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Exact and stable least squares solution to the linear programming problem

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Abstract: A linear programming problem is transformed to the finding an element of polyhedron with the minimal norm. According to A.Cline [6], the problem is equivalent to the least squares problem on positive ortant. An orthogonal method for solving the problem is used. This method was presented earlier by the author and it is based on the highly developed least squares technique. First of all, the method is meant for solving unstable and degenerate problems. A new version of the artifical basis method (M-method) is presented. Also, the solving of linear inequality systems is considered.

Keywords: Linear programming, method of least squares, M-method
MSC (2000): 90C05, 65K05

1 Introduction

The least squares method is used in mechanics, physics, statistics, but not in linear programming. The application of this universal method in mathematical programming is the main purpose of this paper.

We consider the standard linear program

$$\begin{aligned} &\min\{z = (c, x)\}, \\ &s.t. Ax = b, x \geq 0 \end{aligned} \tag{1}$$

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and its dual

$$\max\{w = (b, y)\},\\ s.t. A^T y \leq c^T, \tag{2}$$

where A is an $m \times n$ matrix, b and y are $m-$vectors, c and x are $n-$vectors.

In the papers [1, 2] two finite approximate methods of solving linear programming problem are considered. In one of them the least squares method is applied to the system

$$Ax = b$$

$$\epsilon(c, x) = \epsilon z_0$$

$$x \geq 0,$$

where $z_0 \leq z_{min}, \epsilon > 0$. Another algorithm finds the least squares solution $x(\epsilon)$ to the system

$$Ax = b$$

$$\epsilon x = -c^T$$

$$x \geq 0.$$

It is proved that $x(\epsilon) \rightarrow\ x*$ if $\epsilon \mapsto 0$ where $x*$ is a solution of the problem (1).

In this paper an exact orthogonal method for solving the linear programming problem is presented. This method is based on the author's previous work where the problem is solved by the orthogonal method [3]. This method can be applied directly if the minimum z_{min} of the objective function of the problem (1) is known. In this case the system

$$Ax = b$$

$$(c, x) = z_{min}$$

$$x \geq 0$$

has to be solved using least squares method described in [3].

In Section 2 we show that by shifting coordinates the optimal solution of dual problem has minimal norm.

In Section 3 we describe the solving of the linear programming problem by the method of least squares. A detailed description of algorithm VD is given. It is based on the QR decomposition of coefficients matrix D. If in the solution process of the system with a triangular matrix R some variable proves to be nonpositive, then the column corresponding to this partcular variable is eliminated from the matrix R and other columns are transformed again to the triangular form using Givens rotations.

In Section 4 the similarity of the presented method with the classical M-method is described and the determining of the shift parameter is discussed.

In Section 5 a solution of the system of linear inequalities as an element of polyhedron with minimal norm is found. The solving of the linear programming problem, when optimal value of the objective function is known, is considered. While computing, the linear least squares techniques which is thoroughly described in [4, 5, 6], is used.

2 Proof of the solving method

Assumption 2.1. The right hand sides of the initial problem are non-negative:

$$b(i) \geq 0, i = 1, ..., m \tag{3}$$

Assumption 2.2. The constraints of the problem (1) are not contradictory.

If the constraints of the dual problem (2) are not contradictory, then algorithm VD described below assumes that the optimal solution $y*$ of the dual problem has minimal norm among feasible solutions $\{y : A^T y \leq c^T\}$, $y*$ is the normal solution. If the optimal solution to the dual problem is not unique, then denote $y*$ the optimal solution which has minimal norm. This is always uniquely determined. In order to achieve that the solution $y*$ to the dual problem has minimal norm, transform the coordinates by shifting

$$y_i = y_i^{'} + tb_i, i = 1, ..., m, \tag{4}$$

where t is a sufficiently large positive number. In this case the dual problem takes the form

$$\max\{w^{'} = (b, y^{'})\} \ s.t.(A_{.j}, y^{'}) \leq c_j - t(A_{.j}, b) = c_j^{'} \tag{5}$$

$$j = 1, ..., n, \ t > 0,$$

where $A.j$ denotes a column of the matrix A and constant additive terms in the objective function are dropped (see example 2.1). Let us show that if the dual problem has a bounded optimal solution then for the sufficiently large shift (t) the vector with minimal norm is a solution to this problem. For this reason it is possible to consider as the objective function the expression

$$v = \left\| y^{'} \right\|^2 = \|y - tb\|^2 . \tag{6}$$

Let $y_t^{'}$ be an optimal solution to the problem with such an objective function and $y^{'*}$ an optimal solution to the problem (5). Transfer the objective function (6),

$$max\{-v/t = -(y, y)/t + 2(b, y) - t(b, b)\}.$$

According to [7] (Theorem 1,Part 3, Ch.10) there exists a number t_0 such that for each $t \geq t_0$ the equality $y_t^{'} = y^{'*}$ holds.

