Central European Science Journals
www.cesj.com

Central European Journal of Mathematics
Central European Science Journals
CEJM 3(2) 2005 260–272

Multiple Prime covers of the Riemann sphere

Aaron Wootton*

Department of Mathematics,
University of Arizona,
617 North Santa Rita, Tucson AZ85721, USA

Received 14 June 2004; accepted 8 March 2005

Abstract: A compact Riemann surface X of genus $g \geqslant 2$ which admits a cyclic group of automorphisms C_q of prime order q such that X/C_q has genus 0 is called a cyclic q-gonal surface. If a q-gonal surface X is also p-gonal for some prime $p \neq q$, then X is called a multiple prime surface. In this paper, we classify all multiple prime surfaces. A consequence of this classification is a proof of the fact that a cyclic q-gonal surface can be cyclic p-gonal for at most one other prime p.

Keywords: Automorphism group, compact Riemann surface, hyperelliptic curve
MSC (2000): 14H30, 14H37, 30F10, 30F60, 20H10

1 Introduction

A compact Riemann surface X of genus $g \geqslant 2$ which admits a cyclic group of automorphisms C_q of prime order q such that X/C_q has genus 0 is called a **cyclic q-gonal surface** or a q-gonal surface for brevity. The group C_q is called a **q-gonal group for** X. If in addition C_q is normal in the full automorphism group of X, then we call X a **normal cyclic q-gonal surface** or a normal q-gonal surface. As they are the central focus of this paper, we call a surface X which is both cyclic q-gonal and cyclic p-gonal for primes $p \neq q$ a **multiple prime surface**. The primary aim of this paper is to classify all multiple prime surfaces. By classify, we mean find the full automorphism group and the signature for the normalizer of a surface group for each such surface. There are a number of interesting consequences of this classification, two of the most interesting are the following.

* E-mail: awootton@math.arizona.edu

Theorem. A cyclic q-gonal surface can be p-gonal for at most one other prime p.

Theorem. If X is a multiple prime surface which is cyclic q-gonal and cyclic p-gonal, then any element from a q-gonal group commutes with any element from a p-gonal group.

We start in Section 2 by developing a number of general results regarding automorphism groups of compact Riemann surfaces, uniformization and Fuchsian groups - discrete subgroups of $\mathrm{PSL}(2, \mathbb{R})$. Following this in Section 3, we shall examine a number of preliminary results more specific to the situation we are considering. Section 4 will present one of the main steps needed for this classification. Specifically, we shall show that the group generated by a q-gonal and p-gonal group in the full automorphism group of a multiple prime surface X is cyclic of order pq. With these results, in Section 5 we shall complete the classification and prove explicitly that a cyclic q-gonal surface can be p-gonal for at most one other prime p. The author would like to express his gratitude to the referees for their useful comments and in particular for their input toward a complete proof of Theorem 4.5.

2 Fuchsian Groups and Uniformization

In this section, we develop the necessary theory regarding automorphism groups of compact Riemann surfaces, uniformization and Fuchsian groups. Let X denote a compact Riemann surface of genus $g \geqslant 2$ and G a group of automorphisms of X.

Uniformization implies that X is conformally equivalent to a quotient of the upper half plane $\mathbb{H}$ by a torsion free Fuchsian group Λ called a **surface group** or a **surface kernel** for X. Under such a realization, a group G is a group of automorphisms of X if and only if $G = \Gamma/\Lambda$ for some Fuchsian group Γ containing Λ as a normal subgroup, see [5]. It follows that the full automorphism group $\mathrm{Aut}(X)$ of X is the quotient group $N(\Lambda)/\Lambda$ where $N(\Lambda)$ denotes the normalizer of Λ in $\mathrm{PSL}(2, \mathbb{R})$.

Since G is a group of automorphisms acting on X, we can form the quotient space X/G which can be endowed with a unique structure making the map $\pi_G \colon X \to X/G$ a holomorphic map between compact Riemann surfaces. The group Γ is a group of biholomorphic maps acting on $\mathbb{H}$, so we can form the quotient space $\mathbb{H}/\Gamma$ with a complex structure so that the map $\pi_\Gamma \colon \mathbb{H} \to \mathbb{H}/\Gamma$ is holomorphic. Let $\pi_\Lambda \colon \mathbb{H} \to \mathbb{H}/\Lambda = X$ denote the smooth unramified cover of X by $\mathbb{H}$. Then X/G can be identified with $\mathbb{H}/\Gamma$ and after identification, we have $\pi_\Gamma = \pi_G \circ \pi_\Lambda$:

$$\mathbb{H} \xrightarrow{\pi_\Lambda} \mathbb{H}/\Lambda = X \xrightarrow{\pi_G} \mathbb{H}/\Gamma = X/G, \qquad \pi_\Gamma \colon \mathbb{H} \to \mathbb{H}/\Gamma = X/G$$

Fig. 1 Holomorphic quotient maps and surface identifications.

