infty = \sum_{s=1}^n \Lambda_s - \sum_{i=1}^r l_i \alpha_i$ is dominant integral, see [11].

Conjecture 2.2 ([11]). Every population, associated with a Kac-Moody algebra $\mathfrak{g}$, dominant integral weights $\Lambda_1, \dots, \Lambda_n$, points $z_1, \dots, z_n$, is an algebraic variety isomorphic to the flag variety associated to the Kac-Moody algebra ${}^t\mathfrak{g}$ which is Langlands dual to $\mathfrak{g}$. Moreover, the parts of the family corresponding to tuples of polynomials with fixed degrees are isomorphic to Bruhat cells of the flag variety.

The conjecture is proved for the Lie algebras with root systems of types A_r, B_r, C_r, G_2 in [11, 4]. In Theorems 4.3 and 5.3 we prove this conjecture for any simple Lie algebra.

2.6 Diagonal sequences of polynomials associated with a critical point and a sequence of indices

We introduce notions which will be used in Chapter 6 to construct solutions of differential equations.

Lemma 2.3. Assume that a tuple $\boldsymbol{y}$ of non-zero polynomials represents a critical point of the master function $\Phi(\boldsymbol{t}; \Lambda_\infty)$. Let $\boldsymbol{i} = [i_1, i_2, \dots, i_k]$, $i_j \in \{1, \dots, r\}$, be a sequence of natural numbers. Then there exist tuples $\boldsymbol{y}^{[i_1]} = (y_1^{[i_1]}, \dots, y_r^{[i_1]})$, $\boldsymbol{y}^{[i_1,i_2]} = (y_1^{[i_1,i_2]}, \dots, y_r^{[i_1,i_2]})$, $\dots$, $\boldsymbol{y}^{[i_1,i_2,\dots,i_k]} = (y_1^{[i_1,i_2,\dots,i_k]}, \dots, y_r^{[i_1,i_2,\dots,i_k]})$ in $\boldsymbol{P}(\mathbb{C}[x])^r$ such that

(i)

$$W(y_{i_1}, y_{i_1}^{[i_1]}) \;=\; T_{i_1} \prod_{j,\; j\neq i_1} y_j^{-a_{i_1,j}}$$

and $y_j^{[i_1]} = y_j$ for $j \neq i_1$;

(ii) for $l = 2, \dots, k$, we have

$$W(y_{i_l}^{[i_1,\dots,i_{l-1}]}, y_{i_l}^{[i_1,\dots,i_l]}) \;=\; T_{i_l} \prod_{j,\; j\neq i_l} (y_j^{[i_1,\dots,i_{l-1}]})^{-a_{i_l,j}}$$

and $y_j^{[i_1,\dots,i_l]} = y_j^{[i_1,\dots,i_{l-1}]}$ for $j \neq i_l$. □

The tuples $\boldsymbol{y}^{[i_1]}$, $\boldsymbol{y}^{[i_1,i_2]}$, $\dots$, $\boldsymbol{y}^{[i_1,i_2,\dots,i_k]}$ belong to the population $P_{\boldsymbol{y}}$. The tuple $\boldsymbol{y}^{[i_1]}$ is obtained from $\boldsymbol{y}$ by the i_1-th simple generation procedure and for $l = 2, \dots, k$, the tuple $\boldsymbol{y}^{[i_1,\dots,i_l]}$ is obtained from $\boldsymbol{y}^{[i_1,\dots,i_{l-1}]}$ by the i_l-th simple generation procedure.

The sequence of tuples $\boldsymbol{y}^{[i_1]}$, $\boldsymbol{y}^{[i_1,i_2]}$, $\dots$, $\boldsymbol{y}^{[i_1,i_2,\dots,i_k]}$ satisfying Lemma 2.3 will be called *associated with the critical point* $\boldsymbol{y}$ *and the sequence of indices* $\boldsymbol{i}$. The sequence of polynomials $y_{i_1}^{[i_1]}$, $y_{i_2}^{[i_1,i_2]}$, $\dots$, $y_{i_k}^{[i_1,i_2,\dots,i_k]}$ will be called *the diagonal sequence of polynomials associated with the critical point* $\boldsymbol{y}$ *and the sequence of indices* $\boldsymbol{i}$. For a given $\boldsymbol{y}$ the diagonal sequence of polynomials determine the sequence of tuples $\boldsymbol{y}^{[i_1]}$, $\boldsymbol{y}^{[i_1,i_2]}$, $\dots$, $\boldsymbol{y}^{[i_1,i_2,\dots,i_k]}$ uniquely.

There are many diagonal sequences of polynomials associated with a given critical point and a given sequence of indices.

3 Opers

Let $\mathfrak{g} = \mathfrak{g}(A)$ be a Kac-Moody algebra with simple roots $\alpha_1, \dots, \alpha_r$ and simple coroots $\alpha_1^\vee, \dots, \alpha_r^\vee$. Let ${}^t\mathfrak{g} = \mathfrak{g}({}^tA)$ be the Langlands dual algebra with Chevalley generators ${}^t\mathfrak{h}$, $E_1, \dots, E_r, F_1, \dots, F_r$, simple roots ${}^t\alpha_1, \dots, {}^t\alpha_r$ and simple coroots ${}^t\alpha_1^\vee, \dots, {}^t\alpha_r^\vee$. Set $H_1 = {}^t\alpha_1^\vee$, $\dots$, $H_r = {}^t\alpha_r^\vee$ and

$$I = F_1 + \dots + F_r, \qquad \partial = d/dx\ .$$

A ${}^t\mathfrak{g}$*-oper* is a differential operator of the form

$$D = \partial + I + V + W$$

with $V \in M({}^t\mathfrak{h})$ and $W \in \bar{M}({}^t\mathfrak{n}_+)$. *A Miura* ${}^t\mathfrak{g}$*-oper* is a differential operator of the form

$$D = \partial + I + V$$

with $V \in M({}^t\mathfrak{h})$.

The differential operators of that type were considered by V. Drinfeld and V. Sokolov in their study of the KdV type equations [5]. On opers and Miura opers see [1, 6, 7, 8, 11, 4, 14].

For $u \in \bar{M}({}^t\mathfrak{n}_+)$ and a ${}^t\mathfrak{g}$-oper D, the differential operator

$$e^{\mathrm{ad}\,u} \cdot D = D + [u, D] + \frac{1}{2}[u, [u, D]] + \dots$$

is a ${}^t\mathfrak{g}$-oper. The opers D and $e^{\mathrm{ad}\,u} \cdot D$ are called *gauge equivalent*.

Let X be a ${}^t\mathfrak{g}$-module with locally finite action of ${}^t\mathfrak{n}_+$. Let D be a ${}^t\mathfrak{g}$-oper and $u \in \bar{M}({}^t\mathfrak{n}_+)$. Then D, $e^{\mathrm{ad}\,u} \cdot D$, $e^{\pm u}$ determine linear operators on $M(X)$. Moreover, we have

$$e^{\mathrm{ad}\,u} \cdot D = e^u D e^{-u}\ .$$

Lemma 3.1. Let $D = \partial + I + V$ be a Miura ${}^t\mathfrak{g}$-oper. Let $g \in M(\mathbb{C})$ and $i \in \{1, \dots, r\}$. Then

$$e^{\mathrm{ad}\,(gE_i)} \cdot D = \partial + I + (V + g H_i) - (g' + \langle {}^t\alpha_i, V\rangle g + g^2)\, E_i\ .$$

The proof is straightforward.

Corollary 3.2. Let $D = \partial + I + V$ be a Miura ${}^t\mathfrak{g}$-oper. Then the ${}^t\mathfrak{g}$-oper $e^{\mathrm{ad}\,(gE_i)} \cdot D$ is a Miura oper if and only if the scalar rational function g satisfies the Ricatti equation

$$g' + \langle {}^t\alpha_i, V\rangle g + g^2 = 0\ . \tag{8}$$

We say that the Miura ${}^t\mathfrak{g}$-oper D is *deformable in the i-th direction* if equation (8) has a non-zero solution which is a rational function.

Fix a collection of dominant integral weights $\mathbf{\Lambda} = (\Lambda_1, \dots, \Lambda_n)$ of the Kac-Moody algebra $\mathfrak{g}$, and numbers $\boldsymbol{z} = \{z_1, \dots, z_n\} \subset \mathbb{C}$, $z_i \neq z_j$ if $i \neq j$. Introduce polynomials $T_1(x), \dots, T_r(x)$ by formulas (4).

Let $\boldsymbol{y} = (y_1, \dots, y_r)$ be a tuple of non-zero polynomials. We say that a Miura ${}^t\mathfrak{g}$-oper $D = \partial + I + V$ is *associated with weights* $\mathbf{\Lambda}$*, numbers* $\boldsymbol{z}$, and the tuple $\boldsymbol{y} = (y_1, \dots, y_r)$, if for every $i \in \{1, \dots, r\}$ we have

$$\langle\, {}^t\alpha_i, V\rangle \;=\; -\,\log'\left(T_i \prod_{j=1}^{r} y_j^{-\langle \alpha_j, \alpha_i^\vee\rangle}\right) \;=\; -\,\log'\left(T_i \prod_{j=1}^{r} y_j^{-a_{i,j}}\right), \tag{9}$$

cf. (5).

If a Miura oper D is associated with weights $\mathbf{\Lambda}$, numbers $\boldsymbol{z}$, and a generic tuple $\boldsymbol{y} = (y_1, \dots, y_r) \in \boldsymbol{P}(\mathbb{C}[x])^r$, then the tuple $\boldsymbol{y}$ is determined uniquely. Indeed, the residues of the rational function $\langle\, {}^t\alpha_i, V\rangle$ are positive exactly at the roots of the polynomial y_i and the residues are equal to the multiplicities of roots of y_i multiplied by two.