In the paper [8] the following theorem is proved.

Theorem 2.3. For any matrix A and vector $b \neq 0$ only one of the two following statements is valid:

a) there exists a non-negative solution to the system

$$Ax = b, \tag{7}$$

b) the system of inequalities

$$A^T y \leq 0, \ (y, b) > 0 \tag{8}$$

holds.

Set a least squares problem with the aid of which the problems (1) and (2) are solved:

$$Au = 0, \ (c, u) = -1, \ u \geq 0, \tag{9}$$

where u is $n-$ vector. In more compact way this system is written as follows:

$$Du = f, \ u \geq 0, \tag{10}$$

where

$$D = (A, c)^T, \ f = (0, 0, ..., 0, -1)^T, \ r = Du - f.$$

Due to the formula (7) the following theorem can be proved.

Theorem 2.4. The system (9) has a solution ($r = 0$) if and only if the objective function of the initial problem is unbounded.

Proof. If $r = 0$ then substituting an infinitely small negative number $-M$ for the right side of the last equation of system (9),we obtain a solution which differs from the previous one M times. Therefore, the condition $r = 0$ is equivalent to the unboundedness of the objective function of the problem

$$min\{z = (c, x)\}$$
$$s.t. Ax = 0$$
$$x \geq 0.$$

Let x^0 be an arbitary feasible basic solution. Then substituting $u_j = x_j - x_j^0$ for the basic variables and $u_j = x_j - x_j^0$ for the other variables we get that the objective function of the initial problem is unbounded. □

Remark 2.5. From the proof above it can be seen that in the system (9) the right hand side of the last equation can be taken to be an arbitrary negative number.

Remark 2.6. Substituting matrix D for matrix A in the first theorem, and if (8) is valid, we get for the dual problem

$$A^T y + y_{m+1} c^T \leq 0$$
$$-y_{m+1} > 0.$$

It means that the set of feasible solutions is not empty.

Assumption 2.7. The optimal solution $y*$ of the dual problem is normal (feasible solution with the minimal norm) if $y*$ is unique. If the optimal solution to the problem (2) is not unique then by $y*$ the normal solution is denoted.

Theorem 2.8. Let $\hat{u}$ be the least squares solution of the system (9) and $r = D\hat{u} - f \neq 0$. If the optimal solution to the dual problem is normal, then it can be expressed in the form

$$\hat{y} = -\frac{A\hat{u}}{\|r\|^2} = -\frac{A\hat{u}}{r_{m+1}} \tag{11}$$

Proof. Write the problem (10) in the form

$$min\{\varphi(u) = \|Du - f\|^2 /2\} \tag{12}$$

Find the gradient

$$\varphi' = D^T(Du - f) = D^T r. \tag{13}$$

Suppose that the least squares solution $\hat{u}$ to the problem (9) is determined by the k first components

$$\varphi_i'(\hat{u}) = 0,\ \hat{u_i} \geq 0,\ i = 1, ..., k,\ \varphi_i'(\hat{u}\ ) \geq 0,\ \hat{u_i} = 0,\ i = k+1, ..., n. \tag{14}$$

According to the assumptions $r \neq 0$, due to formulas (13) and (14) we have

$$\|r\|^2 = (r, D\hat{u}) - (r, f) = (\varphi'(\hat{u}), \hat{u}) + r_{m+1} = r_{m+1} > 0. \tag{15}$$

Let us show that $\hat{y}$ determined by (11) satisfies the constraints $1, 2, ..., k$ of the problem (2) as equalities. The system of normal equations of (9) is

$$\sum_{j=1}^{j=k}[(A_{.i}, A_{.j}) + c_i c_j]u_j = -c_i,\ i = 1, ..., k \tag{16}$$

or after some transformation

$$\sum_{j=1}^{j=k}(A_{.i}, A_{.j})u_j = c_i[-1 - \sum_{j=1}^{j=k} c_j u_j],\ i = 1, ..., k$$

