Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 3(2) 2005 183–187

On the weak non-defectivity of Veronese embeddings of projective spaces*

Edoardo Ballico†

Department of Mathematics,
University of Trento,
38050 Povo (TN), Italy

Received 24 November 2004; accepted 10 January 2005

Abstract: Fix integers n, x, k such that $n \geq 3$, $k > 0$, $x \geq 4$, $(n, x) \neq (3, 4)$ and $k(n+1) < \binom{n+x}{n}$. Here we prove that the order x Veronese embedding of $\mathbf{P}^n$ is not weakly $(k-1)$-defective, i.e. for a general $S \subset \mathbf{P}^n$ such that $\sharp(S) = k+1$ the projective space $|\mathcal{I}_{2S}(x)|$ of all degree t hypersurfaces of $\mathbf{P}^n$ singular at each point of S has dimension $\binom{n+x}{n} - 1 - k(n+1)$ (proved by Alexander and Hirschowitz) and a general $F \in |\mathcal{I}_{2S}(x)|$ has an ordinary double point at each $P \in S$ and $\mathrm{Sing}(F) = S$.

Keywords: Veronese variety, weakly defective variety, zero-dimensional scheme, double point, fat point, Veronese embedding
MSC (2000): 14N05

1 Introduction

The main aim of this paper is to use the so-called Horace Method introduced by A. Hirschowitz to prove the following result.

Theorem 1.1. *Fix integers $n \geq 3$, $x \geq 4$, $(n, x) \neq (3, 4)$ and $k \geq 0$ such that*

$$k(n+1) < \binom{n+x}{n} \tag{1}$$

and a general $S \subset \mathbf{P}^n$ such that $\sharp(S) = k$. Let $|\mathcal{I}_{2S}(x)|$ denote the projective space of all degree x hypersurfaces of $\mathbf{P}^n$ singular at each point of S. Then $\dim(|\mathcal{I}_{2S}(t)|) =$

* The author was partially supported by MIUR and GNSAGA of INdAM (Italy).
† E-mail: ballico@science.unitn.it

$\binom{n+x}{n} - k(n+1) - 1$. *A general* $F \in |\mathcal{I}_{2S}(x)|$ *satisfies Sing*$(F) = S$ *and it has an ordinary node at each point of* S.

The computation of $\dim(|\mathcal{I}_{2S}(x)|)$ is an important theorem due to J. Alexander and A. Hirschowitz [1, 2, 3, 4, 6]. With the classical terminology their theorem means that the degree x Veronese embedding of $\mathbf{P}^n$ is not $(k-1)$-defective. They also give the list of all triples (n, x, k) such that the degree $x \geq 3$ Veronese embedding of $\mathbf{P}^n$ is $(k-1)$-defective: the triples $(n, x, k) \in \{(2, 4, 5), (3, 4, 9), (4, 3, 7), (4, 4, 14)\}$ ([6], Th. 1). The new (I hope) part is that $\mathrm{Sing}(F) = S$ and that F has an ordinary node at each point of S for a general $F \in |\mathcal{I}_{2S}(x)|$. For each fixed pair (n, x) there is at most one integer k ($k = [(\binom{n+x}{n} - 1)/(n+1)]$) satisfying (1) and for which Theorem 1.1 is not known to be true by [10], Cor. 4.5. As obvious to everybody working on this topic the case $k = [(\binom{n+x}{n} - 1)/(n+1)]$ is by far the most difficult. To prove it we will use an idea due to M. Mella and used in [10]. For related examples in which the singular locus has positive dimension, see [9] and [8], Remark 6.2, which quotes [11], and [10], Remark 4.4.

We work over an algebraically closed field $\mathbb{K}$ with $\mathrm{char}(\mathbb{K}) = 0$. Our proof of Theorem 1.1 heavily depends on the characteristic zero assumption: a key tool will be [7], Th. 1.4. We borrowed a key idea from [10].

2 The proof

For any scheme A and any $P \in A_{reg}$ let $2P$ (or $2\{P, A\}$ if there is any danger of misunderstanding) denote the first infinitesimal neighborhood of P in A, i.e. the closed subscheme of A with $(\mathcal{I}_P)^2$ as its ideal sheaf. Hence $2P$ is a zero-dimensional subscheme of A and $\mathrm{length}(2P) = \dim(A)+1$. For any finite subset $S \subset A_{reg}$, set $2S := \cup_{P\in S} 2P$ and $2\{S, A\} := \cup_{P\in S} 2\{P, A\}$. For any closed subscheme Z of A and every effective Cartier divisor D of A let $\mathrm{Res}_D(Z)$ denote the residual scheme of of Z with respect to D, i.e. the closed subscheme of A with $\mathcal{I}_{Z,A} : \mathcal{I}_{D,A}$ as its ideal sheaf. For any effective Cartier divisor D of A such that $P \in D_{reg}$ we have $2\{P, A\} \cap D = 2\{P, D\}$ and $\mathrm{Res}_D(2\{P, A\}) = \{P\}$.