If $\mathfrak{g}$ is a simple Lie algebra, then D determines $\boldsymbol{y}$ uniquely even if $\boldsymbol{y}$ is not generic. That fact follows from the invertibility of the Cartan matrix of $\mathfrak{g}$.

Theorem 3.3. Let the Miura ${}^t\mathfrak{g}$-oper $D = \partial + I + V$ be associated with weights $\mathbf{\Lambda}$, numbers $\boldsymbol{z}$, and the tuple $\boldsymbol{y} = (y_1, \dots, y_r)$. Let $i \in \{1, \dots, r\}$. Then D is deformable in the i-th direction if and only if there exists a polynomial $\tilde{y}_i$ satisfying (5). Moreover, in that case any non-zero rational solution g of the Ricatti equation (8) has the form $g = \log'(\tilde{y}_i/y_i)$ where $\tilde{y}_i$ is a solution of (5). If $g = \log'(\tilde{y}_i/y_i)$, then the Miura ${}^t\mathfrak{g}$-oper

$$e^{\mathrm{ad}\,(gE_i)} \cdot D = \partial + I + (V \,+\, g\, H_i) \tag{10}$$

is associated with weights $\mathbf{\Lambda}$, numbers $\boldsymbol{z}$, and the tuple $\boldsymbol{y}^{[i]} = (y_1, \dots, \tilde{y}_i,\ \dots, y_r)$, where the tuple $\boldsymbol{y}^{[i]}$ is called in Section 2.3 an immediate descendant of $\boldsymbol{y}$ in the i-th direction, see (7).

Proof 3.4. Write (8) as

$$g'/g + g = \log'\left(T_i \prod_{j=1}^{r} y_j^{-a_{i,j}}\right)\ . \tag{11}$$

If g is a rational function, then $g \to 0$ as $x \to \infty$ and all poles of g are simple. Moreover, the residue of g at any point is an integer. Hence $g = c'/c$ for a suitable rational function c. Then

$$c \;=\; \int T_i(x) \prod_{j=1}^{r} y_j(x)^{-a_{i,j}} dx \tag{12}$$

and equation (5) has a polynomial solution $\tilde{y}_i = -cy_i$. Conversely if equation (5) has a polynomial solution $\tilde{y}_i$, then the function c in (12) is rational. Then $g = c'/c$ is a rational solution of equation (8).

Let $g = \log'(c) = \log'(\tilde{y}_i/y_i)$, where $\tilde{y}_i$ is a solution of (5). Then

$$e^{\mathrm{ad}\,(gE_i)} \cdot D = \partial + I + (V \;+\; \log'(\tilde{y}_i/y_i)\; H_i)$$

and

$$\langle\, {}^t\alpha_k, V\rangle \;+\; \log'(\tilde{y}_i/y_i)\,\langle\, {}^t\alpha_k,\; {}^t\alpha_i^\vee\rangle \;=\; -\,\log'\left(T_k\, \tilde{y}_i^{\,-\langle \alpha_i,\alpha_k^\vee\rangle} \prod_{j=1,\; j\neq i}^{r} y_j^{-\langle \alpha_j,\alpha_k^\vee\rangle}\right).$$

Note that if equation (8) has one non-zero rational solution $g = c'/c$ with rational c, then other non-zero (rational) solutions have the form $g = c'/(c + \mathrm{const})$.

Corollary 3.5. Let the Miura ${}^t\mathfrak{g}$-oper $D = \partial + I + V$ be associated with weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and the tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$. Then D is deformable in all directions from 1 to r if and only if the tuple $\boldsymbol{y}$ is fertile.

Corollary 3.6. Let the Miura ${}^t\mathfrak{g}$-oper $D = \partial + I + V$ be associated with weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and the tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$. Let the tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$ be generic in the sense of Section 2.3. Then D is deformable in all directions from 1 to r if and only if the tuple $\boldsymbol{y}$ represents a critical point of the master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$.

Let the Miura ${}^t\mathfrak{g}$-oper $D = \partial + I + V$ be associated with weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and the tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$. Let the tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$ represent a critical point of the master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$. Let $\mathrm{Om_D}^0$ be the variety of all Miura opers which can be obtained from D by a sequence of deformations in directions $i_1, \ldots, i_N$ where N is any positive integer and all i_j lie in $\{1, \ldots, r\}$.

Corollary 3.7. For a simple Lie algebra $\mathfrak{g}$ the variety $\mathrm{Om_D}^0$ is isomorphic to the population of critical points originated at $\boldsymbol{y}$.

4 Miura opers and flag varieties

In this section we assume that $\mathfrak{g}$ and ${}^t\mathfrak{g}$ are simple Lie algebras although most of considerations can be extended to Kac-Moody algebras.

Let ${}^tG$ be the complex simply connected Lie group with Lie algebra ${}^t\mathfrak{g}$. Let ${}^tB_\pm$, ${}^tN_\pm$, ${}^tH$ be the subgroups with Lie algebras ${}^t\mathfrak{b}_\pm, {}^t\mathfrak{n}_\pm, {}^t\mathfrak{h}$, respectively.

4.1 Triviality of the monodromy

Let $D = \partial + I + V$ be a Miura ${}^t\mathfrak{g}$-oper. Let $\mathbb{P}^1$ be the complex projective line. Consider D as a ${}^tG$-connection ∇_D on the trivial principal ${}^tG$-bundle $p : {}^tG \times \mathbb{P}^1 \to \mathbb{P}^1$. The connection has singularities at the set Sing $\subset \mathbb{P}^1$ where the function V has poles. Choose a regular point $x_0 \in \mathbb{P}^1 - \infty$ of the connection. Parallel translations with respect to the connection define the monodromy representation $\pi(\mathbb{P}^1 - \mathrm{Sing}) \to {}^tG$. Its image is called *the monodromy group.*

Theorem 4.1. Assume that the Miura ${}^t\mathfrak{g}$-oper D is associated with weights $\mathbf{\Lambda}$, numbers $\boldsymbol{z}$, and a tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ as in Section 3. Assume that the tuple $\boldsymbol{y} = (y_1, \dots, y_r)$ is generic in the sense of Section 2.3. Assume that the Miura oper D is deformable in all directions from 1 to r. Then the monodromy group of ∇_D belongs to the center of ${}^tG$.

Proof 4.2. It is known that the intersection of all of the Borel subgroups in ${}^tG$ is the center of ${}^tG$, see [3, 9]. We show that the monodromy of ∇_D lies in the intersection of all of the Borel subgroups.

Let id $\in {}^tG$ be the identity element. Let $\bar{Y}(x) \in {}^tG$ be the (possibly multi-valued) solution of the equation $DY = (\partial + I + V)Y = 0$ such that $Y(x_0) = \mathrm{id}$. Since D is a Miura oper we have for any regular x the equality of sets $\bar{Y}(x)\,{}^tB_- = {}^tB_-$. Hence if $m \in {}^tG$ is an element of the monodromy group of ∇_D, then $m\,{}^tB_- = {}^tB_-$ and hence $m \in {}^tB_-$.

Let $i \in \{1, \dots, r\}$ and let $g_i \in M(\mathbb{C})$ be a solution of the Ricatti equation (8). Assume that g_i is regular at x_0. Then the ${}^tG$-valued function

$$e^{g_i(x)E_i}\ \bar{Y}(x)\ e^{-g_i(x_0)E_i}$$

is the solution of the equation $(e^{\mathrm{ad}\,(g_i(x)E_i)} \cdot D)\,Y = 0$ such that $Y(x_0) = \mathrm{id}$. Since $e^{\mathrm{ad}\,(g_i(x)E_i)} \cdot D$ is a Miura oper we have an equality of sets $Y^{[i]}_{g_i}(x)\,{}^tB_- = {}^tB_-$ for any x at which $e^{\mathrm{ad}\,(g_i(x)E_i)} \cdot D$ is regular. Hence for any element m of the monodromy group of ∇_D we have

$$e^{g_i(x_0)E_i}\ m\ e^{-g_i(x_0)E_i}\ {}^tB_- \ \subset\ {}^tB_-$$

or $m \in\ e^{-g_i(x_0)E_i}\ {}^tB_-\ e^{g_i(x_0)E_i}$.

Now consider the Miura oper

$$e^{\mathrm{ad}\,(g_i(x)E_i)} \cdot D = \partial + I + V_{i;g_i}$$

where $V_{i;g_i}$ is the ${}^t\mathfrak{h}$-part of $e^{\mathrm{ad}\,(g_i(x)E_i)} \cdot D$. Let $j \in \{1, \dots, r\}$ and let $g_{i,j;g_i} \in M(\mathbb{C})$ be a solution of the j-th Ricatti equation

$$g' + \langle {}^t\alpha_i, V_{i;g_i} \rangle g + g^2 = 0$$

associated with the Miura oper $e^{\mathrm{ad}\,(g_i(x)E_i)} \cdot D$. Assume that $g_{i,j;g_i}$ is regular at x_0. Then the ${}^tG$-valued function

$$e^{g_{i,j;g_i}(x)E_j} e^{g_i(x)E_i}\ \bar{Y}(x)\ e^{-g_i(x_0)E_i} e^{-g_{i,j;g_i}(x_0)E_j}$$

is the solution of the equation $(e^{\mathrm{ad}\,(g_{i,j;g_i}(x)E_j)} e^{\mathrm{ad}\,(g_i(x)E_i)} \cdot D)\, Y = 0$ such that $Y(x_0) = \mathrm{id}$. Repeating the previous argument we conclude that any element m of the monodromy group of ∇_D lies in the Borel subgroup

$$e^{-g_i(x_0)E_i} e^{-g_{i,j;g_i}(x_0)E_j}\ {}^tB_-\ e^{g_{i,j;g_i}(x_0)E_j} e^{g_i(x_0)E_i}.$$

Every $u \in\ {}^tN_+$ is a product of elements of the form $e^{c_iE_i}$ for $i \in \{1, \ldots, r\}$ and $c_i \in \mathbb{C}$. Every c_i can be taken as the initial condition for a solution of the suitable i-th Ricatti equation. Therefore the iteration of the previous reason shows that every element of the monodromy group of ∇_D lies in every Borel subgroup of the form $u^{-1}({}^tB_-)u$, $u \in\ {}^tN_+$. The Borel subgroups in ${}^tG$ of the form $u^{-1}({}^tB_-)u$, $u \in\ {}^tN_+$, form an open dense subset in the flag variety of all of the Borel subgroups. Hence the monodromy lies in the intersection of all of the Borel subgroups.