Substituting $\hat{y}$ which is determined by (11) to the ith constraint of the dual problem (2) we have due to the last equation and the formula $r = Du - f$

$$a_{1i}\hat{y_i} + ... + a_{mi}\hat{y_m} = -\frac{1}{r_{m+1}}[a_{1i}r_1 + ... + a_{mi}r_m] = -\frac{1}{1 + c_1u_1 + ... + c_ku_k}\times$$
$$\times[a_{1i}(a_{11}u_1 + ... + a_{mi}(a_{m1}u_1 + ... + a_{mk}(a_{m1}u_1 + ... + a_{mk}u_k)] =$$
$$= -\frac{1}{1 + c_1u_1 + ... + c_ku_k}[c_i(-1 - c_1u_1 - ... - c_ku_k)] = c_i,\ i = 1, ...k.$$

It means that k first constraints of the dual problem are satisfied as equations.The rest of constraints are satisfied due to the formulas (13)-(15), because

$$(c^T - A^T\hat{y})\,\|r\|^2 = D^T r = \varphi'(\hat{u}) \geq 0.$$

The vector determined by (11) $-\hat{y}$ is a linear combination of the rows of the matrix A^T:

$$-\hat{y} = \frac{A\hat{u}}{r_{m+1}} = \frac{A\hat{u}}{\|r\|^2} \tag{17}$$

Let us consider a least squares problem

$$min\{\|y\|^2/2\}, s.t.\ A^T y \leq c^T \tag{18}$$

The antigradient of this function $-y$ can be due to (17) presented as a linear combination of the rows of the matrix A^T where all the coefficients of this combination are non-negative. In addition the conditions of complementary slackness are fulfilled. So the vector $\hat{y}$ is a least squares solution to the problem (18) and at the same time under the assumption 2.2 the optimal solution to the LP problem (2). □

Remark 2.9. The optimal solution to the initial problem (1) can be found by solving the least squares problem

$$A_{.1}x_1 + ... + A_{.k}x_k = b. \tag{19}$$

Here x_j are the variables for which $\varphi_j'(\hat{u}) = 0$. Such variables x_j and $\hat{u_j}$ are called active.

Remark 2.10. In the problem (18) at least k first constraints are equalities, so $1 \leq k \leq m$.
If $k = m$ then the problem (19) can be solved by the Gaussian elimination method.

Remark 2.11. If the least squares solution $\hat{u}$ is not unique, then in (11) an arbitrary solution can be used (see Example 5.2). The normal solution $\hat{y}$ is always unique.

Example 2.12.

$$min\{2x_1 + 4x_2 + x_3 + 4x_4 = z\}$$

$$-x_1 + 2x_2 + x_3 + x_4 = 1$$

$$x_1 - x_2 - 2x_3 + 2x_4 = 2$$

$$x \geq 0.$$

The optimal solution to the initial problem $x* = (0, 0, 0, 1)^T, z_{min} = 4$. The optimal solution to the dual problem is $y* = (4 - 2p, p)^T, 4/5 \leq p \leq 2$. In the transformed problem choose the shifting parameter $t = 2$, new origin of coordinates $O' = (2, 4)^T$, then $y_1 = y_1' + 2, y_2 = y_2' + 4$. Consider the transformed dual problem according to the formula (5),

$$
\begin{aligned}
&\max\{y_1' + 2y_2' = w'\}\\
&-y_1' + y_2' \le 2 - t = 0\\
&2y_1' - y_2' \le 4 - 0t = 4\\
&y_1' - 2y_2' \le 1 + 3t = 7\\
&y_1' + 2y_2' \le 4 - 5t = -6.
\end{aligned}
\tag{20}
$$

Consider the least squares problem (9):

$$
\begin{aligned}
-u_1 + 2u_2 + u_3 + u_4 &= 0\\
u_1 - u_2 - 2u_3 + 2u_4 &= 0\\
4u_2 + 7u_3 - 6u_4 &= -1\\
u &\ge 0.
\end{aligned}
$$

