

# On the weak non-defectivity of Veronese embeddings of projective spaces\*

Edoardo Ballico<sup>†</sup>

*Department of Mathematics,  
University of Trento,  
38050 Povo (TN), Italy*

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**Abstract:** Fix integers  $n, x, k$  such that  $n \geq 3$ ,  $k > 0$ ,  $x \geq 4$ ,  $(n, x) \neq (3, 4)$  and  $k(n+1) < \binom{n+x}{n}$ . Here we prove that the order  $x$  Veronese embedding of  $\mathbf{P}^n$  is not weakly  $(k-1)$ -defective, i.e. for a general  $S \subset \mathbf{P}^n$  such that  $\sharp(S) = k+1$  the projective space  $|\mathcal{I}_{2S}(x)|$  of all degree  $t$  hypersurfaces of  $\mathbf{P}^n$  singular at each point of  $S$  has dimension  $\binom{n+x}{n} - 1 - k(n+1)$  (proved by Alexander and Hirschowitz) and a general  $F \in |\mathcal{I}_{2S}(x)|$  has an ordinary double point at each  $P \in S$  and  $\text{Sing}(F) = S$ .

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## 1 Introduction

The main aim of this paper is to use the so-called Horace Method introduced by A. Hirschowitz to prove the following result.

**Theorem 1.1.** *Fix integers  $n \geq 3$ ,  $x \geq 4$ ,  $(n, x) \neq (3, 4)$  and  $k \geq 0$  such that*

$$k(n+1) < \binom{n+x}{n} \tag{1}$$

*and a general  $S \subset \mathbf{P}^n$  such that  $\sharp(S) = k$ . Let  $|\mathcal{I}_{2S}(x)|$  denote the projective space of all degree  $x$  hypersurfaces of  $\mathbf{P}^n$  singular at each point of  $S$ . Then  $\dim(|\mathcal{I}_{2S}(t)|) =$*

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<sup>†</sup> E-mail: ballico@science.unitn.it

$\binom{n+x}{n} - k(n+1) - 1$ . A general  $F \in |\mathcal{I}_{2S}(x)|$  satisfies  $\text{Sing}(F) = S$  and it has an ordinary node at each point of  $S$ .

The computation of  $\dim(|\mathcal{I}_{2S}(x)|)$  is an important theorem due to J. Alexander and A. Hirschowitz [1, 2, 3, 4, 6]. With the classical terminology their theorem means that the degree  $x$  Veronese embedding of  $\mathbf{P}^n$  is not  $(k-1)$ -defective. They also give the list of all triples  $(n, x, k)$  such that the degree  $x \geq 3$  Veronese embedding of  $\mathbf{P}^n$  is  $(k-1)$ -defective: the triples  $(n, x, k) \in \{(2, 4, 5), (3, 4, 9), (4, 3, 7), (4, 4, 14)\}$  ([6], Th. 1). The new (I hope) part is that  $\text{Sing}(F) = S$  and that  $F$  has an ordinary node at each point of  $S$  for a general  $F \in |\mathcal{I}_{2S}(x)|$ . For each fixed pair  $(n, x)$  there is at most one integer  $k$  ( $k = [(\binom{n+x}{n} - 1)/(n+1)]$ ) satisfying (1) and for which Theorem 1.1 is not known to be true by [10], Cor. 4.5. As obvious to everybody working on this topic the case  $k = [(\binom{n+x}{n} - 1)/(n+1)]$  is by far the most difficult. To prove it we will use an idea due to M. Mella and used in [10]. For related examples in which the singular locus has positive dimension, see [9] and [8], Remark 6.2, which quotes [11], and [10], Remark 4.4.

We work over an algebraically closed field  $\mathbb{K}$  with  $\text{char}(\mathbb{K}) = 0$ . Our proof of Theorem 1.1 heavily depends on the characteristic zero assumption: a key tool will be [7], Th. 1.4. We borrowed a key idea from [10].