4.2 Gauge equivalent Miura opers

As in Section 4.1, let D be the Miura ${}^tG$-oper associated with weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and a tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$. We assume that the tuple $\boldsymbol{y} = (y_1, \ldots, y_r)$ is generic in the sense of Section 2.3 and the Miura oper D is deformable in all directions from 1 to r.

Consider the variety $\mathrm{Om_D}$ of all Miura opers gauge equivalent to D. If $D' \in \mathrm{Om_D}$, then there exists a rational N_+-valued function v on $\mathbb{P}^1$ such that $D' = v\, D\, v^{-1}$. In that case we denote D' by D^v.

Let $\mathrm{Om_D}^0 \subseteq \mathrm{Om_D}$ be the subvariety of all Miura opers which can be obtained from D by a sequence of deformations in directions $i_1, \ldots, i_N$ where N is a non-negative integer and all i_j lie in $\{1, \ldots, r\}$. By Corollary 3.7 the subvariety $\mathrm{Om_D}^0$ is isomorphic to the population of critical points originated at $\boldsymbol{y}$.

The connection ∇_D is regular at $x_0 \in \mathbb{P}^1$ if x_0 does not lie in $\{z_1, \ldots, z_n, \infty\}$ and x_0 is not a root of some of polynomials $y_1, \ldots, y_r$.

Consider the trivial bundle $p' : ({}^tG/{}^tB_-) \times \mathbb{P}^1 \to \mathbb{P}^1$ associated with the bundle p. The fiber of p' is the flag variety ${}^tG/{}^tB_-$. The connection ∇_D induces a connection ∇'_D on p'. The monodromy of ∇'_D is trivial by Theorem 4.1. Thus the variety Γ of global horizontal sections of ∇'_D is identified with the fiber $(p')^{-1}(x_0)$ over any x_0 which is a regular point of the connection. Thus Γ is isomorphic to ${}^tG/{}^tB_-$.

Any ${}^tG$-valued rational function v defines a section

$$S_v\ :\ x\ \mapsto\ v(x)^{-1}\ {}^tB_-\ \times\ x \tag{13}$$

of p' over the set of regular points of v. The section S_v is also well defined over the poles of v since ${}^tG/{}^tB_-$ is a projective variety.

If $D^v \in \mathrm{Om_D}$, then the section S_v is horizontal with respect to ∇'_D, cf. the proof of Theorem 4.1. Thus we have a map

$$S : \mathrm{Om_D} \to \Gamma, \qquad D^v \mapsto S_v \ .$$

Theorem 4.3. The map $S : \mathrm{Om_D} \to \Gamma$ is an isomorphism and $\mathrm{Om_D}^0 = \mathrm{Om_D}$.

Proof 4.4. Let $D^{v_1}, D^{v_2} \in \mathrm{Om_D}$. Assume that the images of D^{v_1} and D^{v_2} under the map S coincide. Assume that v_1, v_2, D are regular at $x_0 \in \mathbb{P}^1$. The equality $S_{v_1}(x_0) = S_{v_2}(x_0)$ means that $v_1(x_0)^{-1}\,{}^tB_- = v_2(x_0)^{-1}\,{}^tB_-$. Then $v_1(x_0) = v_2(x_0)$. Hence $v_1 = v_2$ and $D^{v_1} = D^{v_2}$. That proves the injectivity of S.

Let x_0 be a regular point of D in $\mathbb{P}^1 - \infty$. For any $u \in {}^tN_+$ there exists a rational ${}^tN_+$-valued function v such that $v(x_0) = u$ and $D^v \in \mathrm{Om_D}$ and D^v is obtained from D by a sequence of deformations in some directions $i_1, \dots, i_N$, see the proof of Theorem 4.1. Thus the set

$$Im(x_0) \ = \ \{S_v(x_0) \in \ ({}^tG/{}^tB_-) \times x_0 \mid D^v \in \mathrm{Om_D}^0\}$$

contains the set $(({}^tN_+\,{}^tB_-)/{}^tB_-) \times x_0 \ \subset \ ({}^tG/{}^tB_-) \times x_0$. The set $Im(x_0)$ is the image with respect to S of a population of critical points. Hence it is closed as the image of a closed variety. On the other hand the set $(({}^tN_+\,{}^tB_-)/{}^tB_-) \times x_0$ is dense in $({}^tG/{}^tB_-) \times x_0$. Hence $Im(x_0) = ({}^tG/{}^tB_-) \times x_0$ and $S(\mathrm{Om_D}^0) = \Gamma$. Therefore $\mathrm{Om_D}^0 = \mathrm{Om_D}$ since the map S is injective.

4.3 Remarks on the isomorphism

Let $\mathfrak{g}$ be a simple Lie algebra. Let $P_{\boldsymbol{y}_0}$ be the population of critical points originated at a tuple $\boldsymbol{y}^0$. We assume that the tuple $\boldsymbol{y}^0$ is generic in the sense of Section 2.3. Theorem 4.3 says that the population $P_{\boldsymbol{y}^0}$ is isomorphic to the flag variety ${}^tG/{}^tB_-$. The isomorphism is constructed in three steps. If $\boldsymbol{y}' \in P_{\boldsymbol{y}}$ is a point of the population, then one assigns to it the associated Miura oper $D_{\boldsymbol{y}'}$ as in Section 3, see also Lemma 6.1. By Theorem 3.3 we have $D_{\boldsymbol{y}'} \ = \ v\ D_{\boldsymbol{y}^0}\ v^{-1}$ for a suitable rational function $v : \mathbb{P}^1 \to \ {}^tN_+$. To the Miura oper D^v one assigns the section $S_v \in \Gamma$ by formula (13). Then one chooses a point $x_0 \in \mathbb{C}$, regular with respect to the connection $\nabla_{D_{\boldsymbol{y}^0}}$, and assigns to a section S^v its value $S^v(x_0) \in ({}^tG/{}^tB_-) \times x_0$. The resulting composition

$$\phi_{\boldsymbol{y}^0, x_0} \ : \ P_{\boldsymbol{y}} \ \to \ {}^tG/{}^tB_-$$

is an isomorphism according to Theorem 4.3.

Lemma 4.5. If $x_0, x_1 \in \mathbb{C}$ are points regular with respect to $\nabla_{D_{\boldsymbol{y}^0}}$, then there exists an element $g \in {}^tB_-$ such that $\ \phi_{\boldsymbol{y}^0, x_1} \ = \ g\ \phi_{\boldsymbol{y}^0, x_0}$.

Proof 4.6. Let Y be the ${}^tG$-valued solution of the equation $D_{\boldsymbol{y}^0} Y = 0$ such that $Y(x_0) =$ id. Then $Y(x) \in {}^tB_-$ for all x. If $D^v \in \mathrm{Om}_{\mathrm{D}_{\boldsymbol{y}^0}}$, then S^v is a horizontal section of

$\nabla'_{D_{\boldsymbol{y}^0}}$. Thus it has the form $x \mapsto (Y(x)\, u\ {}^tB_-) \times x$ for a suitable element $u \in {}^tG$. Hence $\phi_{\boldsymbol{y}^0,x_0}(\boldsymbol{y}') = Y(x_0)\, u\ {}^tB_-$ and $\phi_{\boldsymbol{y}^0,x_1}(\boldsymbol{y}') = Y(x_1)\, u\ {}^tB_-$. We conclude that $\phi_{\boldsymbol{y}^0,x_1} = Y(x_1)Y(x_0)^{-1}\, \phi_{\boldsymbol{y}^0,x_0}$.

Let $\boldsymbol{y}^1$ be a point of $P_{\boldsymbol{y}^0}$. Let $P_{\boldsymbol{y}^1}$ be the population originated at $\boldsymbol{y}^1$. We have $P_{\boldsymbol{y}^0} = P_{\boldsymbol{y}^1}$.

Lemma 4.7. Let $x_0 \in \mathbb{C}$ be regular with respect to both connections $\nabla_{D_{\boldsymbol{y}^0}}$ and $\nabla_{D_{\boldsymbol{y}^1}}$. Then there exists an element $g \in {}^tB_+$ such that $\phi_{\boldsymbol{y}^1,x_0} = g\ \phi_{\boldsymbol{y}^0,x_0}$.

Proof 4.8. We have $D_{\boldsymbol{y}^1} = w\ D_{\boldsymbol{y}^0}\ w^{-1}$ for a suitable rational function $w : \mathbb{P}^1 \to {}^tN_+$. If $Y_0(x)$ is the ${}^tG$-valued solution of the equation $D_{\boldsymbol{y}^0}Y = 0$ such that $Y_0(x_0) = \mathrm{id}$, then $Y_1(x) = w(x)\, Y(x)\, w(x_0)^{-1}$ is the ${}^tG$-valued solution of the equation $D_{\boldsymbol{y}^1}Y = 0$ such that $Y_1(x_0) = \mathrm{id}$.

Let $\boldsymbol{y}' \in P_{\boldsymbol{y}^0}$ and $D_{\boldsymbol{y}'} = v\ D_{\boldsymbol{y}^0}\ v^{-1}$ for a suitable rational function $v : \mathbb{P}^1 \to {}^tN_+$. Then $D_{\boldsymbol{y}'} = vw^{-1}\ D_{\boldsymbol{y}^1}\ wv^{-1}$. Hence $\phi_{\boldsymbol{y}^0,x_0}(\boldsymbol{y}') = v(x_0)^{-1}\, {}^tB_-$ and $\phi_{\boldsymbol{y}^1,x_0}(\boldsymbol{y}') = w(x_0)v(x_0)^{-1}\, {}^tB_-$. Therefore, $\phi_{\boldsymbol{y}^1,x_0} = w(x_0)\, \phi_{\boldsymbol{y}^0,x_0}$.