The least squares solution to this problem is $\hat{u} = (0, 0, 0, 6/41)^T$. According to the formulas (10) and (11) find the optimal solution to the transformed dual problem

$$r = D\hat{u} - f = (6/41, 12/41, 5/41)^T, \hat{y}' = (-6/5, -12/5)^T.$$

General form of the optimal solution to the transformed dual problem is

$y'^* = (2 - 2p, p - 4), 4/5 \le p \le 2$. The norm of this vector achieves its minimum at $p = 8/5$, if $y'^* = \hat{y}$. Shifting all the coordinates $\hat{y}'$ by tb we find the optimal solution $y^* = (4/5, 8/5)^T$ to the initial dual problem, which corresponds to $\hat{y}'$. Using least squares method the optimal solution to the initial problem can be found from the system

$$
\begin{aligned}
x_4 &= 1\\
2x_4 &= 2
\end{aligned}
$$

containing for $k \le m$ variables for which $\varphi_i'(\hat{u}) = 0$. It follows from the conditions of complementary slackness. If in the case of the optimal solution of the dual problem exactly m constraints occur to be equalities, then optimal solution to the initial problem can be found by the Gaussian elimination method from the system (19).

If the initial problem is **contradictory** then the system (19) has no non-negative solution. E.g., this is the case if all $(A_j, b) < 0$, $j = 1, ..., n$.

In Section 5 solving LP problem is considered if the optimal value of the objective function is known.

3 Description of the algorithm VD

Describe the algorithm VD for solving the pair of dual problems (1) and (2). Let us write the least squares problem (9) in the form

$$Du = f, u \geq 0,$$

where D is an $(m+1) \times n$ matrix. In practical computations it is convenient to write the coefficients of the transformed problem found according to (5) into the first row of the matrix D. It quarantees somewhat greater stability of the computing process if $c_j^{'}$ are large.

Algorithm $VD(A, b, c, D, f, IJ, F, G, x, y, m, n, t)$.

1. Choose a sufficiently large shifting parameter t and calculate $c_j^{'}$ according to (5).
2. Evaluate $n-$ vectors F and G with coordinates

$$F_j = -c_j^{'}, G_j = (D_{.j}, D_{.j}),\ j = 1, ..., n.$$

3. Initiate the number of active variables $k = 0$ and $u = 0$.
4. Determine the following active variable $u(j0)$ by solving the problem

$$max\{\frac{F_j^2}{G_j} = \frac{F_{j0}^2}{G_{j0}} = Re,\}$$

where the maximum is found for all passive (i.e. $u_j = 0$) variables satisfying inequality $G_j > 0$.

5. If $Re \leq 0$, then go to Step 23.
6. Increase number of active variables, $k = k + 1$.
7. Write index $j0$ into array IJ (active variables).
8. If $k = m + 1$ then go to Step 10.
9. Fulfil Householder transformation with an $(m + 2 - k)-$vector $v = D_{.j0}$ to D and f, see [6] ch.10.
10. Compute new

$$F_j = F_j - D_{kj} f_k,\ G_j = G_j - D_{kj}^2, j = 1, ..., n.$$

11. Solve the triangular system $Ru = f$ of order k to determine the active variables.
12. Let $L = k + 1$ (where L denotes the number of active variables being verified).
13. Let $L = L - 1$.
14. If $1 \leq L$, then go to step 16.
15. If $k < m + 1$, then go to step 4 else go to step 23.
16. Let $j = IJ(L)$.
17. If $u_j > 0$, then go to Step 13.
18. Let $u_j = 0$ and delete index j from the set IJ.
19. Transform the active columns D into the triangular form by the Givens rotations.
20. Compute new

$$F_j = F_j + D_{kj} f_k,\ G_j = G_j + D_{kj}^2, j = 1, ..., n$$

21. Decrease the number of active variables, $k = k - 1$.
22. Go to Step 11.
23. Compute $r = Du - f$.
24. If $r = 0$ then z is unbounded and the dual problem (2) is contradictory. Stop.
25. Solve by the least squares method system $\hat{Ax} = b$, composed for all active variables $u_j, j \in IJ$.
26. If system $\hat{Ax} = b$ has a solution, satisfying all equations, then it is the solution of the problem (1).
27. If system $\hat{Ax} = b$ has no solution, then problem (1) is contradictory. Stop.
28. Compute solution of the dual problem (2), $y = -r/r_{m+1}$.
29. The problems (1) and (2) are solved.