For a Fuchsian group Γ with compact orbit space $\mathbb{H}/\Gamma$ of genus g, a presentation for

Γ is:

$$\Gamma = \langle a_1, b_1, \ldots, a_g, b_g, c_1, \ldots, c_r | c_1^{m_1}, \ldots, c_r^{m_r}, \prod_{i=1}^{r} c_i \prod_{j=1}^{g} [a_j, b_j] \rangle$$

where the quotient map π_Γ branches over r points with ramification indices m_i for $1 \leqslant i \leqslant r$. The presentation of such a group is described by the tuple $(g; m_1, \ldots, m_r)$ called the signature of Γ, g the orbit genus of Γ and $m_1, \ldots, m_r$ the periods of Γ. Notice that if Γ is a surface group for a surface of genus g, since it is torsion free, it must have signature $(g; -)$. Given a tuple $(g; m_1, \ldots, m_r)$, it is natural to ask when this tuple is the signature for some Fuchsian group Γ. The statement of the answer to this question dates back to Poincaré and a complete rigorous proof was published by Maskit in [6]. It was shown that a tuple $(g; m_1, \ldots, m_r)$ is the signature for a Fuchsian group Γ if and only if it satisfies the inequality $2g - 2 + \sum_{i=1}^{r}(1 - \frac{1}{m_i}) > 0$. Among other interesting consequences, this result can be used to prove that the automorphism group G of a compact Riemann surface of genus $g \geqslant 2$ satisfies $|G| \leqslant 84(g-1)$, a bound often referred to as the Hurwitz bound.

We shall now interpret this information into results we shall be using. If X is a cyclic q-gonal surface, let C_q be a q-gonal group for X and if Λ is a surface group for X, let Γ_q denote the Fuchsian group with $\Gamma_q/\Lambda = C_q$. Let G denote a subgroup of the normalizer in $\mathrm{Aut}\,(X)$ of C_q and let Γ be the Fuchsian group with $\Gamma/\Lambda = G$. Since C_q is normal in G, it follows that the group $K = G/C_q$ acts by automorphism on the quotient space X/C_q. Let π_K denote the quotient map of the surface X/C_q by K. After appropriate identifications, we get the tower of Galois covers illustrated in Figure 2. Since π_Λ is unramified, our remarks imply that Γ_q has signature $(0; \underbrace{q, \ldots, q}_{r \text{ times}})$ where r is the number of branch points of the map π_{C_q}. With a little more work, we can find the the signature of Γ.

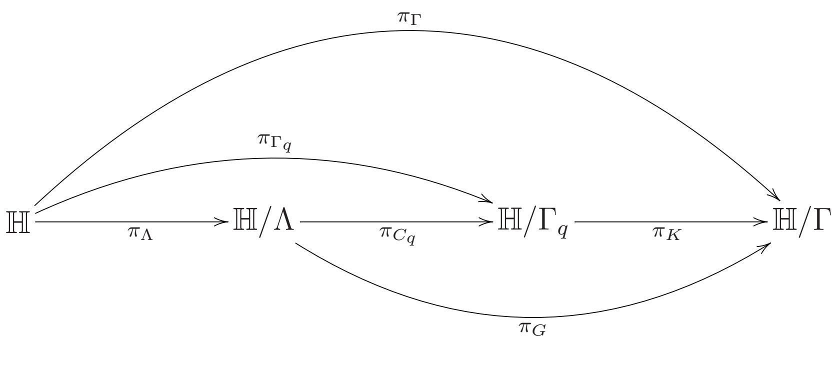

Fig. 2 Holomorphic quotient maps and surface identifications.

The map π_K is a Galois map from the Riemann sphere to itself and the branching properties of such maps are well known. We summarize them in Table 1. The branching data is a vector whose length is the number of branch points of the quotient map π_K and whose entries are the ramification indices of ramification points above these branch points.

To find the possible signatures for Γ, we use the fact that we know complete branching

Group	Branching Data
C_n	(n, n)
D_n	$(2, 2, n)$
A_4	$(2, 3, 3)$
S_4	$(2, 3, 4)$
A_5	$(2, 3, 5)$

Table 1 Groups of automorphisms of the Riemann sphere and branching data.

data of the maps π_K and π_{C_q}. It is then a simple matter of determining whether or not any branch points of π_{C_q} coincide with any ramification points of π_K. We summarize below.

Proposition 2.1. Let K be the group Γ/Γ_q.

