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On the Bochner conformal curvature of Kähler-Norden manifolds

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Abstract: Using the one-to-one correspondence between Kähler-Norden and holomorphic Riemannian metrics, important relations between various Riemannian invariants of manifolds endowed with such metrics were established in my previous paper [19]. In the presented paper, we prove that there is a strict relation between the holomorphic Weyl and Bochner conformal curvature tensors and similarly their covariant derivatives are strictly related. Especially, we find necessary and sufficient conditions for the holomorphic Weyl conformal curvature tensor of a Kähler-Norden manifold to be holomorphically recurrent.

Keywords: Kähler-Norden manifold, holomorphic Riemannian manifold, Bochner conformal curvature, Weyl holomorphic conformal curvature
MSC (2000): 53C15, 53C50, 53C56

1 Preliminaries

Kähler-Norden manifolds. Let M be a real connected $n(=2m)$-dimensional differentiable manifold endowed with an almost complex structure J ($J^2 = -I$, I being the identity transformation) and a pseudo-Riemannian metric g of Norden type (that is, of neutral signature (m,m)) and such that

$$g(JX, JY) = -g(X, Y), \quad (\nabla_X J)Y = 0 \quad \text{for any} \quad X, Y \in \mathfrak{X}(M) \tag{1}$$

where ∇ is the Levi-Civita connection of g and $\mathfrak{X}(M)$ is the Lie algebra of smooth vector fields on M. Then the triple (M, J, g) will be said to be a Kähler-Norden manifold (it is

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called a Kählerian manifold with Norden metric in [6], and an anti-Kählerian manifold in [1, 2]).

Holomorphic Riemannian manifolds. Let M be a complex manifold of complex dimension m. Denote by (M, J) the manifold considered as a real $2m$-dimensional manifold with the induced almost complex structure J. The tangent space to (M, J) at $p \in M$ and its complexification are denoted by T_pM and $T_p^{\mathbb{C}}M$, respectively. The subspaces of $T_p^{\mathbb{C}}M$ consisting of complex vectors of type $(1,0)$ and $(0,1)$ are denoted by $T_p^{(1,0)}M$ and $T_p^{(0,1)}M$, respectively. The Lie algebras of real smooth vector fields, complex vector fields, complex vector fields of type $(1,0)$ and complex vector fields of type $(0,1)$ on M are denoted by $\mathfrak{X}(M)$, $\mathfrak{X}^{\mathbb{C}}(M)$, $\mathfrak{X}^{(1,0)}(M)$ and $\mathfrak{X}^{(0,1)}(M)$, respectively. In the sequel, for any $X \in \mathfrak{X}(M)$, by $\widehat{X}$ we denote the complex vector field defined by $\widehat{X} = \frac{1}{2}(X - \sqrt{-1}\,JX) \in \mathfrak{X}^{(1,0)}(M)$. Any $Z \in \mathfrak{X}^{(1,0)}(M)$ is of this form, that is, $Z = \widehat{X}$ for a certain $X \in \mathfrak{X}(M)$ ([11], Vol. II).

A complex Riemannian metric on M is defined to be a symmetric $(0,2)$-tensor field G, which is nondegenerate at each point of M and such that

$$G\left(\overline{Z}_1, \overline{Z}_2\right) = \overline{G(Z_1, Z_2)} \quad \text{for any } Z_1, Z_2 \in \mathfrak{X}^{\mathbb{C}}(M),$$
$$G(Z_1, Z_2) = 0 \quad \text{for any} \quad Z_1 \in \mathfrak{X}^{(1,0)}(M) \quad \text{and} \quad Z_2 \in \mathfrak{X}^{(0,1)}(M).$$

The second condition of the above is equivalent to

$$G(JZ_1, JZ_2) = -G(Z_1, Z_2) \quad \text{for any} \quad Z_1, Z_2 \in \mathfrak{X}^{\mathbb{C}}(M).$$

Thus, a complex Riemannian metric is completely determined by its values on $\mathfrak{X}^{(1,0)}(M)$.

If M is a complex manifold and G is a complex Riemannian metric on M, then the pair (M, G) is said to be a complex Riemannian manifold ([8] - [10], [21]).