We will often use the following elementary form of the so-called Horace Lemma.

Lemma 2.1. *Let* $H \subset \mathbf{P}^n$ *be a hyperplane and* $Z \subset \mathbf{P}^n$ *a closed subscheme. Then:*
(a) $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^0(\mathbf{P}^n, \mathcal{I}_{Res_H(Z)}(d-1)) + h^0(H, \mathcal{I}_{Z\cap H}(d))$;
(b) $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^1(\mathbf{P}^n, \mathcal{I}_{Res_H(Z)}(d-1)) + h^1(H, \mathcal{I}_{Z\cap H}(d))$.

Proof. By the very definition of a residual scheme with respect to H, there is the following exact sequence

$$0 \to \mathcal{I}_{\mathrm{Res}_H(Z)}(d-1) \to \mathcal{I}_Z(d) \to \mathcal{I}_{Z\cap H}(d) \to 0 \tag{2}$$

whose long cohomology exact sequence proves the lemma. □

The following result is a very particular case of [5], Lemma 2.3 (see in particular Fig. 1 on p. 308).

Lemma 2.2. *Let* $H \subset \mathbf{P}^n$ *be hyperplane,* $Z \subset \mathbf{P}^n$ *a closed subscheme not containing* H *and* s *a positive integer. Let* U *be the union of* Z *and* s *general double points of*

$\mathbf{P}^n$. *Let* S *be the union of* s *general points of* H. *Let* $E \subset H$ *be the union of* s *general double points of* H *(not double points of* $\mathbf{P}^n$*, i.e. each of them has length* n*). To prove* $h^1(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$ *(resp.* $h^0(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$*) it is sufficient to prove* $h^1(H, \mathcal{I}_{(Z\cap H)\cup S}(d)) = h^1(\mathbf{P}^n, \mathcal{I}_{Res_{H}(Z)\cup E}(d-1)) = 0$ *(resp.* $h^0(H, \mathcal{I}_{(Z\cap H)\cup S}(d)) = h^0(\mathbf{P}^n, \mathcal{I}_{Res_{H}(Z)\cup E}(d-1)) = 0$*).*

For all integers $n > 0, t > 0$ define the integers $a_{n,t}, b_{n,t}, c_{n,t}$ and $d_{n,t}$ using the following relations:

$$(n+1)a_{n,t} + b_{n,t} = \binom{n+t}{n},\ 0 \le b_{n,t} \le n \tag{3}$$

$$(n+1)c_{n,t} + d_{n,t} + 1 = \binom{n+t}{n},\ 0 \le d_{n,t} \le n \tag{4}$$

Notice that $c_{n,t} = a_{n,t}$ and $d_{n,t} = b_{n,t} - 1$ if $b_{n,t} > 0$, while $c_{n,t} = a_{n,t} - 1$ and $d_{n,t} = n$ if $b_{n,t} = 0$.

Subtracting (3) from the same equation for the integers $n, t-1$ we obtain the following relation:

$$(n+1)(a_{n,t} - a_{n,t-1}) + b_{n,t} - b_{n-1,t} = \binom{n+t-1}{n-1} \tag{5}$$

Using (3) for the integers $n-1, t$ and (5), we obtain the following relation:

$$(n+1)(a_{n,t} - a_{n,t-1}) + b_{n,t} - b_{n-1,t} = na_{n-1,t} + b_{n-1,t} \tag{6}$$

Lemma 2.3. *Fix a hyperplane* $H \in \mathbf{P}^n$ *and an integer* $y > 0$. *Let* D *be an irreducible* y*-dimensional family of hypersurfaces of* $\mathbf{P}^n$. *Then for a general* $B \subset H$ *such that* $\sharp(B) = y$ *there is* $Y \in D$ *such that* $B \subset Y$.

Proof. Take a general $P \in H$. Since $y > 0$ there are infinitely many hypersurfaces parametrized by D. Hence the set of all $Y \in D$ containing P is non-empty and contains an irreducible subfamily $D_P \subset D$ of dimension $y-1$. If $y = 1$, we are done. If $y \ge 2$ we use induction on y and the family D_P for the integer $y' := y - 1$. $\square$

Remark 2.4. If $x \ge 2$, $\sharp(A) = c_{n,x} - a_{n-1,x} - b_{n-1,x}$ and $h^1(\mathbf{P}^n, \mathcal{I}_{2A}(x-1)) = 0$, then $h^0(\mathbf{P}^n, \mathcal{I}_{2A}(x-1)) = \binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) = \binom{n+x-1}{n} - \binom{n+x}{n} + 1 + d_{n,x} + a_{n-1,x} + (n+1)b_{n-1,x}\binom{n+x-1}{n-1} - b_{n-1,x} = 1 + d_{n,x} + nb_{n-1,x} + a_{n-1,x} \ge 1 + a_{n-1,x} + b_{n-1,x}$.