(i) If $K \neq C_n$ and (m_1, m_2, m_3) is the branching data of the quotient map π_K, the signature of Γ is $(0; am_1, bm_2, cm_3, \underbrace{q, \ldots, q}_{s \text{ times}})$ where a, b, and c are either 1 or q depending upon whether any branch points of π_{C_q} coincide with ramification points of π_K. For such a Γ, the signature of Γ_q is $(0; \underbrace{q \ldots, q}_{r \text{ times}})$ where

$$r = s|K| + \frac{(a-1)|K|}{(q-1)m_1} + \frac{(b-1)|K|}{(q-1)m_2} + \frac{(c-1)|K|}{(q-1)m_3}.$$

(ii) If $K = C_n$, the signature of Γ is $(0; an, bn, \underbrace{q, \ldots, q}_{s \text{ times}})$ where a, and b are either 1 or q depending upon whether any branch points of π_{C_q} coincide with ramification points of π_K . For such a Γ, the signature of Γ_q is $(0; \underbrace{q, \ldots, q}_{r \text{ times}})$ where

$$r = sn + \frac{(a-1)|K|}{(q-1)n} + \frac{(b-1)|K|}{(q-1)n}.$$

Proof. See [8], Proposition 3.1. □

3 Preliminary Results

Automorphism groups of compact Riemann surfaces and in particular, cyclic q-gonal surfaces, have been the focus of much research in the last century. In this section, we shall present such results which are specific to the problem we are considering. The first result we examine restricts the different genera we need to consider when studying such surfaces.

Lemma 3.1. If X is cyclic q-gonal with q-gonal group C_q and the quotient map $\pi_{C_q}: X \to X/C_q$ is branched over r points, then the genus of X is $(\frac{r}{2}-1)(q-1)$. In particular, the smallest genus (greater or equal to 2) for which such a group can exist is $g = \frac{1}{2}(q-1)$ for $q \geqslant 5$ and 2 for $q = 2$ or 3.

Proof. The group Γ_q will have signature $(0; \underbrace{q, \ldots, q}_{r \text{ times}})$. Since the surface group Λ is torsion free, all elliptic generators will have non-trivial image in the quotient group $C_q = \Gamma_q/\Lambda$. Using the Riemann-Hurwitz formula, we get

$$g - 1 = -q + \frac{q}{2}\sum_{i=1}^{r}(1 - \frac{1}{q}) = -q + \frac{q}{2}r(1 - \frac{1}{q}) = -q + \frac{r}{2}(q-1),$$

and thus

$$g = -q + 1 + \frac{r}{2}(q-1) = -(q-1) + \frac{r}{2}(q-1) = (\frac{r}{2} - 1)(q-1).$$

□

The main results we produce rely heavily on the fact that if the size of the automorphism group of a surface is sufficiently large in relation to its genus, then there are a very small number of possibilities for the signature of the normalizer for a surface group for such a surface.

Lemma 3.2. Suppose that G is an automorphism group of a compact Riemann surface X of genus g and $|G| \geqslant 13(g-1)$. If Λ is a surface group for X and Γ the Fuchsian group with $\Gamma/\Lambda = G$, then Γ is a triangle group - a Fuchsian group whose signature has three periods and orbit genus 0 - with one of the signatures tabulated in Table 3.

Case	Signature	Additional Conditions
1	$(0; 3, 3, n)$	$4 \leqslant n \leqslant 5$
2	$(0; 2, 6, 6)$	
3	$(0; 2, 5, 5)$	
4	$(0; 2, 4, n)$	$5 \leqslant n \leqslant 10$
5	$(0; 2, 3, n)$	$7 \leqslant n \leqslant 78$

Table 2 Signatures for large automorphism groups.

Proof. We extend the proof of Lemma 3.18 in [3]. Suppose that Γ has signature $(g_\Gamma; m_1, \ldots, m_r)$. Since $|G| \geqslant 13(g-1)$ we have

$$g - 1 = |G|(g_\Gamma - 1) + \frac{|G|}{2}\sum_{i=1}^{r}(1 - \frac{1}{m_i}) \geqslant 13(g-1)(g_\Gamma - 1) + \frac{13(g-1)}{2}\sum_{i=1}^{r}(1 - \frac{1}{m_i}).$$

Simplifying, we get

$$\frac{2}{13} \geqslant 2(g_\Gamma - 1) + \sum_{i=1}^{r}(1 - \frac{1}{m_i}).$$

From this last inequality, it is clear that $g_\Gamma = 0$. After simplifying, this gives the inequality

$$\frac{r}{2} \geqslant \sum_{i=1}^{r} \frac{1}{m_i} \geqslant r - \frac{28}{13}$$

and thus

$$5 > \frac{56}{13} \geqslant r.$$

If $r = 4$, then

$$4 > \frac{311}{78} = \frac{28}{13} + \frac{3}{2} + \frac{1}{3} \geqslant \frac{28}{13} + \sum_{i=1}^{4} \frac{1}{m_i} \geqslant 4$$

which is clearly not the case. Hence we must have $r = 3$. We now need to consider the different possible signatures.