For a local holomorphic coordinates system $(z^\alpha; 1 \leqslant \alpha \leqslant m)$ of a complex Riemannian manifold, let $z^\alpha = x^\alpha + \sqrt{-1}y^\alpha$, with $x^\alpha = \mathfrak{Re}\, z^\alpha$ and $y^\alpha = \mathfrak{Im}\, z^\alpha$, and next suppose $\partial/\partial z^\alpha = (1/2)(\partial/\partial x^\alpha - \sqrt{-1}\,\partial/\partial y^\alpha)$, $\partial/\partial z^{\overline{\alpha}} = (1/2)(\partial/\partial x^\alpha + \sqrt{-1}\,\partial/\partial y^\alpha)$, and $G_{AB} = G(\partial/\partial z^A, \partial/\partial z^B)$, $A, B = 1, \ldots, m, \overline{1}, \ldots, \overline{m}$. Therefore, we may then express the defining conditions for a complex Riemannian metric G with respect to this system of local coordinates in the form $G_{\overline{\alpha}\overline{\beta}} = \overline{G_{\alpha\beta}}$, $G_{\overline{\alpha}\beta} = G_{\alpha\overline{\beta}} = 0$.

A complex Riemannian manifold (M, G) is said to be holomorphic Riemannian ([8, 9]; also [3, 12, 13, 21]) if additionally the local components $G_{\alpha\beta}$ are holomorphic functions, that is, $\partial G_{\alpha\beta}/\partial z^{\overline{\gamma}} = 0$, or equivalently $\widehat{\nabla} J = 0$, where $\widehat{\nabla}$ is the Levi-Civita connection of G ([8]).

Kähler-Norden vs. holomorphic Riemannian. There exists a one-to-one correspondence between Kähler-Norden manifolds and holomorphic Riemannian manifolds ([1, 2]; compare also [21] and [19]). Below, we sketch the description of this correspondence.

Let (M, J, g) be a Kähler-Norden manifold. Since $\nabla J = 0$, the almost complex structure J is integrable. Therefore, the real manifold M inherits the structure of a complex manifold, which for simplicity will also be denoted by M, and J comes from the complex structure in the usual way. To define a complex Riemannian metric G on the complex manifold M it is sufficient to suppose

$$G(\widehat{X}, \widehat{Y}) = \frac{1}{2}\left(g(X, Y) - \sqrt{-1}\, g(X, JY)\right), \quad X, Y \in \mathfrak{X}(M), \tag{2}$$

and next extend G to be complex Riemannian. Additionally, (M, G) is holomorphic Riemannian.

Conversely, a holomorphic Riemannian manifold (M, G) can be considered as a real $2m$-dimensional Kähler-Norden manifold (M, J, g). Namely, we define J to be the almost complex structure coming from the complex structure of M and suppose

$$g(X, Y) = 2\,\Re e\left(G(\widehat{X}, \widehat{Y})\right), \quad X, Y \in \mathfrak{X}(M). \tag{3}$$

The relations (2) and (3) state the one-to-one correspondence between Kähler-Norden structures (J, g) and holomorphic Riemannian metrics G on M.

2 General formulas [19]

In this section, we recall important formulas concerning the main Riemannian invariants obtained for Kähler-Norden manifolds in our previous paper [19].

Let (M, J, g) be a Kähler-Norden manifold and let $(M, \widehat{g})$ be the corresponding holomorphic Riemannian manifold (in the sense explained in the previous section). Here and in the rest of this paper, we write $\widehat{g}$ instead of G.

Let $\mathfrak{X}^h(M)$ denote the Lie algebra of holomorphic vector fields on M.

Agreement. Throughout the rest of this paper, without loss of generality, $X, Y, \ldots$ will denote arbitrary real smooth vector fields on M such that $\widehat{X}, \widehat{Y}, \ldots \in \mathfrak{X}^h(M)$.