## 2 The proof

For any scheme  $A$  and any  $P \in A_{\text{reg}}$  let  $2P$  (or  $2\{P, A\}$  if there is any danger of misunderstanding) denote the first infinitesimal neighborhood of  $P$  in  $A$ , i.e. the closed subscheme of  $A$  with  $(\mathcal{I}_P)^2$  as its ideal sheaf. Hence  $2P$  is a zero-dimensional subscheme of  $A$  and  $\text{length}(2P) = \dim(A) + 1$ . For any finite subset  $S \subset A_{\text{reg}}$ , set  $2S := \cup_{P \in S} 2P$  and  $2\{S, A\} := \cup_{P \in S} 2\{P, A\}$ . For any closed subscheme  $Z$  of  $A$  and every effective Cartier divisor  $D$  of  $A$  let  $\text{Res}_D(Z)$  denote the residual scheme of  $Z$  with respect to  $D$ , i.e. the closed subscheme of  $A$  with  $\mathcal{I}_{Z,A} : \mathcal{I}_{D,A}$  as its ideal sheaf. For any effective Cartier divisor  $D$  of  $A$  such that  $P \in D_{\text{reg}}$  we have  $2\{P, A\} \cap D = 2\{P, D\}$  and  $\text{Res}_D(2\{P, A\}) = \{P\}$ .

We will often use the following elementary form of the so-called Horace Lemma.

**Lemma 2.1.** *Let  $H \subset \mathbf{P}^n$  be a hyperplane and  $Z \subset \mathbf{P}^n$  a closed subscheme. Then:*

- (a)  $h^0(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^0(\mathbf{P}^n, \mathcal{I}_{\text{Res}_H(Z)}(d-1)) + h^0(H, \mathcal{I}_{Z \cap H}(d));$
- (b)  $h^1(\mathbf{P}^n, \mathcal{I}_Z(d)) \leq h^1(\mathbf{P}^n, \mathcal{I}_{\text{Res}_H(Z)}(d-1)) + h^1(H, \mathcal{I}_{Z \cap H}(d)).$

**Proof.** By the very definition of a residual scheme with respect to  $H$ , there is the following exact sequence

$$0 \rightarrow \mathcal{I}_{\text{Res}_H(Z)}(d-1) \rightarrow \mathcal{I}_Z(d) \rightarrow \mathcal{I}_{Z \cap H}(d) \rightarrow 0 \quad (2)$$

whose long cohomology exact sequence proves the lemma.  $\square$

The following result is a very particular case of [5], Lemma 2.3 (see in particular Fig. 1 on p. 308).

**Lemma 2.2.** *Let  $H \subset \mathbf{P}^n$  be hyperplane,  $Z \subset \mathbf{P}^n$  a closed subscheme not containing  $H$  and  $s$  a positive integer. Let  $U$  be the union of  $Z$  and  $s$  general double points of*

$\mathbf{P}^n$ . Let  $S$  be the union of  $s$  general points of  $H$ . Let  $E \subset H$  be the union of  $s$  general double points of  $H$  (not double points of  $\mathbf{P}^n$ , i.e. each of them has length  $n$ ). To prove  $h^1(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$  (resp.  $h^0(\mathbf{P}^n, \mathcal{I}_U(d)) = 0$ ) it is sufficient to prove  $h^1(H, \mathcal{I}_{(Z \cap H) \cup S}(d)) = h^1(\mathbf{P}^n, \mathcal{I}_{Res_H(Z) \cup E}(d-1)) = 0$  (resp.  $h^0(H, \mathcal{I}_{(Z \cap H) \cup S}(d)) = h^0(\mathbf{P}^n, \mathcal{I}_{Res_H(Z) \cup E}(d-1)) = 0$ ).

For all integers  $n > 0, t > 0$  define the integers  $a_{n,t}, b_{n,t}, c_{n,t}$  and  $d_{n,t}$  using the following relations:

$$(n + 1)a_{n,t} + b_{n,t} = \binom{n + t}{n}, \quad 0 \leq b_{n,t} \leq n \tag{3}$$

$$(n + 1)c_{n,t} + d_{n,t} + 1 = \binom{n + t}{n}, \quad 0 \leq d_{n,t} \leq n \tag{4}$$

Notice that  $c_{n,t} = a_{n,t}$  and  $d_{n,t} = b_{n,t} - 1$  if  $b_{n,t} > 0$ , while  $c_{n,t} = a_{n,t} - 1$  and  $d_{n,t} = n$  if  $b_{n,t} = 0$ .

Subtracting (3) from the same equation for the integers  $n, t-1$  we obtain the following relation:

$$(n + 1)(a_{n,t} - a_{n,t-1}) + b_{n,t} - b_{n-1,t} = \binom{n + t - 1}{n - 1} \tag{5}$$

Using (3) for the integers  $n - 1, t$  and (5), we obtain the following relation:

$$(n + 1)(a_{n,t} - a_{n,t-1}) + b_{n,t} - b_{n-1,t} = na_{n-1,t} + b_{n-1,t} \tag{6}$$

**Lemma 2.3.** Fix a hyperplane  $H \in \mathbf{P}^n$  and an integer  $y > 0$ . Let  $D$  be an irreducible  $y$ -dimensional family of hypersurfaces of  $\mathbf{P}^n$ . Then for a general  $B \subset H$  such that  $\sharp(B) = y$  there is  $Y \in D$  such that  $B \subset Y$ .