Order the periods of Γ so that $m_1 \leqslant m_2 \leqslant m_3$. Since $r = 3$, we know that

$$\frac{1}{m_1} + \frac{1}{m_2} + \frac{1}{m_3} \geqslant \frac{11}{13}.$$

By simple calculation, this implies that $m_1 \leqslant 3$. Assume that $m_1 = 3$. It follows that

$$\frac{1}{m_2} + \frac{1}{m_3} \geqslant \frac{20}{39}.$$

Further simple calculations show that $m_2 = 3$, and under these circumstances, we have $m_3 < 6$. If we assume $m_1 = 2$ it follows that

$$\frac{1}{m_2} + \frac{1}{m_3} \geqslant \frac{9}{26}$$

and consequently $m_2 \leqslant 6$. If $m_2 = 6$, then m_3 must also be 6, if $m_2 = 5$, then m_3 is also 5, if $m_2 = 4$, then $5 \leqslant m_3 \leqslant 10$ and if $m_2 = 3$ then $7 \leqslant m_3 \leqslant 78$. □

The following result specifies conditions on the periods of the normalizer of a cyclic q-gonal surface.

Lemma 3.3. Suppose that X is a cyclic q-gonal surface and Γ is the normalizer of a surface group for X. Then the signature for Γ must have periods divisible by q and have orbit genus 0.

Proof. Since X is cyclic q-gonal, there will exist an intermediate subgroup Γ_q of Γ and a surface group Λ for X with signature $(0; q, \ldots, q)$. Since Λ is torsion free, the periods of Γ_q must be induced by (conjugates to powers of) elliptic generators of Γ. Hence Γ must have elements whose order is divisible by q and in particular must have periods divisible by q. It must have orbit genus 0 because it contains a Fuchsian group, Γ_q, whose orbit genus is 0.

The next two results are important because they relate information about the genus of a compact Riemann surface X and the structure of groups which can act on X. Both results are due to Accola, see [1] and [2].

Theorem 3.4. Let X be a compact Riemann surface of genus g. Suppose that X admits a finite group of automorphisms G with subgroups $G_1 \ldots, G_n$ such that $G = \cup_{i=1}^{n} G_i$ and $G_i \cap G_j$ is trivial for $i \neq j$. Let g_i be the genus of the surface X/G_i for $1 \leqslant i \leqslant n$ and let g_0 be the genus of the surface X/G. Then

$$(n-1)g + |G|g_0 = \sum_{i=1}^{n} |G_i| g_i.$$

Theorem 3.5. If X of genus g is cyclic q-gonal and $g > (q-1)^2$, then X is normal cyclic q-gonal.

4 The Group Generated by a p-gonal and q-gonal Group

Suppose X is a multiple prime surface which is cyclic p-gonal and q-gonal for primes $p \neq q$ and let C_p and C_q denote p-gonal and q-gonal groups for X respectively. We shall show that the group generated by C_p and C_q is cyclic of order pq through a series of Lemmas. We shall first show that the group G generated by C_p and C_q has order pq. To prove this, we shall make essential use of the fact that if G has order greater than pq, then neither C_p nor C_q can be normal in G, and in fact $C_p \cap N_G(C_q) = 1$ and $C_q \cap N_G(C_p) = 1$. Following this, we shall show that any group of order pq generated by C_p and C_q is necessarily cyclic. Without loss of generality, we shall henceforth assume that $p > q$.

Lemma 4.1. Suppose X is a multiple prime surface which is q-gonal for $q \in \{2, 3, 5, 7\}$. If X is p-gonal for $p \neq q$, then the group G generated by a p-gonal group and a q-gonal group has order pq.

Proof. By Theorem 3.5, if $g > (q-1)^2$, then X is normal q-gonal and so C_q will be normal in the full automorphism group $\mathrm{Aut}(X)$ of X. In particular, it will be normal in G and consequently the order of G must be pq. Therefore given q, by Lemma 3.1, we just need to consider surfaces of genus

$$g = \frac{n}{2}(q-1)$$

where $1 \leqslant n \leqslant 2(q-1)$. Since we are assuming $q \in \{2, 3, 5, 7\}$, this means we only need consider surfaces of genus $g \leqslant 36$. For all such genera, Breuer developed lists of all automorphism groups and corresponding signatures for Fuchsian groups in [3]. Therefore, we can proceed through these lists to explicitly show that no surfaces exist admitting automorphism groups with the specified properties. To illustrate, we shall examine the case $q = 7$ in more detail.