Lemma 2.5. *Assume either* $x = 3$ *and* $n \ge 10$ *or* $x \ge 4$ *and* $n \ge 4$. *Then*

$$(n+1)(a_{n,x} - a_{n-1,x}) + n + 1 \le \binom{n+x-1}{n} \tag{7}$$

Proof. Since $(n+1)a_{n,x} \le \binom{n+x}{n}$ and $na_{n-1,x} \ge \binom{n+x-1}{n-1}$, the inequality (7) is satisfied if $2n - 1 \ge a_{n-1,x}$, i.e. if

$$(2n-1)(n+1) \le \binom{n+x-1}{n-1} \tag{8}$$

which is satisfied if either $x = 3$ and $n \geq 10$ or $x \geq 4$ and $n \geq 4$. $\square$

Lemma 2.6. *We have* $\binom{n+x-1}{n} \geq c_{n,x}$ *for all* $n \geq 2$ *and* $x \geq 3$

Proof. Since $(n+1)c_{n,x} < \binom{n+x}{n}$, it is sufficient to use the obvious inequality $(n+1)x \geq (n+x)$. $\square$

Lemma 2.7. *We have* $\binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) \geq a_{n-1,x}$ *for all* $n \geq 3$ *and* $x \geq 3$.

Proof. By (3) and (4) this inequality is equivalent to the inequality $\binom{n+x-1}{n} - \binom{n+x}{n} + 1 + d_{n,x} + \binom{n+x-1}{n-1} + nb_{n-1,x}$, which is obviously satisfied. $\square$

Proof (of Theorem 1.1)**.** Fix a general $S \subset \mathbf{P}^n$ such that $\sharp(S) = c_{n,x} - a_{n-1,x}$. By Lemma 2.5 and the inequality $c_{n,x} \leq a_{n,x}$ we have $(n+1)\sharp(S) + n + 1 \leq \binom{n+x-1}{n}$. Hence $h^1(\mathbf{P}^n, \mathcal{I}_{2S}(x-1)) = 0$ and a general $Y \in |\mathcal{I}_{2S}(x-1)|$ has an ordinary node at each point of S and $\mathrm{Sing}(Y) = S$ ([10], Cor. 4.5). Let $H \subset \mathbf{P}^n$ be a general hyperplane. Hence $H \cap S = \emptyset$ and H is transversal to Y. Now we fix H and S and move Y. Since $n \geq 3$, $Y \cap H$ is irreducible. We fix a general $B \subset H \cap Y$ such that $\sharp(B) = a_{n-1,x}$. $\square$

Claim 2.8. We have $h^1(\mathbf{P}^n, \mathcal{I}_{2S\cup 2B}(x)) = 0$

Proof (of the Claim)**.** Since $(n-1, x) \notin \{(2,4), (3,4), (4,3), (4,4)\}$ and B is general in an integral degree $x-1$ hypersurface of H, we have $h^1(H, \mathcal{I}_{2B}(x)) = 0$, i.e. $h^0(H, \mathcal{I}_{2B}(x)) = b_{n-1,x}$. Notice that $\sharp(S) \geq b_{n-1,x}$ (Lemma 2.3). Fix $S'' \subseteq S$ such that $\sharp(S'') = b_{n-1,x}$ and set $S' := S \backslash S''$. S'' may be seen as a set of $b_{n-1,x}$ general points of $\mathbf{P}^n$. We degenerate it (keeping fixed $S' \cup B$) into a union E of $b_{n-1,x}$ general points of H. Hence $h^0(H, \mathcal{I}_{2B\cup E}(x)) = h^1(H, \mathcal{I}_{2B\cup E}(x)) = 0$. Since $S \cap H = \emptyset$, then $\mathrm{Res}_{2S\cup 2B} = 2S \cup B$. The local deformation space of an ordinary nodal hypersurface singularity is one-dimensional. Since $h^1(\mathbf{P}^n, \mathcal{I}_{2S}(x-1)) = 0$, we obtain that, moving S, the set of the possible nodal hypersurfaces Y is (near Y) at least of dimension $\binom{n+x-1}{n} - 1 - \sharp(S) \geq a_{n-1,x}$ (Lemma 2.6). Hence by Lemma 2.2 to prove the Claim it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2\{E,H\}\cup B}(x-1)) = 0$. First, we will check that $h^1(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2\{E,H\}}(x-1)) = 0$. To prove this vasnishing it is sufficient to prove $h^1(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2E}(x-1)) = 0$. Since $\sharp(E) \leq n$, any n points of $\mathbf{P}^n$ are contained in a hyperplane and S' is chosen independently from H and E, we may consider $S' \cup E$ as a general union of $c_{n,x} - a_{n-1,x}$ points of $\mathbf{P}^n$. Hence $h^1(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2E}(x-1)) = 0$ and thus $h^1(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2\{E,H\}}(x-1)) = 0$. By Remark 2.4 and Lemma 2.3 for fixed S' we may take $B \cup E$ general in H. Hence for fixed $S' \cup E$ we may take B general in H. By the generality of B, $S' \cap H = \emptyset$ and $h^1(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2\{E,H\}}(x-1)) = 0$, to prove $h^1(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2\{E,H\}\cup B}(x-1)) = 0$ and hence the Claim it is sufficient to prove the inequality $h^0(\mathbf{P}^n, \mathcal{I}_{2S'\cup 2\{E,H\}}(x-1)) - h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) \geq \sharp(B)$ (see e.g. [6], Lemma 3), i.e. the inequality