**Proof.** Take a general  $P \in H$ . Since  $y > 0$  there are infinitely many hypersurfaces parametrized by  $D$ . Hence the set of all  $Y \in D$  containing  $P$  is non-empty and contains an irreducible subfamily  $D_P \subset D$  of dimension  $y - 1$ . If  $y = 1$ , we are done. If  $y \geq 2$  we use induction on  $y$  and the family  $D_P$  for the integer  $y' := y - 1$ . □

**Remark 2.4.** If  $x \geq 2$ ,  $\sharp(A) = c_{n,x} - a_{n-1,x} - b_{n-1,x}$  and  $h^1(\mathbf{P}^n, \mathcal{I}_{2A}(x-1)) = 0$ , then  $h^0(\mathbf{P}^n, \mathcal{I}_{2A}(x-1)) = \binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) = \binom{n+x-1}{n} - \binom{n+x}{n} + 1 + d_{n,x} + a_{n-1,x} + (n+1)b_{n-1,x} - b_{n-1,x} = 1 + d_{n,x} + nb_{n-1,x} + a_{n-1,x} \geq 1 + a_{n-1,x} + b_{n-1,x}$ .

**Lemma 2.5.** Assume either  $x = 3$  and  $n \geq 10$  or  $x \geq 4$  and  $n \geq 4$ . Then

$$(n + 1)(a_{n,x} - a_{n-1,x}) + n + 1 \leq \binom{n + x - 1}{n} \tag{7}$$

**Proof.** Since  $(n + 1)a_{n,x} \leq \binom{n+x}{n}$  and  $na_{n-1,x} \geq \binom{n+x-1}{n-1}$ , the inequality (7) is satisfied if  $2n - 1 \geq a_{n-1,x}$ , i.e. if

$$(2n - 1)(n + 1) \leq \binom{n + x - 1}{n - 1} \tag{8}$$

which is satisfied if either  $x = 3$  and  $n \geq 10$  or  $x \geq 4$  and  $n \geq 4$ .  $\square$

**Lemma 2.6.** *We have  $\binom{n+x-1}{n} \geq c_{n,x}$  for all  $n \geq 2$  and  $x \geq 3$*

**Proof.** Since  $(n+1)c_{n,x} < \binom{n+x}{n}$ , it is sufficient to use the obvious inequality  $(n+1)x \geq (n+x)$ .  $\square$

**Lemma 2.7.** *We have  $\binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) \geq a_{n-1,x}$  for all  $n \geq 3$  and  $x \geq 3$ .*

**Proof.** By (3) and (4) this inequality is equivalent to the inequality  $\binom{n+x-1}{n} - \binom{n+x}{n} + 1 + d_{n,x} + \binom{n+x-1}{n-1} + nb_{n-1,x}$ , which is obviously satisfied.  $\square$

**Proof** (of Theorem 1.1). Fix a general  $S \subset \mathbf{P}^n$  such that  $\sharp(S) = c_{n,x} - a_{n-1,x}$ . By Lemma 2.5 and the inequality  $c_{n,x} \leq a_{n,x}$  we have  $(n+1)\sharp(S) + n + 1 \leq \binom{n+x-1}{n}$ . Hence  $h^1(\mathbf{P}^n, \mathcal{I}_{2S}(x-1)) = 0$  and a general  $Y \in |\mathcal{I}_{2S}(x-1)|$  has an ordinary node at each point of  $S$  and  $\text{Sing}(Y) = S$  ([10], Cor. 4.5). Let  $H \subset \mathbf{P}^n$  be a general hyperplane. Hence  $H \cap S = \emptyset$  and  $H$  is transversal to  $Y$ . Now we fix  $H$  and  $S$  and move  $Y$ . Since  $n \geq 3$ ,  $Y \cap H$  is irreducible. We fix a general  $B \subset H \cap Y$  such that  $\sharp(B) = a_{n-1,x}$ .  $\square$