For $q = 7$, we need to consider surfaces of genus $3k$ where $1 \leqslant k \leqslant 12$. Assuming the result for $q \in \{2, 3, 5\}$, we can loop through all these possible genera and use Lemma 3.1 to find all primes in addition to $q = 7$ which occur for that genus. For genus $g = 6$, there are cyclic 7-gonal and cyclic 13-gonal surfaces. By Lemma 3.2, if a surface were cyclic 7-gonal and cyclic 13-gonal, the normalizer of a surface group for X would have orbit genus 0 and periods divisible by 7 and 13. By observation of Breuer's list for genus $g = 6$, we see that no such signature exists and hence there exists no surface of genus 6 which is 7-gonal and 13-gonal. Similar arguments holds for all genera $g \neq 36$ for which there exist 7-gonal and 13-gonal surfaces. For genus $g = 36$, there does exist a surface which is 7-gonal and 13-gonal, but in this case $G = \mathrm{Aut}(X)$ is cyclic of order pq. Identical arguments hold for all other possible choices of p and each corresponding choice for g. □

Lemma 4.2. There does not exist G with $|G| \geqslant 13pq$ for any choice of p and q.

Proof. If $|G| \geqslant 13pq$, X cannot be normal cyclic q-gonal, so it follows that

$$|G| \geqslant 13pq > 13q^2 > 13(q-1)^2 \geqslant 13g > 13(g-1).$$

Therefore if Λ is a surface group for X and Γ is the Fuchsian group with $\Gamma/\Lambda = G$, then Γ must have one of the signatures given in Table 3. For each of these signatures, since we are assuming that $p > q$, the only possible choices for q are 2, 3, 5, and 7. However, by Lemma 4.1, if $q \in \{2, 3, 5, 7\}$, then $|G| = pq$. Thus there exists no surface X with $|G| \geqslant 13pq$. □

Lemma 4.3. There does not exist G with $q = 11$ and $|G| = 121p$ or $|G| = 132p$ for any choice of $p > 11$.

Proof. If $|G| \neq pq$, then $|G| = apq$ for some integer $a > 1$. Assuming $|G| = apq$ for some $a > 1$, it follows that $C_p \cap N_G(C_q) = 1$ and $C_q \cap N_G(C_p) = 1$. Using the Sylow Theorems, this implies there exist integers a_1 and a_2, both divisors of a, and b_1 and b_2 such that

$$a_1 q = b_1 p + 1 \qquad (1)$$

and

$$a_2 p = b_2 q + 1. \qquad (2)$$

Since $p > q$, it also follows that

$$a_1 > b_1 \qquad (3)$$

and

$$a_2 < b_2. \qquad (4)$$

If $a = 11$, then $a_1 = 1$ or 11, so (1) implies that either $121 = b_1 p + 1$ or $11 = b_1 p + 1$. In the latter case, $b_1 p = 10$ so the only possible choices for p are 2 or 5. Both choices contradict our assumption that $p > q$. In the former case, $b_1 p = 120$. This implies the only possible choices for p are 2, 3 and 5 which also contradicts our assumption that $p > q$.

Now suppose $a = 12$. In this case, we can have $a_1 = 1$, 2, 3, 4, 6, or 12. For $a_1 = 1$, 2 or 3, the only possibilities for p are less than 11 contradicting our assumption that $p > q$. For $a_1 = 4$, we get $p = 43$, but there are no possible values of $g \leqslant 100 = (11-1)^2$ for which both 11 and 43 are admissible. For $a_1 = 6$, we get $p = 13$ and the possible genera are $g = 30$, 60 and 90. However, for each of these choices of g, $|G| = 1716 > 13(g-1)$ so they each reduce to the cases considered in Lemma 4.1. Finally, if $a_1 = 12$, we get $p = 131$ and $g = 65$. In this case, $|G| = 17292 > 84(g-1)$ which contradicts the Hurwitz bound. Thus we cannot have $q = 11$ and $|G| = 121p$ or $|G| = 132p$ for any choice of $p > 11$. □

Lemma 4.4. There does not exist G with $|G| = apq$ for $2 \leqslant a \leqslant 12$ and $p > q > a$.

Proof. Let a_1, a_2, b_1 and b_2 be as defined in (1) and (2) of the proof of Lemma 4.3. By Lemma 4.1, we may assume that $q \geqslant 11$. (1) and (2) imply that

$$q(a_1a_2 - b_1b_2) = a_2 + b_1. \tag{5}$$

Also, we get

$$\begin{aligned} |G| = apq &> \begin{pmatrix} \text{number of elements} \\ \text{of order } p \text{ and } q \end{pmatrix} \\ &= a_1q(p-1) + a_2p(q-1) \\ &= (a_1+a_2)pq - (a_1q + a_2p) \\ &> (a_1+a_2)pq - 2pq \\ &= (a_1+a_2+2)pq \end{aligned} \tag{6}$$

which implies that $a_1 + a_2 < 14$. Therefore, since $b_1 < a_1$, we get

$$q(a_1a_2 - b_1b_2) = a_2 + b_1 < a_1 + a_2 < 14. \tag{7}$$

It follows that since $q \geqslant 11$, we only need consider the two cases $q = 11$ and $q = 13$.