Remark 3.1. If, for example, all products $(A_j, b) < 0, j = 1, ..., n$, then by sufficient big shifting parameter t in (5) system $\hat{Ax} = b$ has no solution (Step 25).

Remark 3.2. The advantage of the algorithm VD is the following: there is no need to calculate matrices of orthogonal transformations.

Remark 3.3. In the algorithm VD the number of steps is finite as at each step the calculations according to (12) give minima of squares' sum in a subspace, which number is finite [3, 6].

Example 3.4.

$$min\{z = -x_1 - 6x_2 + x_3\}$$

$$s.t. \quad 8x_2 + 2x_3 = 16$$

$$2x_1 \qquad + 2x_3 = 8$$

$$x \geq 0.$$

Consider the least squares problem (9):

$$8u_2 + 2u_3 = 0$$

$$2u_1 \qquad + 2u_3 = 0$$

$$-u_1 - \ 6u_2 + \ u_3 = -1$$

$$u \geq 0.$$

The solution in least squares is $\hat{u} = (8/58, 3/58, 0)^T$. According to the formulas (10) and (11) find the optimal solution to the dual problem, $r = (24/58, 16/58, 32/58)^T, y* = (-0.75, -0.50)^T$. In this problem shifting is not needed, because $y*$ is a normal solution. On the first step the variable u_2 and on the second step u_1 will be active.

Iteration	u_1	u_2	u_3	f
1	0	8	2	0
	2	0	2	0
	-1	-6	1	-1
F	1	6	-1	
G	5	100	9	
u	0	0	0	
2	-0,600	-10	-1	-0,600
	2	0	2	0
	-0,800	0	2	-0,800
F	0,640	0	-1,600	
G	4,64	0	8	
u	0	0,06	0	
3	-0,600	-10	-1	-0,600
	-2,154	0	-1,116	-0,297
	0	0	2,600	-0,742
F	0	0	-1,930	
G	0	0	2,600	
u	0,138	0,052	0	

4 The artifical-basis and shifting

In the former section we shifted coordinates to minimize the norm of the optimal solution to the dual problem. Now, we demonstrate the correspondence between the shift and the transformation of the primal problem in case of classical M-method. To explain this we reconsider Example 2.12. We use the penalty coefficient vector tb and artifical variables v.

Example 4.1.

$$min\{2x_1 + 4x_2 + x_3 + 4x_4 + tv_1 + 2tv_2 = z\}$$

$$-x_1 + 2x_2 + x_3 + x_4 + v_1 \quad = 1$$

$$x_1 - x_2 - 2x_3 + 2x_4 \quad + v_2 \; = 2$$

$$x \geq 0, v \geq 0.$$

Let's eliminate the artifical variables v from the objective function,

$$z(t) = (2-t)x_1 + (4-2t)x_2 + (1+3t)x_3 + (4-5t)x_4 - 5t.$$

At each value of the parameter t the coefficients of this function equal to the right sides of the dual problem (20). Therefore the penalty coefficients tb_i correspond to the coordinate shift in the dual problem. In case of penalty coefficients tb_i (in contrast to equal penalty M) the number of steps of the simplex method may be smaller as the coefficients of the objective function $z(t)$ depend both on primal objective function and the right sides of the constraints. This notion is affirmed by the solved examples. Knowing the classical penalty coefficient M and assuming positivity of all right sides $b_i > 0, i = 1, ..., m$, the parameter t has to be chosen large enough to satisfy the conditions $tb_i > M, i = 1, ..., m$. There is no universal method for determining the most suitable value, neither for M nor t. When a feasible basic solution to the primal problem is known, the dual problem includes constraints $y \leq 0$. As according to the assumption dual objective vector is $b \geq 0$, i.e. in "opposite direction" to the negative ortant, the shift parameter should not be very large. In most cases of random coefficients no shift in the dual problem was necessary if a feasible basic solution existed. Solution of (10) using algorithm VD is voluminous as all coefficients in the system are transforming at each step. There is a similarity to the primal simplex method. It is possible to derive an algorithm similar to the revised simplex method where orthogonal transformations are used and memorized as products [6, ch 24]. The presented algorithm VD is slower than the widely used revised simplex method, that has been perfected for over 50 years. The main disadvantage of the algorithm VD is the large amount of memory capacity needed. One can use it to solve comparatively small unstable problems. For large-size sparse problems the well-known least squares technique should be used. The biggest advantage of the algorithm VD is precision as seen in the following example 4.2. The optimal solution even to the degenerate and unstable problem could be found according to (19) using least squares method.