Under the above agreement, the considered vector fields on M are always infinitesimal automorphisms of the almost complex structure J. Therefore (cf. e.g. [11], Vol. II),

$$[JX, Y] = [X, JY] = J[X, Y], \quad [JX, JY] = -[X, Y], \quad [\widehat{X}, \widehat{Y}] = \widehat{[X, Y]}.$$

One notes that for a holomorphic function f and a vector field $\widehat{W}$, we have

$$f\,\widehat{W} = ((\Re e\, f)\, W + (\Im m\, f)\, JW)\widehat{\ }. \tag{4}$$

By $(e_1, e_2, \ldots, e_{2m})$ we denote a frame of a tangent space T_pM, which is adapted to the structure (J, g) in the sense that it consists of real vectors such that $g(e_\alpha, e_\beta) = -g(e_{\alpha'}, e_{\beta'}) = \delta_{\alpha\beta}$, $g(e_\alpha, e_{\beta'}) = g(e_{\alpha'}, e_\beta) = 0$, $Je_\alpha = e_{\alpha'}$, $Je_{\alpha'} = -e_\alpha$, where the Greek indices take on values $1, \ldots, m$ and $\alpha' = \alpha + m$. Then assuming $\widehat{e}_\alpha = (1/2)(e_\alpha - \sqrt{-1}\, Je_\alpha)$, we have a frame $(\widehat{e}_1, \ldots, \widehat{e}_m)$ of the space $T_p^{(1,0)}M$ for which $\widehat{g}(\widehat{e}_\alpha, \widehat{e}_\beta) = (1/2)\,\delta_{\alpha\beta}$.

Let ∇ and $\widehat{\nabla}$ be the Levi-Civita connections of the Kähler-Norden metric g and the holomorphic Riemannian metric $\widehat{g}$, respectively. The connection $\widehat{\nabla}$ is holomorphic, that is, $\widehat{\nabla}_{\widehat{X}}\widehat{Y} \in \mathfrak{X}^h(M)$ for any $\widehat{X}, \widehat{Y} \in \mathfrak{X}^h(M)$ [3, 12, 13]. For the Levi-Civita connections ∇ and $\widehat{\nabla}$, we have the following basic relation

$$\widehat{\nabla}_{\widehat{X}}\widehat{Y} = \widehat{\nabla_X Y}. \tag{5}$$

Let R be the Riemann curvature tensor field connected with ∇, and let $\widehat{R}$ be the holomorphic Riemann curvature tensor field connected with $\widehat{\nabla}$,

$$R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]}, \quad \widehat{R}(\widehat{X}, \widehat{Y}) = [\widehat{\nabla}_{\widehat{X}}, \widehat{\nabla}_{\widehat{Y}}] - \widehat{\nabla}_{[\widehat{X}, \widehat{Y}]}.$$

The Riemann curvature tensors R and $\widehat{R}$ are related by

$$\widehat{R}(\widehat{X},\widehat{Y})\widehat{Z} = (R(X,Y)Z)\widehat{\ }. \tag{6}$$

Let S, $\widehat{S}$ be the Ricci and the holomorphic Ricci curvature tensor fields, respectively,

$$S(X,Y) = \mathrm{Tr}\,\{Z \mapsto R(Z,X)Y\}, \quad \widehat{S}(\widehat{X},\widehat{Y}) = \mathrm{Tr}\,\{\widehat{Z} \mapsto \widehat{R}(\widehat{Z},\widehat{X})\widehat{Y}\}.$$

Denote by Q and $\widehat{Q}$ the Ricci and the holomorphic Ricci operators, respectively,

$$g(QX,Y) = S(X,Y), \quad \widehat{g}(\widehat{Q}\widehat{X},\widehat{Y}) = \widehat{S}(\widehat{X},\widehat{Y}).$$

For S and Q, we have

$$S(JX,JY) = -S(X,Y), \quad S(JX,Y) = S(X,JY), \quad QJ = JQ. \tag{7}$$

The Ricci curvature tensors $S, \widehat{S}$ and the Ricci operators $Q, \widehat{Q}$ are related by

$$\widehat{S}(\widehat{X},\widehat{Y}) = \frac{1}{2}\left(S(X,Y) - \sqrt{-1}\,S(X,JY)\right), \qquad \widehat{Q}\widehat{X} = \widehat{QX}. \tag{8}$$