5 Bruhat cells

5.1 Properties of Bruhat cells

Let $\mathfrak{g}$ be a simple Lie algebra. For an element w of the Weyl group W, the set

$$B_w = {}^tB_-\, w\, {}^tB_- \subset {}^tG/{}^tB_-$$

is called *the Bruhat cell* associated to w. The Bruhat cells form a cell decomposition of the flag variety ${}^tG/{}^tB_-$.

For $w \in W$ denote $l(w)$ the length of w. We have $\dim B_w = l(w)$.

Let $s_1, \dots, s_r \in W$ be the generating reflections of the Weyl group.

For $v \in {}^tG/{}^tB_-$ and $i \in \{1, \dots, r\}$ consider the rational curve

$$\mathbb{C} \to {}^tG/{}^tB_-, \qquad c \mapsto e^{cE_i}\, v\ .$$

The limit of $e^{cE_i}\, v$ is well defined as $c \to \infty$, since ${}^tG/{}^tB_-$ is a projective variety.

We need the following standard property of Bruhat cells.

Lemma 5.1. Let $s_i, w \in W$ be such that $l(s_iw) = l(w) + 1$. Then

$$B_{s_iw} = \{\ e^{cE_i}\, v \mid v \in B_w,\ c \in \{\mathbb{P}^1 - 0\}\ \}\ .$$

□

Corollary 5.2. Let $w = s_{i_1} \cdots s_{i_k}$ be a reduced decomposition of $w \in W$. Then

$$B_w = \{ \lim_{c_1 \to c_1^0} \dots \lim_{c_k \to c_k^0} e^{c_1 E_{i_1}} \cdots e^{c_k E_{i_k}}\, {}^tB_- \ \in\ {}^tG/{}^tB_- \mid c_1^0, \dots, c_k^0 \in \{\mathbb{P}^1 - 0\}\}.$$

Introduce the map

$$f_{i_1,\dots,i_k} : (\mathbb{C}-0)^k \to B_{s_{i_1}\cdots s_{i_k}}, \qquad (c_1,\dots,c_k) \mapsto e^{c_1 E_{i_1}} \cdots e^{c_k E_{i_k}}\ {}^tB_- \ .$$

5.2 Populations and Bruhat cells

Let P be a population of critical points associated with weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$. Let $T_1,\dots,T_r$ be the polynomials defined by (4). Let $\boldsymbol{y}^0 = (y_1^0,\dots,y_r^0) \in P$ with $l_i = \deg y_i^0$ for $i \in \{1,\dots,r\}$. Assume that the weight at infinity of $\boldsymbol{y}^0$,

$$\Lambda_\infty \ = \ \sum_{i=1}^n \Lambda_i \ - \ \sum_{i=1}^r \ l_i\, \alpha_i \ ,$$

is integral dominant, see Section 2. Such $\boldsymbol{y}^0$ exists according to [11]. For $w \in W$ consider the weight $w \cdot \Lambda_\infty$, where $w\cdot$ is the shifted action of w on $\mathfrak{h}^*$. Write

$$w \cdot \Lambda_\infty \ = \ \sum_{i=1}^n \Lambda_i \ - \ \sum_{i=1}^r \ l_i^w\, \alpha_i.$$

Set

$$P_w \ = \ \{ \ \boldsymbol{y} = (y_1,\dots,y_r) \ \in \ P \ | \ \deg y_i = l_i^w, \ i = 1,\dots,r \ \} \ .$$

Consider the trivial bundle $p' : ({}^tG/{}^tB_-) \times \mathbb{P}^1 \to \mathbb{P}^{\mathbf{1}}$ with connection $\nabla'_{D_{\boldsymbol{y}^0}}$. Consider the Bruhat cell decomposition of fibers of p'.

Let $x_0 \in \mathbb{C}$ be such that $T_i(x_0) \neq 0$ and $y_i^0(x_0) \neq 0$ for $i = 1,\dots,r$. The point $x_0 \in \mathbb{C}$ is a regular point of the connection $\nabla'_{D_{\boldsymbol{y}^0}}$. Let

$$\phi_{\boldsymbol{y}^0, x_0} \ : \ P \ \to \ {}^tG/{}^tB_-$$

be the isomorphism defined in Section 4.3.

Theorem 5.3. For $w \in W$ we have

$$\phi_{\boldsymbol{y}^0,\, x_0}(P_w) \ = \ B_{w^{-1}} \ .$$

Corollary 5.4. Let $\Lambda_1,\dots,\Lambda_n,\Lambda_\infty$ be integral dominant $\mathfrak{g}$-weights. Let $z_1,\dots,z_n$ be distinct complex numbers. Let $w \in W$. Consider the master function $\Phi(\boldsymbol{t};\boldsymbol{z},\boldsymbol{\Lambda}, w\cdot\Lambda_\infty)$. Let K be a connected component of the critical set of the master function. For each $\boldsymbol{t} \in K$ consider the tuple $\boldsymbol{y_t} \in (\mathbb{C}[x])^r$ of monic polynomials representing the critical point $\boldsymbol{t}$. Then the closure of the set $\{ \ \boldsymbol{y_t} \mid \boldsymbol{t} \in K \ \}$ is an $l(w)$-dimensional cell.

5.3 Proof of Theorem 5.3

Lemma 5.5. For $w \in W$, the subset $B_w \times \mathbb{P}^1 \ \subset \ ({}^tG/{}^tB_-) \times \mathbb{P}^1$ is invariant with respect to the connection $\nabla'_{D_{\boldsymbol{y}^0}}$.

Proof 5.6. Let Y be the ${}^tG$-valued solution of the equation $D_{\boldsymbol{y}^0} Y = 0$ such that $Y(x_0) = \mathrm{id}$. Then $Y(x) \in {}^tB_-$ for all x. The horizontal sections of $\nabla'_{D_{\boldsymbol{y}^0}}$ have the form $x \mapsto (Y(x)\, u\, {}^tB_-) \times x$ for a suitable element $u \in {}^tG$. Hence if $u\, {}^tB_- \in B_w$, then $Y(x) u\, {}^tB_- \in B_w$ for all x.

Let $w = s_{i_k} \cdots s_{i_1}$ be a reduced decomposition of $w \in W$. For $b = 1, \dots, k$ set

$$(s_{i_b} \cdots s_{i_1}) \cdot \Lambda_\infty \;=\; \sum_{i=1}^{n} \Lambda_i \;-\; \sum_{i=1}^{r} l_i^b\, \alpha_i .$$

From [2] it follows that $l_{i_1}^1 > l_{i_1}$ and $l_{i_b}^b > l_{i_b}^{b-1}$ for $b = 2, \dots, k$.

For $b = 1, \dots, k$ define by induction on b a family of tuples of polynomials depending on complex parameters $c_1, \dots, c_b$. Namely, let $\tilde y_{i_1}$ be a polynomial satisfying equation

$$W(\, y_{i_1}^0, \tilde y_{i_1}\,) \;=\; T_{i_1} \prod_{j,\ j \neq i_1} (\, y_j^0\,)^{-a_{i_1,j}} \;.$$

We fix $\tilde y_{i_1}$ assuming that the coefficient of $x^{l_{i_1}}$ in $\tilde y_{i_1}$ is equal to zero. Set $\boldsymbol{y}^{1;\, c_1} = (y_1^{1;\, c_1}, \dots, y_r^{1;\, c_1})$, where

$$y_{i_1}^{1;\, c_1}(x) \;=\; \tilde y_{i_1}(x) \;+\; c_1\, y_{i_1}^0(x) \qquad \text{and } y_j^{1;\, c_1}(x) \;=\; y_j^0(x) \text{ for } j \neq i_1 \;.$$

Assume that the family $\boldsymbol{y}^{b-1;\, c_1, \dots, c_{b-1}}$ is already defined. Let $\tilde y_{i_b}^{\,b-1;\, c_1, \dots, c_{b-1}}$ be a polynomial satisfying equation

$$W(\, y_{i_b}^{b-1;\, c_1, \dots, c_{b-1}}, \tilde y_{i_b}^{\,b-1;\, c_1, \dots, c_{b-1}}\,) \;=\; T_{i_b} \prod_{j,\ j \neq i_b} (\, y_j^{b-1;\, c_1, \dots, c_{b-1}}\,)^{-a_{i_b,j}} \;.$$

We fix $\tilde y_{i_d}^{\,d-1;\, c_1, \dots, c_{d-1}}$ assuming that the coefficient of $x^{l_{i_d}^{d-1}}$ in $\tilde y_{i_d}^{\,d-1;\, c_1, \dots, c_{d-1}}$ is equal to zero. Set $\boldsymbol{y}^{b;\, c_1, \dots, c_b} = (y_1^{b;\, c_1, \dots, c_b}, \dots, y_r^{b;\, c_1, \dots, c_b})$, where

$$y_{i_b}^{b;\, c_1, \dots, c_b}(x) \;=\; \tilde y_{i_b}^{\,b-1;\, c_1, \dots, c_b}(x) \;+\; c_b\, y_{i_b}^{b-1;\, c_1, \dots, c_{b-1}}(x)$$

and

$$y_j^{b;\, c_1, \dots, c_b}(x) \;=\; y_j^{b-1;\, c_1, \dots, c_{b-1}}(x) \;\text{ for }\; j \neq i_b \;.$$

The b-th family is obtained from the $(b-1)$-st family by the generation procedure in the i_b-th direction, see Section 2.5. For any $c_1, \dots, c_k$ the tuple $\boldsymbol{y}^{k;\, c_1, \dots, c_k}$ lies in P.