$$\binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) \geq h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) + a_{n-1,x} \quad (9)$$

First assume $c_{n,x} - a_{n-1,x} - b_{n-1,x} \leq a_{n,x-2}$. Then $h^1(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = 0$ and hence $h^0(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E,H\}}(x-1)) - h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = \binom{n+x-2}{n-1} - nb_{n-1,x}$. Hence the inequality (9) is satisfied in this case. Now assume $c_{n,x} - a_{n-1,x} - b_{n-1,x} > a_{n,x-2}$. Hence $h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = 0$. Hence the inequality (9) is satisfied by Lemma 2.7, proving the Claim. Set $A := Y \cup H$. Since $S \cap H = \emptyset$ and $c_{n,x} > a_{n-1,x}$, at least one of the points of S is an isolated singular point of A. Hence a general $E \in |\mathcal{I}_{2S \cup 2B}(x)|$ has at least one isolated singular point and this point is contained in $S \cup B$, i.e. in a set T such that $h^1(\mathbf{P}^n, \mathcal{I}_{2T}(x)) = 0$ and $\sharp(T) = c_{n,x}$. Hence we may apply the semicontinuity theorem for cohomology and the openness of smoothness to obtain that for a general $G \subset \mathbf{P}^n$ such that $\sharp(G) = c_{n,x}$, the linear system $|\mathcal{I}_{2G}(x)|$ has the expected dimension and a general member of it has an isolated singularity at one point of G. We conclude by [7], Th. 1.4. $\square$

References

[1] J. Alexander: "Singularités imposables en position générale aux hypersurfaces de $\mathbb{P}^n$", *Compositio Math.*, Vol. 68, (1988), pp. 305–354.

[2] J. Alexander and A. Hirschowitz: "Un lemme d'Horace différentiel: application aux singularité hyperquartiques de $\mathbb{P}^5$", *J. Algebraic Geom.*, Vol. 1, (1992), pp. 411–426.

[3] J. Alexander and A. Hirschowitz: "La méthode d'Horace éclaté: application à l'interpolation en degré quatre", *Invent. Math.*, Vol. 107, (1992), pp. 585–602.

[4] J. Alexander and A. Hirschowitz: "Polynomial interpolation in several variables", *J. Algebraic Geom.*, Vol. 4, (1995), pp. 201–222.

[5] J. Alexander and A. Hirschowitz: "An asymptotic vanishing theorem for generic unions of multiple points", *Invent. Math.*, Vol. 140, (2000), pp. 303–325.

[6] K. Chandler: "A brief proof of a maximal rank theorem for generic double points in projective space", *Trans. Amer. Math. Soc.*, Vol. 353(5), (2000), pp. 1907–1920.

[7] L. Chiantini and C. Ciliberto: "Weakly defective varieties", *Trans. Amer. Math. Soc.*, Vol. 454(1), (2002), pp. 151–178.

[8] C. Ciliberto: "Geometric aspects of polynomial interpolation in more variables and of Waring's problem", In: *European Congress of Mathematics (Barcelona, 2000)*, Progress in Math., Vol. 201, Birkhäuser, Basel, 2001, pp. 289–316.

[9] C. Ciliberto and A. Hirschowitz: "Hypercubique de $\mathbb{P}^4$ avec sept points singulieres génériques", *C. R. Acad. Sci. Paris*, Vol. 313(I), (1991), pp. 135–137.

[10] M. Mella: *Singularities of linear systems and the Waring problem*, arXiv mathAG/0406288.

[11] A. Terracini: "Sulla rappresentazione delle coppie di forme ternarie mediante somme di potenze di forme lineari", *Ann. Mat. Pura e Appl.*, Vol. 24, (1915), pp. 91–100.