To finish the problem, since $b_1 < a_1$ and $a_1 \leqslant 12$, we can loop over all possibilities for $p = \frac{a_1q-1}{b_1}$ with $q = 11$ or 13. For each such pair (p, q), we can calculate the genus of each surface with $g \leqslant (q-1)^2$ which is admissible for both p and q. The only possibility we obtain is $q = 13$, $p = 103$, $a = 8$ and $g = 102$. In this case however, $|G| = 10712 > 84 * 101 = 84(g-1)$ which contradicts the Hurwitz bound. Therefore there does not exist G with $|G| = apq$ for $2 \leqslant a \leqslant 12$ and $p > q > a$. □

We now have the necessary tools to prove the main result of this section.

Theorem 4.5. Suppose X is a multiple prime surface which is cyclic p-gonal and cyclic q-gonal for primes $p > q$. If C_p and C_q are a cyclic p-gonal and a cyclic q-gonal group for X respectively, then the group of automorphisms G generated by C_p and C_q is cyclic of order pq.

Proof. By Lemma's 4.1-4.4, we know that $|G| = pq$. Therefore we just need to show that it is cyclic. Since p and q are distinct primes, any group of order pq will either be cyclic of order pq or a semi-direct product $C_p \rtimes C_q$ provided $p \equiv 1 \mod (q)$. Assuming the latter case, such a group admits a partition into p groups of order q and 1 group of order p. Since all groups of order q are conjugate in G, the quotient space X/H will have the same genus for any such group H. Applying Theorem 3.4, it follows that X/C_q must have genus strictly greater than 0 contrary to our assumption that C_q is a q-gonal group for X. Therefore G must be cyclic of order pq. □

5 The Classification of Multiple Prime Surfaces

We now have the necessary results to find all multiple prime surfaces. We start by fixing some notation. Let X denote a multiple prime surface which is p-gonal and q-gonal for primes $q < p$. Let C_p and C_q denote a p-gonal group and a q-gonal group respectively for X and let $G \leqslant \mathrm{Aut}(X)$ denote the group generated by C_p and C_q. Let Λ denote some fixed surface group for X and Γ_p, Γ_q and Γ_G the Fuchsian groups with $\Gamma_p/\Lambda = C_p$, $\Gamma_q/\Lambda = C_q$ and $\Gamma_G/\Lambda = G$ respectively. Before we calculate the possible full automorphism groups for X, we need the following result.

Lemma 5.1. If $p > 3$ then the signature of Γ_G is either $(0; p, p, q, q)$ or $(0; p, q, pq)$. If $p = 3$ and $q = 2$, the only possible signature for Γ_G is $(0; 2, 2, 3, 3)$.

Proof. Since X/C_p has genus 0 and $G/C_p = C_q$, Proposition 2.1 implies that Γ_G has signature $(0; aq, bq, p, \ldots, p)$ for a and b either 1 or p. Likewise, since X/C_q has genus 0 and $G/C_q = C_p$, Proposition 2.1 implies that Γ_G has signature $(0; cp, dp, q, \ldots, q)$ for c and d either 1 or q. This implies that there are exactly two periods divisible by p and two periods divisible by q. Since the number of periods of a Fuchsian group with orbit genus 0 has to be at least 3, the only possible signatures are $(0; p, p, q, q)$ and $(0; p, q, pq)$. If $p = 3$ and $q = 2$, there is no Fuchsian group with signature $(0; 2, 3, 6)$, so in this case the only possibility is $(0; 2, 2, 3, 3)$. □

Theorem 5.2. Either X has full automorphism group $C_p \times C_q$ and the normalizer of Λ has signature $(0; pq, p, q)$, or the dihedral group D_{pq} is an automorphism group of X and either:

(i) D_{pq} is the full automorphism group of X and the normalizer of Λ has signature $(0; 2, 2, p, q)$,

(ii) $C_q \rtimes D_{2p}$ is the full automorphism group of X and the normalizer of Λ has signature

$(0; 2, 2p, 2q)$ where $D_{2q} \ltimes C_p$ has presentation

$$\langle x, y, z | x^2, y^{2q}, z^p, xyxy, xzxz, yzy^{-1}z \rangle,$$

(*iii*) $GL(2,3)$ is the full automorphism group of X, which has genus 2, and the normalizer of Λ has signature $(0; 2, 3, 8)$.

Proof. We examine the two cases individually.

(*i*) Γ_G has signature $(0; p, q, pq)$.

In this case, Γ_G is a Fuchsian triangle group. For such groups, Singerman's list, [7], gives us complete knowledge regarding the signatures for each Fuchsian group Γ with $\Gamma \geqslant \Gamma_G$ and $[\Gamma : \Gamma_G] < \infty$. By inspection of this list, the only instances in which there exists a Fuchsian group $\Gamma > \Gamma_G$ is when either $q = 2$ or $q = 3$. Therefore, if $q \geqslant 5$, it follows that Γ_G must be the normalizer of Λ and hence the full automorphism group of X is $C_p \times C_q$.