For any $c_1, \dots, c_k$ and any $i \in \{1, \dots, r\}$, we have

$$\deg\; y_i^{k;\, c_1, \dots, c_k}(x) \;=\; l_i^w \;.$$

Set

$$P^{[i_1, \dots, i_k]} \;=\; \{\; \boldsymbol{y}^{k;\, c_1, \dots, c_k} \mid c_1, \dots, c_k \in \mathbb{C} \;\} \;.$$

Proposition 5.7. We have

$$\phi_{\boldsymbol{y}^0,x_0}(P^{[i_1,\dots,i_k]}) \;=\; B_{w^{-1}}\;.$$

Proof of Proposition 5.7. Let $D_{\boldsymbol{y}^{k;\,c_1,\dots,c_k}}$ be the Miura oper associated with the tuple $\boldsymbol{y}^{k;\,c_1,\dots,c_k}$, then

$$D_{\boldsymbol{y}^{k;\,c_1,\dots,c_k}} = \exp\left(\operatorname{ad}\log'\left(\frac{y_{i_k}^{k;\,c_1,\dots,c_k}}{y_{i_k}^{k-1;\,c_1,\dots,c_{k-1}}}\right)E_{i_k}\right)\dots$$
$$\exp\left(\operatorname{ad}\log'\left(\frac{y_{i_2}^{2;\,c_1,c_2}}{y_{i_2}^{1;\,c_1}}\right)E_{i_2}\right)\exp\left(\operatorname{ad}\log'\left(\frac{y_{i_1}^{1;\,c_1}}{y_{i_1}^{0}}\right)E_{i_1}\right)\;\cdot\;D_{\boldsymbol{y}^0}\;=$$
$$\exp\left(-\operatorname{ad}\left(\frac{T_{i_k}\;\prod_{j,\;j\neq i_k}(y_j^{k-1;\,c_1,\dots,c_{k-1}})^{-a_{i_k,j}}}{y_{i_k}^{k;\,c_1,\dots,c_k}\;y_{i_k}^{k-1;\,c_1,\dots,c_{k-1}}}\right)E_{i_k}\right)\dots$$
$$\exp\left(-\operatorname{ad}\left(\frac{T_{i_2}\;\prod_{j,\;j\neq i_2}(y_j^{1;\,c_1})^{-a_{i_2,j}}}{y_{i_2}^{2;\,c_1,c_2}y_{i_2}^{1;\,c_1}}\right)E_{i_2}\right)\exp\left(-\operatorname{ad}\left(\frac{T_{i_1}\prod_{j,\;j\neq i_1}(y_j^0)^{-a_{i_1,j}}}{y_{i_1}^{1;\,c_1}y_{i_1}^0}\right)E_{i_1}\right)\cdot D_{\boldsymbol{y}^0}, \tag{14}$$

see Theorem 3.3.

Introduce the rational map

$$g:\mathbb{C}^{k+1}\to\mathbb{C}^k,\quad (x;c_1,\dots,c_k)\mapsto(g_1(x;c_1),\dots,g_k(x;c_1,\dots,c_k))$$

where

$$g_1(x;c_1)=\frac{T_{i_1}(x)\prod_{j,\;j\neq i_1}(\,y_j^0(x)\,)^{-a_{i_1,j}}}{y_{i_1}^{1;\,c_1}(x)\;y_{i_1}^0(x)},$$

$$g_b(x;c_1,\dots,c_b)=\frac{T_{i_b}(x)\;\prod_{j,\;j\neq i_b}(\,y_j^{b-1;\,c_1,\dots,c_{b-1}}(x)\,)^{-a_{i_b,j}}}{y_{i_b}^{b;\,c_1,\dots,c_b}(x)\;y_{i_b}^{b-1;\,c_1,\dots,c_{b-1}}(x)}$$

for $b=2,\dots,k$. From (14) it follows that the tuple $\boldsymbol{y}^{k;\,c_1,\dots,c_k}$ corresponds to the rational section

$$S_{(c_1,\dots,c_k)}\;:\;x\;\mapsto\;f_{i_1,\dots,i_k}(g(x,c_1,\dots,c_k))\times x$$

of the bundle p'. This section is horizontal with respect to the connection $\nabla'_{D_{\boldsymbol{y}^0}}$, and we have

$$\phi_{\boldsymbol{y}^0,\,x}(\boldsymbol{y}^{k;\,c_1,\dots,c_k})\;=\;f_{i_1,\dots,i_k}(g(x,c_1,\dots,c_k))\;.$$

This means that

$$\phi_{\boldsymbol{y}^0,x_0}(P^{[i_1,\dots,i_k]}) \ \subset \ B_{w^{-1}} \ .$$

It remains to show that every point in $B_{w^{-1}}$ is the limit of points of $\phi_{\boldsymbol{y}^0,x_0}(P^{[i_1,\dots,i_k]})$, where the limit is taken in the sense of the limit in Corollary 5.2, but that statement follows from

Lemma 5.8. Let $x_0 \in \mathbb{C}$ be such that $T_i(x_0) \neq 0$ and $y_i^0(x_0) \neq 0$ for $i = 1, \dots, r$. Then for any $(c_1^1, \dots, c_k^1) \in (\mathbb{C} - 0)^k$ there exists a unique $(c_1^2, \dots, c_k^2) \in \mathbb{C}^k$ such that

$$(c_1^1, \dots, c_k^1) = g(x_0; c_1^2, \dots, c_k^2) \ .$$

□

The proposition is proved. □

Theorem 5.3 is a direct corollary of Proposition 5.7.

6 Solutions of differential equations

As we observed earlier, the Miura opers, associated with a population of critical points, help to study the structure of the population. In addition to that it turns out that for a Miura oper D associated with a critical point of a population, all solutions of the differential equation $DY = 0$ with values in the corresponding group can be written explicitly in terms of critical points composing the population.

First we give formulas for solutions of the equation $DY = 0$ for opers associated with Lie algebras of types A_r, B_r, and then consider more general formulas for solutions which do not use the structure of the Lie algebra.

In this section $\mathfrak{g} = \mathfrak{g}\,(A)$ is a simple Lie algebra with Cartan matrix $A = (a_{i,j})$.

6.1 Elimination of polynomials T_i

Let $B = (b_{i,j})$ be the matrix inverse to A.

Let D be the Miura ${}^tG$-oper associated with weights $\boldsymbol{\Lambda}$, numbers $\boldsymbol{z}$, and a tuple $\boldsymbol{y} = (y_1, \dots, y_r)$. Introduce polynomials $T_1(x), \dots, T_r(x)$ by formulas (4). Introduce a tuple of functions $\bar{\boldsymbol{y}} = (\bar{y}_1, \dots, \bar{y}_r)$ by

$$\bar{y}_i \ = \ y_i \prod_{l=1}^{r} T_l^{-b_{i,l}} \ . \tag{15}$$

Lemma 6.1. We have $D = \partial + I + V$ where

$$V \ = \ \sum_{j=1}^{r} \ \log'(\bar{y}_j) \ H_j \ . \tag{16}$$

Proof 6.2. If V is given by (16), then

$$\langle\ {}^t\alpha_i, V\rangle = \sum_{j=1}^{r} \log'(\bar{y}_j)\ \langle\ {}^t\alpha_i,\ {}^t\alpha_j^\vee\rangle = \sum_{j=1}^{r} \log'(\ \bar{y}_j^{\ a_{i,j}}\ )\ =\ \log'(\ T_i^{-1}\ \prod_{j=1}^{r} y_j^{\ a_{i,j}}\ )\ .$$

Assume that $\boldsymbol{y}$ represents a critical point of the master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$.

Let $\boldsymbol{i} = [i_1, \dots, i_k],\ i_j \in \{1, \dots, r\}$, be a sequence of natural numbers. Let $\boldsymbol{y}^{[i_1]} = (y_1^{[i_1]}, \dots, y_r^{[i_1]}),\ \boldsymbol{y}^{[i_1,i_2]} = (y_1^{[i_1,i_2]}, \dots, y_r^{[i_1,i_2]}),\ \dots,\ \boldsymbol{y}^{[i_1,\dots,i_k]} = (y_1^{[i_1,\dots,i_k]}, \dots, y_r^{[i_1,\dots,i_k]})$ be a sequence of tuples associated with the critical point $\boldsymbol{y}$ and the sequence of indices $\boldsymbol{i}$, see Section 2.5. Introduce functions $\bar{y}_i^{[i_1,\dots,i_l]}$ by

$$\bar{y}_i^{[i_1,\dots,i_l]}\ =\ y_i^{[i_1,\dots,i_l]}\ \prod_{l=1}^{r} T_l^{-b_{i,l}}\ . \tag{17}$$

Lemma 6.3. We have

$$W(\bar{y}_{i_1}, \bar{y}_{i_1}^{[i_1]})\ =\ \prod_{j,\ j\neq i_1} \bar{y}_j^{\,-a_{i_1,j}} \tag{18}$$

and $\bar{y}_j^{[i_1]} = \bar{y}_j$ for $j \neq i_1$; for $l = 2, \dots, k$, we have

$$W(\bar{y}_{i_l}^{[i_1,\dots,i_{l-1}]}, \bar{y}_{i_l}^{[i_1,\dots,i_l]})\ =\ \prod_{j,\ j\neq i_l} (\bar{y}_j^{[i_1,\dots,i_{l-1}]})^{-a_{i_l,j}} \tag{19}$$

and $\bar{y}_j^{[i_1,\dots,i_l]} = \bar{y}_j^{[i_1,\dots,i_{l-1}]}$ for $j \neq i_l$. □

The sequence of tuples $\bar{\boldsymbol{y}}^{[i_1]} = (\bar{y}_1^{[i_1]}, \dots, \bar{y}_r^{[i_1]}),\ \bar{\boldsymbol{y}}^{[i_1,i_2]} = (\bar{y}_1^{[i_1,i_2]}, \dots, \bar{y}_r^{[i_1,i_2]}),\ \dots,\ \bar{\boldsymbol{y}}^{[i_1,i_2,\dots,i_k]} = (\bar{y}_1^{[i_1,i_2,\dots,i_k]}, \dots, \bar{y}_r^{[i_1,i_2,\dots,i_k]})$ will be called *the sequence of reduced tuples associated with the critical point $\boldsymbol{y}$ and the sequence of indices $\boldsymbol{i}$.*

The sequence of functions $\bar{y}_{i_1}^{[i_1]}, \bar{y}_{i_2}^{[i_1,i_2]}, \dots, \bar{y}_{i_k}^{[i_1,i_2,\dots,i_k]}$, each defined up to multiplication by a non-zero number, will be called *the reduced diagonal sequence of functions associated with the critical point $\boldsymbol{y}$ and the sequence of indices $\boldsymbol{i}$.*

The reduced diagonal sequence of functions determine the sequence of tuples $\bar{\boldsymbol{y}}^{[i_1]}$, $\bar{\boldsymbol{y}}^{[i_1,i_2]}, \dots, \bar{\boldsymbol{y}}^{[i_1,i_2,\dots,i_k]}$ uniquely.