**Claim 2.8.** We have  $h^1(\mathbf{P}^n, \mathcal{I}_{2S \cup 2B}(x)) = 0$

**Proof** (of the Claim). Since  $(n-1, x) \notin \{(2, 4), (3, 4), (4, 3), (4, 4)\}$  and  $B$  is general in an integral degree  $x-1$  hypersurface of  $H$ , we have  $h^1(H, \mathcal{I}_{2B}(x)) = 0$ , i.e.  $h^0(H, \mathcal{I}_{2B}(x)) = b_{n-1,x}$ . Notice that  $\sharp(S) \geq b_{n-1,x}$  (Lemma 2.3). Fix  $S'' \subseteq S$  such that  $\sharp(S'') = b_{n-1,x}$  and set  $S' := S \setminus S''$ .  $S''$  may be seen as a set of  $b_{n-1,x}$  general points of  $\mathbf{P}^n$ . We degenerate it (keeping fixed  $S' \cup B$ ) into a union  $E$  of  $b_{n-1,x}$  general points of  $H$ . Hence  $h^0(H, \mathcal{I}_{2B \cup E}(x)) = h^1(H, \mathcal{I}_{2B \cup E}(x)) = 0$ . Since  $S \cap H = \emptyset$ , then  $\text{Res}_{2S \cup 2B} = 2S \cup B$ . The local deformation space of an ordinary nodal hypersurface singularity is one-dimensional. Since  $h^1(\mathbf{P}^n, \mathcal{I}_{2S}(x-1)) = 0$ , we obtain that, moving  $S$ , the set of the possible nodal hypersurfaces  $Y$  is (near  $Y$ ) at least of dimension  $\binom{n+x-1}{n} - 1 - \sharp(S) \geq a_{n-1,x}$  (Lemma 2.6). Hence by Lemma 2.2 to prove the Claim it is sufficient to prove  $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\} \cup B}(x-1)) = 0$ . First, we will check that  $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) = 0$ . To prove this vanishing it is sufficient to prove  $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2E}(x-1)) = 0$ . Since  $\sharp(E) \leq n$ , any  $n$  points of  $\mathbf{P}^n$  are contained in a hyperplane and  $S'$  is chosen independently from  $H$  and  $E$ , we may consider  $S' \cup E$  as a general union of  $c_{n,x} - a_{n-1,x}$  points of  $\mathbf{P}^n$ . Hence  $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2E}(x-1)) = 0$  and thus  $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) = 0$ . By Remark 2.4 and Lemma 2.3 for fixed  $S'$  we may take  $B \cup E$  general in  $H$ . Hence for fixed  $S' \cup E$  we may take  $B$  general in  $H$ . By the generality of  $B$ ,  $S' \cap H = \emptyset$  and  $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) = 0$ , to prove  $h^1(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\} \cup B}(x-1)) = 0$  and hence the Claim it is sufficient to prove the inequality  $h^0(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E, H\}}(x-1)) - h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) \geq \sharp(B)$  (see e.g. [6], Lemma 3), i.e. the

inequality

$$\binom{n+x-1}{n} - (n+1)c_{n,x} + (n+1)(a_{n-1,x} + b_{n-1,x}) \geq h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) + a_{n-1,x} \quad (9)$$

First assume  $c_{n,x} - a_{n-1,x} - b_{n-1,x} \leq a_{n,x-2}$ . Then  $h^1(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = 0$  and hence  $h^0(\mathbf{P}^n, \mathcal{I}_{2S' \cup 2\{E,H\}}(x-1)) - h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = \binom{n+x-2}{n-1} - nb_{n-1,x}$ . Hence the inequality (9) is satisfied in this case. Now assume  $c_{n,x} - a_{n-1,x} - b_{n-1,x} > a_{n,x-2}$ . Hence  $h^0(\mathbf{P}^n, \mathcal{I}_{2S'}(x-2)) = 0$ . Hence the inequality (9) is satisfied by Lemma 2.7, proving the Claim. Set  $A := Y \cup H$ . Since  $S \cap H = \emptyset$  and  $c_{n,x} > a_{n-1,x}$ , at least one of the points of  $S$  is an isolated singular point of  $A$ . Hence a general  $E \in |\mathcal{I}_{2S \cup 2B}(x)|$  has at least one isolated singular point and this point is contained in  $S \cup B$ , i.e. in a set  $T$  such that  $h^1(\mathbf{P}^n, \mathcal{I}_{2T}(x)) = 0$  and  $\sharp(T) = c_{n,x}$ . Hence we may apply the semicontinuity theorem for cohomology and the openness of smoothness to obtain that for a general  $G \subset \mathbf{P}^n$  such that  $\sharp(G) = c_{n,x}$ , the linear system  $|\mathcal{I}_{2G}(x)|$  has the expected dimension and a general member of it has an isolated singularity at one point of  $G$ . We conclude by [7], Th. 1.4.  $\square$

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