Let r and r^* be the scalar and $*$-scalar curvatures of g, and let $\widehat{r}$ be the holomorphic scalar curvature of $\widehat{g}$, $r = \mathrm{Tr}\,Q$, $r^* = \mathrm{Tr}(JQ)$, $\widehat{r} = \mathrm{Tr}\,\widehat{Q}$. For them, it holds

$$\widehat{r} = \frac{1}{2}\left(r - \sqrt{-1}\,r^*\right). \tag{9}$$

3 Additional operators

Let A be a symmetric $(0,2)$-tensor field on M. For $X, Y \in \mathfrak{X}(M)$, define an operator $X \wedge_A Y$ acting on $\mathfrak{X}(M)$ by

$$(X \wedge_A Y)Z = A(Y,Z)X - A(X,Z)Y.$$

Let A satisfy the additional condition (it should be noted that by (1) and (7) the relation of this type is fulfilled by the metric tensor g and the Ricci curvature tensor S)

$$A(JX,JY) = -A(X,Y).$$

Define $\widehat{A}$ to be the complex $(0,2)$-tensor field which is completely determined by its values on $\mathfrak{X}^{(1,0)}(M)$ and for which

$$\widehat{A}(\widehat{X},\widehat{Y}) = \frac{1}{2}\left(A(X,Y) - \sqrt{-1}\,A(X,JY)\right).$$

For $\widehat{X},\widehat{Y} \in \mathfrak{X}^{(1,0)}(M)$, define $\widehat{X} \wedge_{\widehat{A}} \widehat{Y}$ to be the operator acting on $\mathfrak{X}^{(1,0)}(M)$ by

$$(\widehat{X} \wedge_{\widehat{A}} \widehat{Y})\widehat{Z} = \widehat{A}(\widehat{Y},\widehat{Z})\widehat{X} - \widehat{A}(\widehat{X},\widehat{Z})\widehat{Y}.$$

It is now a straightforward verification that

$$(\widehat{X} \wedge_{\widehat{A}} \widehat{Y})\widehat{Z} = \frac{1}{2}\left((X \wedge_A Y)\,Z - (JX \wedge_A JY)\,Z\right)\widehat{\ }. \tag{10}$$

One notes that if $\widehat{A}$ is additionally a holomorphic tensor field, then $(\widehat{X} \wedge_{\widehat{A}} \widehat{Y})\widehat{Z} \in \mathfrak{X}^h(M)$ for any $\widehat{X},\widehat{Y},\widehat{Z} \in \mathfrak{X}^h(M)$.

4 The holomorphic Weyl conformal curvature

Properties of the real Weyl conformal curvature tensor C in the class of Kähler-Norden manifolds were studied by the author in the papers [17, 18].

In the paper [7], the (real) Bochner curvature tensor field B is defined as a conformal invariant on conformally Kähler manifolds with B-metric. Following [7], on a Kähler-Norden manifold of real dimension $\geqslant 6$, the Bochner tensor appears as

$$\begin{aligned} B(X,Y) = R(X,Y) &- \frac{1}{n-4}(X \wedge_g QY + QX \wedge_g Y - JX \wedge_g QJY - QJX \wedge_g JY) \\ &+ \frac{1}{(n-2)(n-4)}\big(r(X \wedge_g Y - JX \wedge_g JY) - r^*(JX \wedge_g Y + X \wedge_g JY)\big). \end{aligned} \tag{11}$$

On the other hand, in the presented paper, we will also treat with the holomorphic Weyl (H-Weyl in short) conformal curvature tensor $\widehat{C}$, which is defined by the standard formula (see [3, 8, 12])

$$\widehat{C}(\widehat{X},\widehat{Y}) = \widehat{R}(\widehat{X},\widehat{Y}) - \frac{1}{m-2}\Big(\widehat{Q}\widehat{X} \wedge_{\widehat{g}} \widehat{Y} + \widehat{X} \wedge_{\widehat{g}} \widehat{Q}\widehat{Y} - \frac{\widehat{r}}{m-1}\widehat{X} \wedge_{\widehat{g}} \widehat{Y}\Big). \tag{12}$$

We will show that the Bochner curvature tensor and the holomorphic Weyl conformal curvature tensor are strictly related. At first, we have the following result.