In the next sections we use the following lemma.

Lemma 6.4.

- Consider the product $\prod_{j=1}^{r} \bar{y}_j^{\ -H_j}$ as a function of x with values in the group ${}^tG$. Then

$$\prod_{j=1}^{r} \bar{y}_j^{\ -H_j}\ =\ \prod_{j=1}^{r}\ y_j^{\ -H_j} T_j^{\ w_j}\ ,$$

where $w_1, \dots, w_r \in\ {}^t\mathfrak{h}$ are the fundamental coweights, i.e. $\langle\ {}^t\alpha_i, w_j\rangle = \delta_{i,j}$.

- Let $D = \partial + I + V$ be the Miura oper with V given by formula (16). Define

$$\bar{D} \; = \; \partial \; + \; \sum_{j=1}^{r} \, (\, \prod_{l=1}^{r} \, \bar{y}_l^{\;-a_{j,l}} \;) \; F_j \; .$$

Then

$$D \; (\, \prod_{j=1}^{r} \bar{y}_j^{\;-H_j}) \; = \; (\, \prod_{j=1}^{r} \bar{y}_j^{\;-H_j}) \; \bar{D} \; .$$

□

6.2 The A_r critical points and A_r opers

The Lie algebra sl_{r+1} is of A_r-type. The Langlands dual to sl_{r+1} is sl_{r+1}. Let $F_1, \ldots, F_r$, $H_1, \ldots, H_r, E_1, \ldots, E_r$ be the Chevalley generators of sl_{r+1}. Let $w_1, \ldots, w_r$ be the fundamental coweights of sl_{r+1}.

We start with two examples.

Let $\mathfrak{g} = sl_2$. Let $\boldsymbol{y} = (y_1)$ represent a critical point of the sl_2 master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$. Introduce the function $\bar{y}_1 = y_1 T_1^{\;-1/2}$, see (15). Then the Miura oper associated with $\boldsymbol{y}$ has the form

$$D = \partial + F_1 + \log'(\bar{y}_1) H_1 \; .$$

Let $\bar{y}_1^{[1]}$ be the reduced diagonal sequence of functions associated with $\boldsymbol{y}$ and the sequence of indices $[1]$, in other words, $W(\bar{y}_1, \bar{y}_1^{[1]}) = 1$. Then

$$Y \; = \; \bar{y}_1^{\;-H_1} \; e^{\frac{\bar{y}_1^{[1]}}{\bar{y}_1} F_1}$$

is a solution of the differential equation $DY = 0$ with values in $\mathrm{SL}\,(2, \mathbb{C})$. Indeed,

$$DY = \bar{y}_1^{\;-H_1} (\partial + \frac{1}{(\bar{y}_1)^2} F_1) \; e^{\frac{\bar{y}_1^{[1]}}{\bar{y}_1} F_1} \; = Y (\partial + (\left(\frac{\bar{y}_1^{[1]}}{\bar{y}_1} \right)' + \frac{1}{(\bar{y}_1)^2}) F_1) \; \mathrm{id} = \; Y \; \partial \; \mathrm{id} = 0.$$

Let $\mathfrak{g} = sl_3$. Let $\boldsymbol{y} = (y_1, y_2)$ represent a critical point of the sl_3 master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$. Introduce the functions $\bar{y}_1 = y_1 T_1^{\;-2/3} T_2^{\;-1/3}$, $\bar{y}_2 = y_2 T_1^{\;-1/3} T_2^{\;-2/3}$, see (15). Then the Miura oper associated with $\boldsymbol{y}$ has the form

$$D = \partial + F_1 + F_2 + \log'(\bar{y}_1) H_1 + \log'(\bar{y}_2) H_2 \; .$$

Let $\bar{y}_1^{[1]}, \bar{y}_2^{[1,2]}$ be the reduced diagonal sequence of functions associated with $\boldsymbol{y}$ and the sequence of indices $[1, 2]$, in other words,

$$W(\bar{y}_1, \bar{y}_1^{[1]}) = \bar{y}_2, \qquad W(\bar{y}_2, \bar{y}_2^{[1,2]}) = \bar{y}_1^{[1]}.$$

Let $\bar y_2^{[2]}$ be the reduced diagonal sequence of functions associated with $\boldsymbol{y}$ and the sequence of indices [2], in other words, $W(\bar y_2, \bar y_2^{[2]}) = \bar y_1$. Then

$$Y \ = \ \bar y_1^{\,-H_1}\ \bar y_2^{\,-H_2}\ e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}\ e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}\ e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}$$

is a solution of the differential equation $DY = 0$ with values in $\mathrm{SL}\,(3,\mathbb{C})$. Indeed, by Lemma 6.4 it suffices to show that

$$\bar Y \ = \ e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}\ e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}\ e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}$$

is a solution of the differential equation $\bar D Y = 0$ where

$$\bar D \ = \ \partial + \frac{\bar y_2}{(\bar y_1)^2}F_1 + \frac{\bar y_1}{(\bar y_2)^2}F_2\ .$$

Indeed,

$$\begin{aligned}
\bar D\bar Y &= e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}(\partial + (\left(\frac{\bar y_1^{[1]}}{\bar y_1}\right)' + \frac{1}{(\bar y_1)^2})F_1 + \frac{\bar y_1^{[1]}}{(\bar y_2)^2}[F_2,F_1] + \frac{\bar y_1}{(\bar y_2)^2}F_2)\ e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}\ e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}\\
&= e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}(\partial + \frac{\bar y_1^{[1]}}{(\bar y_2)^2}[F_2,F_1] + \frac{\bar y_1}{(\bar y_2)^2}F_2)\ e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}\ e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}\\
&= e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}(\partial + (\left(\frac{\bar y_2^{[1,2]}}{\bar y_2}\right)' + \frac{\bar y_1^{[1]}}{(\bar y_2)^2})[F_2,F_1] + \frac{\bar y_1}{(\bar y_2)^2}F_2)\ e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}\\
&= e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}(\partial + \frac{\bar y_1}{(\bar y_2)^2}F_2)\ e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}\\
&= e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}(\partial + (\left(\frac{\bar y_2^{[2]}}{\bar y_2}\right)' + \frac{\bar y_1}{(\bar y_2)^2})F_2)\ \mathrm{id}\\
&= e^{\frac{\bar y_1^{[1]}}{\bar y_1}F_1}e^{\frac{\bar y_2^{[1,2]}}{\bar y_2}[F_2,F_1]}e^{\frac{\bar y_2^{[2]}}{\bar y_2}F_2}\ \partial\ \mathrm{id} = 0\ .
\end{aligned}$$

Now consider the general case. Let $\mathfrak{g} = sl_{r+1}$. Let $\boldsymbol{y} = (y_1, \dots, y_r)$ represent a critical point of the sl_{r+1} master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$. Introduce the functions $\bar y_1, \dots, \bar y_r$ by formula (15), where $B = (b_{i,j})$ is the matrix inverse to the Cartan matrix of sl_{r+1}. Then the Miura oper associated with $\boldsymbol{y}$ has the form

$$D \ = \ \partial \ + \ \sum_{j=1}^{r}\ F_j \ + \ \sum_{j=1}^{r}\ \log'(\bar y_j)\ H_j\ . \tag{20}$$

For $i = 1, \dots, r$, let $y_i^{[i]},\ y_{i+1}^{[i,i+1]},\ \dots,\ y_r^{[i,\dots,r]}$ be the diagonal sequence of polynomials associated with $\boldsymbol{y}$ and the sequence of indices $[i, i+1, \dots, r]$, in other words,

$$W(y_i, y_i^{[i]}) \ = \ T_i\ y_{i-1}\ y_{i+1}, \qquad W(y_{i+1}, y_{i+1}^{[i,i+1]}) \ = \ T_{i+1}\ y_i^{[i]}\bar y_{i+2}, \quad \dots,$$

$$W(y_{r-1}, y_{r-1}^{[i,\dots,r-1]}) \ = \ T_{r-1}\ y_{r-2}^{[i,\dots,r-2]}\ y_r, \qquad W(y_r, y_r^{[i,\dots,r]}) \ = \ T_r\ y_{r-1}^{[i,\dots,r-1]}.$$

Define $r+1$ functions $Y_0, Y_1, \dots, Y_r$ of x with values in $\mathrm{SL}\,(r+1, \mathbb{C})$ by the formulas

$$Y_0 \;=\; \prod_{j=1}^{r} y_j^{\;-H_j} T_j^{\;w_j}, \qquad Y_i \;=\; \prod_{j=i}^{r} e^{\frac{y_j^{[i,\dots,j]}}{y_j}\;[F_j,[F_{j-1},[\dots,[F_{i+1},F_i]\dots]]]}, \qquad \text{for } i>0.$$

Note that inside each product the factors commute.