When $q = 2$, Γ_G is contained in a Fuchsian group Γ with signature $(0; 2, 3, 2p)$. Since $q = 2$ and the genus g of X satisfies $g \geqslant 2$, Theorem 3.5 implies that X will be normal cyclic 2-gonal. If Λ is normal in Γ, then Γ_2 must also be normal in Γ and so the group $\Gamma/\Gamma_2 = K$ will be a group of automorphisms of the Riemann sphere. However, for all possibles choices of K and a, b, and c (or a and b when $K = C_n$), the signature $(0; 2, 3, 2p)$ does not satisfy Proposition 2.1 (since $p > 3$). Consequently, Λ cannot be normal in Γ and so Γ_G must be the normalizer of Λ and hence $C_p \times C_2$ must be the full automorphism group of X.

When $q = 3$, Γ_G is contained in a Fuchsian group Γ with signature $(0; 2, 3, 3p)$. By inspection of Breuer's lists, [3], for genera $g = 2$, 3, and 4, there is no choice of p or g for which the signature $(0; 2, 3, 3p)$ occurs. Therefore, if there exists Λ with $\Lambda \lhd \Gamma_G$ and $\Lambda \lhd \Gamma$ where Γ has signature $(0; 2, 3, 3p)$, the surface $X = \mathbb{H}/\Lambda$ must have genus $g \geqslant 5$. In particular, Theorem 3.5 implies that X is normal cyclic 3-gonal. As with the case $q = 2$, if Λ is normal in Γ, then Γ_3 must also be normal in Γ and so the group $\Gamma/\Gamma_3 = K$ will be a group of automorphisms of the Riemann sphere. For all possibles choices of K and a, b, and c (or a and b when $K = C_n$), the signature $(0; 2, 3, 3p)$ does not satisfy Proposition 2.1 so Γ_G must be the normalizer of Λ and hence $C_p \times C_3$ is the full automorphism group of X.

(*ii*) Γ_G has signature $(0; p, p, q, q)$.

If $q = 2$, then X is normal cyclic q-gonal. If $q > 2$, applying the Riemann-Hurwitz formula to the map $\pi_G \colon X \to X/G$ and using the fact that $p > q$, we get

$$\begin{aligned} g - 1 &= -pq + \frac{pq}{2}\left(2\frac{(p-1)}{p} + 2\frac{(q-1)}{q}\right) \\ &= pq - p - q = (p-1)(q-1) > (q-1)^2 + 1. \end{aligned}$$

Consequently Theorem 3.5 implies that X is a normal cyclic q-gonal surface for any choice of q. In particular, if N is the normalizer of Λ, then Γ_q is also normal in N and $N/\Gamma_q = K$ is a group of automorphisms of the Riemann sphere. We shall first show that K necessarily contains a dihedral subgroup.

The signature of Γ_G is $(0; p, p, q, q)$, and this signature appears in Singerman's list, [7]. Specifically, there exists a Fuchsian group Γ with signature $(0; 2, 2, p, q)$ in which Γ_G is normal of index 2. It is easy to show that the only epimorphism from Γ_G onto $C_p \times C_q$ with torsion free kernel maps the first two periods to elements of order p which are inverse and the second two periods to similar elements of order q. Applying Theorem 5.1 of [4], it follows that the kernel Λ will also be normal in Γ with signature $(0; 2, 2, p, q)$ and quotient group $\Gamma/\Lambda = D_{pq}$. Therefore, since $\Lambda \leqslant \Gamma_q \leqslant \Gamma \leqslant N$ and $D_p = \Gamma/\Gamma_q \leqslant K$, it follows that K contains a dihedral subgroup.

This immediately implies that K cannot be cyclic. Moreover, since $p > q$, we cannot have $p = 2$, and unless $p = 3$ or 5, the only possibility for K is a dihedral group. Therefore, we shall first consider the cases where $p = 3$ or $p = 5$.

When $p = 5$, we must have $q = 2$ or $q = 3$. If $q = 2$, then Γ_G has signature $(0; 2, 2, 5, 5)$ and X has genus 4. If we assume that K is not dihedral, the only possibility for K is A_5, so the order of Γ/Λ would be divisible by 120. Checking Breuer's list for genus 4, there is no signature for $K = A_5$ and $q = 2$ satisfying Proposition 2.1 and so no surface exists whose automorphism group has these properties. If $q = 3$ then Γ_G has signature $(0; 3, 3, 5, 5)$ and X has genus 8. An identical argument works in this case.