Example 4.2. Let us consider a linear programming problem with Hilbert matrix $a(i,j) = 1/(i+j)$, $b(i) = 1/(i+1) + 1/(i+2) + ... + 1/(i+m)$, $a(i, m+i) = 1$, the rest of elements $a(i,j) = 0$, $c(i) = -b(i) - 1/(i+1)$, $c(m+i) = 0$, $x*(i) = 1, x*(m+i) = 0, i, j, = 1, ..., m$. Well-known programs solve this problem only for $m \leq 8$. The algorithm VD found solution to this problem for $m \leq 12$.

5 Solving the system of linear inequalities by the least squares method

In the system of inequalities

$$Ey \leq h \tag{21}$$

E is a $n \times m$ matrix, h a $n-$vector and y an $m-$vector, $m \leq n$ or $m > n$. We will consider first whether this system of inequalities holds. If the vector $y = 0$ is not a solution to this system apply the algorithm VD. For this purpose we set a dual problem taking the coefficients of the objective function of the initial problem equal to zero.

$$min\{z = (h, u)\},\ E^T u = 0,\ u \geq 0. \tag{22}$$

It is known that the system (21) does not hold if and only if the goal function of the problem (22) is unbounded. Set the least squares problem (10)

$$Du = f,\ u \geq 0, \tag{23}$$

$$D = (E^T h^T), f = (0, 0, ..., 0, -1)^T, r = Du - f.$$

Theorem 5.1. The system of inequalities (21) does not hold if and only if the problem (23) has a solution $\hat{u}$ satisfying all equations, $D\hat{u} - f = r = 0$. If $r \neq 0$ and $\hat{u}$ is the solution of the problem (23) in least squares then $\hat{y} = -E^T\hat{u}/r_{m+1}$ is the solution with minimal norm to the system (21).

Proof follows directly from the theorems 2.4 and 2.8.

Example 5.2. The least squares problem

$$u_1 - 2u_2 = 0$$
$$u_1 - 2u_2 = 0$$
$$2u_1 - 8u_2 = -1$$
$$u \geq 0$$

corresponds to the inequalities

$$y_1 + y_2 \leq 2$$
$$-2y_1 - 2y_2 \leq -8$$

A solution of the least squares problem is $\hat{u} = (2/4, 1/4)^T, r = 0$ and the system of inequalities is contradictory. If the right hand side of the second inequality is taken to be -4 then the least squares solution is not unique, $\hat{u} = (c, c/2 + 1/6)^T, r = (-1/3, -1/3, 1/3)^T, \hat{y} = (1, 1)^T$ is the normal solution. Finally let us consider solving the LP problem if the maximum value of the objective function is known,

$$max\{w = (b, y)\}$$
$$A^T y \leq c^T.$$

Set the system of inequalities

$$A^T y \leq c^T$$
$$-(b, y) \leq -w_{max}$$

and find with the aid of the formula (11) a solution to this system.

Example 5.3.

$$max\{w = 2y_1 + y_2\}$$

$$-y_1 + y_2 \leq 1$$

$$y_1 \quad \leq 1.$$

Add the inequality $-2y_1 - y_2 \leq -4$ to the constraints and find the least squares solution $\hat{u} = (0, 3/6, 2/6)^T, r = (-1/6, -2/6, 1/6)^T$ to the system

$$-u_1 + u_2 - 2u_3 = 0$$

$$u_1 \qquad - u_3 = 0$$

$$u_1 + u_2 - 4u_3 = -1$$

$$u \geq 0.$$

Due to the formula (11) $y* = (1, 2)^T$.

6 Conclusions

In this paper some relations between linear programming and least squares method are considered. The algorithm VD presented uses orthogonal transformations and there is no need to calculate the respective matrices. Based on examples solved it is clear that the number of steps of the algorithm is not significantly larger than the number of constraints and negative variables occur quite seldom. Thus it was in the solved examples for $m \leq 200$. There is some need for additional consideration of multiplying the constraints of the LP problem by constants and the influence of this procedure to the stability of the least squares solution and shift parameter.

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