Theorem 6.5. The product $Y_0 Y_1 \dots Y_r$ is a solution of the differential equation $DY = 0$ with values in $\mathrm{SL}\,(r+1, \mathbb{C})$ where D is given by (20).

Note that if $Y(x)$ is a solution of the equation $DY = 0$ and $g \in \mathrm{SL}\,(r+1, \mathbb{C})$, then $Y(x)g$ is a solution too.

The proof of the theorem is straightforward. One uses Lemma 6.4 and then shows that

$$(\partial \;+\; \sum_{j=i}^{r} \frac{\bar{y}_{j-1}\bar{y}_{j+1}}{\bar{y}_j^{\;2}}\, F_j)\; Y_i \;=\; Y_i\; (\partial \;+\; \sum_{j=i+1}^{r} \frac{\bar{y}_{j-1}\bar{y}_{j+1}}{\bar{y}_j^{\;2}}\, F_j)$$

for $i = 1, \dots, r$. In this formula we set $\bar{y}_0 = \bar{y}_{r+1} = 1$.

6.3 The B_r critical points and C_r opers

Consider the root system of type B_r. Let $\alpha_1, \dots, \alpha_{r-1}$ be the long simple roots and α_r the short one. We have

$$(\alpha_r, \alpha_r) \;=\; 2, \qquad (\alpha_i, \alpha_i) \;=\; 4, \qquad (\alpha_i, \alpha_{i+1}) \;=\; -2, \qquad i = 1, \dots, r-1,$$

and all other scalar products are equal to zero. The root system B_r corresponds to the Lie algebra so_{2r+1}. Let $\mathfrak{h}_B$ be its Cartan subalgebra.

Consider the root system of type C_r. The root system C_r corresponds to the Lie algebra sp_{2r}. Let $F_1, \dots, F_r,\; H_1, \dots, H_r,\; E_1, \dots, E_r$ be its Chevalley generators and $w_1, \dots, w_r$ the fundamental coweights. The symplectic group $\mathrm{Sp}(2r, \mathbb{C})$ is the simply connected group with Lie algebra sp_{2r}.

The Lie algebras so_{2r+1} and sp_{2r} are Langlands dual.

We consider also the root system of type A_{2r-1} with simple roots $\alpha_1^A, \dots, \alpha_{2r-1}^A$. The root system A_{2r-1} corresponds to the Lie algebra sl_{2r}. We denote $\mathfrak{h}_A$ its Cartan subalgebra.

We have a map $\mathfrak{h}_B^* \to \mathfrak{h}_A^*,\; \Lambda \mapsto \Lambda^A$, where Λ^A is defined by

$$\langle \Lambda^A \;,\; (\alpha_i^A)^\vee \rangle \;=\; \langle \Lambda^A \;,\; (\alpha_{2r-i}^A)^\vee \rangle \;=\; \langle \Lambda \;,\; (\alpha_i)^\vee \rangle, \qquad i = 1, \dots, r.$$

Let $\Lambda_1, \dots, \Lambda_n \in \mathfrak{h}_B^*$ be dominant integral so_{2r+1}-weights, $z_1, \dots, z_n$ complex numbers. Let the polynomials $T_1, \dots, T_r$ be given by (4). Remind that an r-tuple of polynomials

$\boldsymbol{y}$ represents a critical point of a master function associated with so_{2r+1}, $\Lambda_1, \dots, \Lambda_n$, $z_1, \dots, z_n$, if and only if $\boldsymbol{y}$ is generic with respect to weights $\Lambda_1, \dots, \Lambda_n$ of so_{2r+1}, and points $z_1, \dots z_n$ and there exist polynomials $\tilde{y}_i$, $i = 1, \dots, r$, such that

$$\begin{aligned} W(y_i\ ,\ \tilde{y}_i) &= T_i\ y_{i-1}\ y_{i+1}, \qquad i\ =\ 1, \dots, r-1, \\ W(y_r\ ,\ \tilde{y}_r) &= T_r\ y_{r-1}^2. \end{aligned}$$

For an r-tuple of polynomials $\boldsymbol{y} = (y_1, \dots, y_r)$, let $\boldsymbol{u}$ be the $2r-1$-tuple of polynomials $(u_1, \dots, u_{2r-1}) = (y_1, \dots, y_{r-1}, y_r, y_{r-1}, \dots, y_1)$.

Lemma 6.6 ([11])**.** An r-tuple $\boldsymbol{y}$ represents a critical point of the so_{2r+1} master function associated with $\Lambda_1, \dots, \Lambda_n$, $z_1, \dots, z_n$, if and only if the $2r-1$-tuple of polynomials $\boldsymbol{u}$ represents a critical point of the sl_{2r} master function associated with $\Lambda_1^A, \dots, \Lambda_n^A$, $z_1, \dots, z_n$. □

We start with an example. Let $\boldsymbol{y} = (y_1, y_2)$ represent a critical point of the so_3 master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$. Set $\bar{y}_1 = y_1 {T_1}^{-1} {T_2}^{-1/2}$, $\bar{y}_2 = y_2 {T_1}^{-1} {T_2}^{-1}$, see (15).

Let $\boldsymbol{u} = (u_1, u_2, u_3) = (y_1, y_2, y_1)$ be the tuple representing the corresponding sl_4 critical point. Set $\bar{\boldsymbol{u}} = (\bar{u}_1, \bar{u}_2, \bar{u}_3) = (\bar{y}_1, \bar{y}_2, \bar{y}_1)$.

Let $\bar{y}_1^{[1]}, \bar{y}_2^{[1,2]}$ be the so_3 reduced diagonal sequence of functions associated with $\boldsymbol{y}$ and the sequence of indices $[1, 2]$, in other words,

$$W(\bar{y}_1, \bar{y}_1^{[1]}) = \bar{y}_2\ , \qquad W(\bar{y}_2, \bar{y}_2^{[1,2]}) = (\bar{y}_1^{[1]})^2\ .$$

Let $\bar{y}_2^{[2]}$ be the so_3 reduced diagonal sequence of functions associated with $\boldsymbol{y}$ and the sequence of indices $[2]$, in other words,

$$W(\bar{y}_2, \bar{y}_2^{[2]}) = (\bar{y}_1)^2\ .$$

Let $\bar{u}_1^{[1]} = \bar{y}_1^{[1]}$, $\bar{u}_2^{[1,2]}$ be the sl_4 reduced diagonal sequence of functions associated with $\boldsymbol{u}$ and the sequence of indices $[1, 2]$, in other words,

$$W(\bar{y}_1, \bar{y}_1^{[1]}) = \bar{y}_2\ , \qquad W(\bar{y}_2, \bar{u}_2^{[1,2]}) = \bar{y}_1^{[1]} \bar{y}_1\ .$$

Then

$$Y\ =\ {\bar{y}_1}^{-H_1}\ {\bar{y}_2}^{-H_2}\ e^{\frac{\bar{y}_1^{[1]}}{\bar{y}_1} F_1} e^{\frac{1}{2} \frac{\bar{y}_2^{[1,2]}}{\bar{y}_2} [[F_2, F_1], F_1]}\ e^{\frac{\bar{u}_2^{[1,2]}}{\bar{y}_2} [F_2, F_1]}\ e^{\frac{\bar{y}_2^{[2]}}{\bar{y}_2} F_2}$$

is an $\mathrm{Sp}\,(4, \mathbb{C})$-valued solution of the differential equation $DY = 0$ where

$$D\ =\ \partial\ +\ F_1\ +\ F_2\ +\ \log'(\bar{y}_1)\, H_1\ +\ \log'(\bar{y}_2)\, H_2\ .$$

Indeed, denote the factors of Y by $P_1, \dots, P_6$ counting from the left. By Lemma 6.4 it suffices to show that the product $P_3 P_4 P_5 P_6$ is a solution of the equation $\bar{D} Y = 0$ where

$$\bar{D} = \partial + \frac{\bar{y}_2}{{\bar{y}_2}^2} F_1 + \frac{{\bar{y}_1}^2}{{\bar{y}_2}^2} F_2\ .$$

We have

$$\begin{aligned}\bar{D}\ P_3P_4P_5P_6 &= P_3\ (\partial + \frac{\bar{y}_1\bar{y}_1^{[1]}}{\bar{y}_2^{\,2}}[F_2,F_1] + \frac{1}{2}\frac{(\bar{y}_1^{[1]})^2}{\bar{y}_2^{\,2}}[[F_2,F_1],F_1] + \frac{\bar{y}_1^{\,2}}{\bar{y}_2^{\,2}}F_2)\ P_4P_5P_6 \\ &= P_3P_4\ (\partial + \frac{\bar{y}_1\bar{y}_1^{[1]}}{\bar{y}_2^{\,2}}[F_2,F_1] + \frac{\bar{y}_1^{\,2}}{\bar{y}_2^{\,2}}F_2)\ P_5P_6 \\ &= P_3P_4P_5\ (\partial + \frac{\bar{y}_1^{\,2}}{\bar{y}_2^{\,2}}F_2)\ P_6 \ = \ P_3P_4P_5P_6\ \partial\ \mathrm{id} \ = \ 0\ .\end{aligned}$$

Now consider the general case. Let $\boldsymbol{y} = (y_1, \ldots, y_r)$ represent a critical point of the so_{2r+1} master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$. Introduce the functions $\bar{y}_1, \ldots, \bar{y}_r$ by formula (15), where $B = (b_{i,j})$ is the matrix inverse to the Cartan matrix of so_{2r+1}. Then the sp_{2r} Miura oper associated with $\boldsymbol{y}$ has the form

$$D \ = \ \partial \ + \ \sum_{j=1}^{r}\ F_j \ + \ \sum_{j=1}^{r}\ \log'(\bar{y}_j)\ H_j\ . \tag{21}$$

Let $\boldsymbol{u} = (u_1, \ldots, u_{2r-1}) = (y_1, \ldots, y_r, \ldots, y_1)$ be the tuple representing the corresponding sl_{2r} critical point.