Proposition 4.1. For a Kähler-Norden manifold, the conformal curvature tensor fields B and $\widehat{C}$ are related by

$$\widehat{C}(\widehat{X},\widehat{Y})\widehat{Z} = (B(X,Y)Z)\widehat{\ }. \tag{13}$$

Proof. By applying (8), (10) and (7), one gets

$$(\widehat{Q}\widehat{X} \wedge_{\widehat{g}} \widehat{Y})\widehat{Z} = \frac{1}{2}\big((QX \wedge_g Y - QJX \wedge_g JY)Z\big)\widehat{\ }, \tag{14}$$

$$(\widehat{X} \wedge_{\widehat{g}} \widehat{Q}\widehat{Y})\widehat{Z} = \frac{1}{2}\big((X \wedge_g QY - JX \wedge_g QJY)Z\big)\widehat{\ }. \tag{15}$$

In the sequel, we will also need the following formula

$$J(X \wedge_g Y - JX \wedge_g JY) = JX \wedge_g Y + X \wedge_g JY, \tag{16}$$

which can be checked by a direct computation.

To transform $\widehat{r}(\widehat{X} \wedge_{\widehat{g}} \widehat{Y})$, using (10) with $A = g$, we find at first

$$\widehat{r}(\widehat{X} \wedge_{\widehat{g}} \widehat{Y})\widehat{Z} = \frac{1}{2}\widehat{r}((X \wedge_g Y)Z - (JX \wedge_g JY)Z)\widehat{\ }.$$

By virtue of (4) with $f = \widehat{r}$, $W = (X \wedge_g Y)Z - (JX \wedge_g JY)Z$, and (9), (16), the above expression turns into

$$\widehat{r}(\widehat{X} \wedge_{\widehat{g}} \widehat{Y})\widehat{Z} = \frac{1}{4}\big(\big(r(X \wedge_g Y - JX \wedge_g JY) - r^*(JX \wedge_g Y + X \wedge_g JY)\big)Z\big)\widehat{\ }. \tag{17}$$

We now apply the formula (12) into the left hand side of (13), and next we use (6) and the obtained equalities (14), (15), (17). Then after certain calculations, regarding the definition of B (11), we obtain the right hand side of (13). □

Remark 4.2. In view of (13), for a Kähler-Norden manifold (M, J, g), we have the following two statements.
(i) In case of $\dim M = 6$, $B = 0$. In fact by a pure algebraic reason, $\widehat{C} = 0$ for a holomorphic Riemannian manifold of complex dimension 3.
(ii) In case of $\dim M > 6$, $B = 0$ if and only if $\widehat{C} = 0$.

Proposition 4.3. For a Kähler-Norden manifold of real dimension $\geqslant 8$,

$$(\widehat{\nabla}_{\widehat{U}}\widehat{C})(\widehat{X}, \widehat{Y})\widehat{Z} = ((\nabla_U B)(X, Y)Z)\widehat{\ }. \tag{18}$$

Proof. At first, we write

$$(\widehat{\nabla}_{\widehat{U}}\widehat{C})(\widehat{X}, \widehat{Y})\widehat{Z} = \widehat{\nabla}_{\widehat{U}}(\widehat{C}(\widehat{X}, \widehat{Y})\widehat{Z}) - \widehat{C}(\widehat{\nabla}_{\widehat{U}}\widehat{X}, \widehat{Y})\widehat{Z} - \widehat{C}(\widehat{X}, \widehat{\nabla}_{\widehat{U}}\widehat{Y})\widehat{Z} - \widehat{C}(\widehat{X}, \widehat{Y})\widehat{\nabla}_{\widehat{U}}\widehat{Z}.$$

Next, we apply the formulas (5), (13) and reduce the right hand side of the above equality to the form $((\nabla_U B)(X, Y)Z)\widehat{\ }$. □

Lemma 4.4. For the covariant derivative of the Bochner curvature tensor, it holds

$$(\nabla_{JU} B)(X, Y) = J(\nabla_U B)(X, Y). \tag{19}$$

Proof. For an arbitrary vector field U, we have

$$\widehat{JU} = J\widehat{U} = \sqrt{-1}\,\widehat{U}. \tag{20}$$

Using (18) and the first equality of (20), we get

$$(\widehat{\nabla}_{J\widehat{U}}\widehat{C})(\widehat{X}, \widehat{Y})\widehat{Z} = ((\nabla_{JU} B)(X, Y)Z)\widehat{\ }.$$