When $p = 3$, we must have $q = 2$, the signature of Γ_G is $(0; 2, 2, 3, 3)$ and the genus of X is 2. If we assume that K is not dihedral, the possibilities for K are S_4, A_4 and A_5. If $K = A_5$, the order of Γ/Λ would be divisible by 120 and no such group exists for genus 2. If K is either A_4 or S_4, then the order of Γ/Λ would be divisible by 24. Checking Breuer's list for genus 2, the signature $(0; 3, 3, 4)$ with $K = A_4$ and automorphism group $SL(2, 3)$ occurs and the signature $(0; 2, 3, 8)$ with $K = S_4$ and automorphism group $GL(2, 3)$ occurs. Using Theorem 5.1 of [4], it can be shown that any surface kernel of orbit genus 2 normal in a Fuchsian group with signature $(0; 3, 3, 4)$ is also normal in a Fuchsian group with signature $(0; 2, 3, 8)$. In particular, since we are trying to find the full automorphism of all multiple prime surfaces, we only need consider the signature $(0; 2, 3, 8)$. Therefore, suppose X is a surface of genus 2 with automorphism group $GL(2, 3)$ and a surface group of X is normal in a Fuchsian group with signature $(0; 2, 3, 8)$. Such a surface is necessarily cyclic 2-gonal as all surfaces of genus 2 are cyclic 2-gonal. It is cyclic 3-gonal since the only elements of order 3 in the automorphism group of a genus 2 surface are generators of cyclic 3-gonal groups. As there are no larger automorphism groups for genus $g = 2$, $GL(2, 3)$ must be the full automorphism group of X. Hence there exists a genus 2 multiple prime surface with full automorphism group $GL(2, 3)$ and the normalizer for such a surface has signature $(0; 2, 3, 8)$.

As we remarked previously, if $p \geqslant 7$, then $K = D_n$ for some n divisible by p. Through simple calculation, we see that the only possible choice satisfying Proposition 2.1 is $K = D_{2p}$ with corresponding signature $(0; 2, 2p, 2q)$. For any choice of p and q, this signature never appears in Singerman's list, [7], and so there is no Fuchsian group containing a group with this signature of finite index. Consequently, if Λ is normal in a Fuchsian group with signature $(0; 2, 2p, 2q)$, then it must be the normalizer N of Λ. Using Theorem 5.3.1 of [9], the only possible quotient group of N by a surface group Λ of a normal cyclic q-gonal surface is $C_q \rtimes D_{2p}$. It is easy to see that such a surface is also p-gonal (and in fact normal p-gonal), hence the result. □

We finish by proving one of the most interesting consequences of our work - the fact that a multiple prime surface can admit cyclic prime covers of the Riemann sphere for at most two different primes. The result is a simple consequence of our analysis.

Theorem 5.3. Suppose X is a cyclic q-gonal surface. Then X is cyclic p-gonal for at most one other prime p.

Proof. Suppose X is a multiple prime surface of a fixed genus g that is cyclic q-gonal and cyclic p-gonal. By results in the proof of Theorem 5.2, if Γ_G has signature $(0; p, q, pq)$, then C_{pq} is the full automorphism group of X, so the result follows. Therefore, we shall assume that the signature of Γ_G is $(0; p, p, q, q)$. In this case, by the proof of Theorem 5.2, we may asumme that X is normal cyclic q-gonal. Then the genus g satisfies $g = (p-1)(q-1)$, so $p = g/(q-1)+1$. Likewise, if we assume that X is cyclic r-gonal, we get $r = g/(q-1)+1$ and thus $r = p$. The result follows. □

References

[1] R.D.M. Accola: "Strongly Branched Covers of Closed Riemann Surfaces", *Proc. of the AMS*, Vol. 26(2), (1970), pp. 315–322.

[2] R.D.M. Accola: "Riemann Surfaces with Automorphism Groups Admitting Partitions", *Proc. Amer. Math. Soc.*, Vol. 21, (1969), pp. 477–482.

[3] T. Breuer: *Characters and Automorphism Groups of Compact Riemann Surfaces*, Cambridge University Press, 2001.

[4] E. Bujalance, F.J. Cirre and M.D.E. Conder: "On Extendability of Group Actions on Compact Riemann Surfaces", *Trans. Amer. Math. Soc.*, Vol. 355, (2003), pp. 1537–1557.

[5] A.M. Macbeath: "On a Theorem of Hurwitz", *Proceedings of the Glasgow Mathematical Association*, Vol. 5, (1961), pp. 90–96.

[6] B. Maskit: "On Poincaré's Theorem for Fundamental Polygons", *Advances in Mathematics*, (1971), Vol. 7, pp. 219–230.

[7] D. Singerman: "Finitely Maximal Fuchsian Groups", *J. London Math. Soc.*, Vol. 2(6), (1972), pp. 29–38.

[8] A. Wootton: "Non-Normal Belyĭ p-gonal Surfaces", In: *Computational Aspects of Algebraic Curves, Lect. Notes in Comp.*, (2005), to appear.

[9] A. Wootton: "Defining Algebraic Polynomials for Cyclic Prime Covers of the Riemann Sphere", *Dissertation*, (2004).