For $i = 1, \ldots, r$, let $y_i^{[i]}$, $y_{i+1}^{[i,i+1]}$, $\ldots$, $y_r^{[i,\ldots,r]}$ be the so_{2r+1} diagonal sequence of polynomials associated with $\boldsymbol{y}$ and the sequence of indices $[i, i+1, \ldots, r]$, in other words,

$$W(y_i, y_i^{[i]}) \ = \ T_i\ y_{i-1}\ y_{i+1}, \qquad W(y_{i+1}, y_{i+1}^{[i,i+1]}) \ = \ T_{i+1}\ y_i^{[i]}\ y_{i+2},\ \ldots,$$

$$W(y_{r-1}, y_{r-1}^{[i,\ldots,r-1]}) \ = \ T_{r-1}\ y_{r-2}^{[i,\ldots,r-2]}\ y_r, \qquad W(y_r, y_r^{[i,\ldots,r]}) \ = \ T_r\ (y_{r-1}^{[i,\ldots,r-1]})^2.$$

For $i = 1, \ldots, r-1$, let

$$u_i^{[i]} = y_i^{[i]}, \qquad u_{i+1}^{[i,i+1]} = y_{i+1}^{[i,i+1]}, \qquad \ldots\ , \qquad u_{r-1}^{[i,\ldots,r-1]} = y_{r-1}^{[i,\ldots,r-1]},$$

$$u_r^{[i,\ldots,r]}, \qquad u_{r+1}^{[i,\ldots,r+1]}, \qquad \ldots\ , \qquad u_{2r-i-1}^{[i,\ldots,2r-i-1]}$$

be the sl_{2r} diagonal sequence of polynomials associated with $\boldsymbol{u}$ and the sequence of indices $[i, i+1, \ldots, 2r-i-1]$, in other words,

$$W(y_r, u_r^{[i,\ldots,r]}) \ = \ T_r\ y_{r-1}^{[i,\ldots,r-1]}\ y_{r-1}, \qquad W(y_{r-1}, u_{r+1}^{[i,\ldots,r+1]}) \ = \ T_{r-1}\ u_r^{[i,\ldots,r]}\ y_{r-2}\ ,$$

$$W(y_{r-l}, u_{r+l}^{[i,\ldots,r+l]}) \ = \ T_{r-l}\ u_{r+l-1}^{[i,\ldots,r+l-1]}\ y_{r-l-1}\ , \qquad \text{for}\ \ l = 2, \ldots, r-i-1.$$

For $i \in \{1, \ldots, r\}$ set $F_{i,i} = F_i$. For $1 \le i < j < r$ set

$$F_{i,j} \ = \ [F_j, [F_{j-1}, [\ldots, [F_{i+1}, F_i]\ldots]]]\ .$$

Set $F_{i,r}^* = [F_r, F_{i,r-1}]$ and for $1 \le i < j < r$ set

$$F_{i,j}^* \ = \ [F_j, [F_{j+1}, [\ldots[F_{r-2}, [F_{r-1}, F_{i,r}^*]]\ldots]]]\ .$$

Define $r+1$ functions $Y_0, Y_1, \ldots, Y_r$ of x with values in $\mathrm{Sp}\,(2r, \mathbb{C})$ by the formulas

$$Y_0 \;=\; \prod_{j=1}^{r} y_j^{\,-H_j} T_j^{\,w_j}\;,$$

$$Y_i \;=\; \left(\prod_{j=i}^{r-1} e^{\frac{y_j^{[i,\ldots,j]}}{y_j}\; F_{i,j}}\right) e^{\frac{1}{2}\frac{y_r^{[i,\ldots,r]}}{y_r}\;[[F_r, F_{i,r-1}], F_{i,r-1}]} \left(\prod_{j=r}^{2r-i-1} e^{\frac{u_j^{[i,\ldots,j]}}{y_{2r-j}}\; F^*_{i,2r-j}}\right)$$

for $i \in \{1, \ldots, r-1\}$, and $Y_r = e^{\frac{y_r^{[r]}}{y_r}\; F_r}$.

Note that inside each product the factors commute.

Theorem 6.7. The product $Y_0 Y_1 \ldots Y_r$ is a solution of the differential equation $DY = 0$ with values in $\mathrm{Sp}\,(2r, \mathbb{C})$ where D is given by (21).

The proof is straightforward. One uses Lemma 6.4 and then shows that

$$(\partial \;+\; \frac{\bar{y}_{r-1}^{\;2}}{\bar{y}_r^{\;2}} F_r \;+\; \sum_{j=i}^{r-1} \frac{\bar{y}_{j-1}\bar{y}_{j+1}}{\bar{y}_j^{\;2}}\, F_j)\; Y_i \;=\; Y_i\; (\partial \;+\; \frac{\bar{y}_{r-1}^{\;2}}{\bar{y}_r^{\;2}} F_r \;+\; \sum_{j=i+1}^{r-1} \frac{\bar{y}_{j-1}\bar{y}_{j+1}}{\bar{y}_j^{\;2}}\, F_j)$$

for $i = 1, \ldots, r-1$.

Remark. Theorems 6.5 and 6.7 give explicit formulas for solutions of the differential equation $DY = 0$ where D is the Miura oper associated to a critical point of type A_r or B_r. In a similar way one can construct explicit formulas for solutions in the case of the Miura oper associated to a critical point of type C_r, cf. Section 7 in [11].

6.4 General formulas for solutions

Let $\mathfrak{g}$ be a simple Lie algebra with Cartan matrix A. Let ${}^t\mathfrak{g}$ be its Langlands dual with Chevalley generators $F_1, \ldots, F_r$, $H_1, \ldots, H_r$, $E_1, \ldots, E_r$. Let $w_1, \ldots, w_r$ be the fundamental coweights of ${}^t\mathfrak{g}$. Let ${}^tG$ be the complex simply connected Lie group with Lie algebra ${}^t\mathfrak{g}$.

Let V be a complex finite dimensional representation of ${}^tG$. Let v_{low} be a lowest weight vector of V, ${}^t\mathfrak{n}_-\, v_{\mathrm{low}} = 0$.

Let $\boldsymbol{y} = (y_1, \ldots, y_r)$ represent a critical point of the $\mathfrak{g}$ master function (1) associated with parameters $\boldsymbol{z}, \boldsymbol{\Lambda}, \Lambda_\infty$. Let $D_{\boldsymbol{y}}$ be the ${}^t\mathfrak{g}$ Miura oper associated with $\boldsymbol{y}$.

Let $\boldsymbol{i} = [i_1, \ldots, i_k]$, $i_j \in \{1, \ldots, r\}$, be a sequence of natural numbers. Let $\boldsymbol{y}^{[i_1]} = (y_1^{[i_1]}, \ldots, y_r^{[i_1]})$, $\boldsymbol{y}^{[i_1,i_2]} = (y_1^{[i_1,i_2]}, \ldots, y_r^{[i_1,i_2]})$, $\ldots$, $\boldsymbol{y}^{[i_1,\ldots,i_k]} = (y_1^{[i_1,\ldots,i_k]}, \ldots, y_r^{[i_1,\ldots,i_k]})$ be a sequence of tuples associated with the critical point $\boldsymbol{y}$ and the sequence of indices $\boldsymbol{i}$, see Section 2.5.

Theorem 6.8. The V-valued function

$$Y \;=\; \exp\left(-\log'\left(\frac{y_{i_1}^{[i_1]}}{y_{i_1}}\right)E_{i_1}\right)\exp\left(-\log'\left(\frac{y_{i_2}^{[i_1,i_2]}}{y_{i_2}^{[i_1]}}\right)E_{i_2}\right)\,\dots$$

$$\exp\left(-\log'\left(\frac{y_{i_k}^{[i_1,\dots,i_k]}}{y_{i_k}^{[i_1,\dots,i_{k-1}]}}\right)E_{i_k}\right)\;\prod_{j=1}^{r}\,(\,y_j^{[i_1,\dots,i_k]}\,)^{-H_j}T_j^{\;w_j}\;v_{\mathrm{low}}$$

is a solution of the differential equation

$$D_{\boldsymbol{y}}\,Y \;=\; 0\;.$$

The proof is straightforward and follows from the identity

$$D_{\boldsymbol{y}^{[i_1,\dots,i_j]}} \;=\; \exp\left(\mathrm{ad}\log'\left(\frac{y_{i_j}^{[i_1,\dots,i_j]}}{y_{i_j}^{[i_1,\dots,i_{j-1}]}}\right)E_{i_j}\right)\cdot D_{\boldsymbol{y}^{[i_1,\dots,i_{j-1}]}}\;,$$

see Theorem 3.3.

Let d be the determinant of the Cartan matrix of $\mathfrak{g}$.

Corollary 6.9. Every coordinate of every solution of the equation $D_{\boldsymbol{y}}Y \;=\; 0$ with values in a finite dimensional representation of ${}^tG$ can be written as a rational function $R(f_1,\dots,f_N;\;T_1^{1/d},\dots,T_r^{1/d}\,)$ of functions $T_1^{1/d},\dots,T_r^{1/d}$ and suitable polynomials $f_1,\dots,f_N$ which appear as coordinates of tuples in the $\mathfrak{g}$ population $P_{\boldsymbol{y}}$ originated at $\boldsymbol{y}$.

Since ${}^tG$ has a faithful finite dimensional representation, the solutions of the differential equation $D_{\boldsymbol{y}}Y \;=\; 0$ with values in ${}^tG$ also can be written as rational functions of functions $T_1^{1/d},\dots,T_r^{1/d}$ and coordinates of tuples of $P_{\boldsymbol{y}}$, cf. Sections 6.2 and 6.3.

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