On the other hand, using (18) and the second equality of (20), we find

$$(\widehat{\nabla}_{J\widehat{U}}\widehat{C})(\widehat{X}, \widehat{Y})\widehat{Z} = (J(\nabla_U B)(X, Y)Z)\widehat{\ }.$$

Comparing the last two equalities, we obtain (19), as required. □

A Kähler-Norden manifold is called (i) of parallel Bochner curvature if $\nabla B = 0$; (ii) of recurrent Bochner curvature if B is non-identically zero and for a 1-form ψ,

$$\nabla B = \psi \otimes B. \tag{21}$$

At the moment, it is useful to recall the theorem stating that a tensor field fulfilling a recurrence condition (like for instance B realizing (21)) is either everywhere zero or nowhere zero on the manifold (see [20, Th. 1.2]).

Theorem 4.5. Any Kähler-Norden manifold of recurrent Bochner curvature and of real dimension $\geqslant 8$ is necessarily of parallel Bochner curvature.

Proof. By applying (21) into (19), we obtain

$$\psi(JU)B(X,Y)Z - \psi(U)JB(X,Y)Z = 0. \tag{22}$$

Under our assumption, B does not vanish at any point of M. Therefore, at every point $p \in M$, $B(X,Y)Z$ and $JB(X,Y)Z$ are linearly independent for certain $X, Y, Z \in T_p(M)$. Now, from (22), it follows that $\psi(U) = 0$ for any $U \in T_p(M)$. Consequently, $\psi = 0$ on M, which by (21) gives the desired assertion. □

A Kähler-Norden manifold will be called (i) of parallel H-Weyl conformal curvature if $\widehat{\nabla}\widehat{C} = 0$; (ii) of holomorphically recurrent (H-recurrent in short) H-Weyl conformal curvature if $\widehat{C}$ is non-identically zero and for a certain holomorphic 1-form $\widehat{\varphi}$,

$$\widehat{\nabla}\widehat{C} = \widehat{\varphi} \otimes \widehat{C}. \tag{23}$$

Remark 4.6. In [19], we have investigated the H-recurrence of the holomorphic Riemann curvature. It is clear that the H-recurrence of the holomorphic Riemann curvature implies the H-recurrence of the H-Weyl conformal curvature. But the converse implication does not hold in general, as it is pointed out in the next section.

We extend J to act on real 1-forms φ by assuming $(J\varphi)(X) = \varphi(JX)$ for any $X \in \mathfrak{X}(M)$. Then J is compatible with the musical isomorphisms acting between tangent and cotangent bundles (cf. [19]).

Theorem 4.7. Let (M, J, g) be a Kähler-Norden manifold of dimension $\geqslant 8$.

(i) (M, J, g) is of H-recurrent H-Weyl conformal curvature if and only if the Bochner tensor B is non-identically zero and

$$(\nabla_U B)(X,Y) = \varphi(U)B(X,Y) - \varphi(JU)JB(X,Y) \tag{24}$$

for a certain real 1-form φ such that $\widehat{\varphi} = \varphi - \sqrt{-1}\,J\varphi$ is a holomorphic 1-form;

(ii) (M, J, g) is of parallel H-Weyl conformal curvature if and only if it is of parallel Bochner curvature.

Proof. (i) We write down the defining condition (23) in the following explicit way

$$(\widehat{\nabla}_{\widehat{U}}\widehat{C})(\widehat{X},\widehat{Y})\widehat{Z} = \widehat{\varphi}(\widehat{U})\widehat{C}(\widehat{X},\widehat{Y})\widehat{Z}. \tag{25}$$

To the left hand side of (25) we can apply the formula (18). We should transform now the right hand side of (25). To do it write the holomorphic 1-form $\widehat{\varphi}$ as $\widehat{\varphi} = \varphi - \sqrt{-1}J\varphi$ with φ being a certain real 1-form. Then we see that

$$\widehat{\varphi}(\widehat{U}) = \varphi(U) - \sqrt{-1}\,\varphi(JU). \tag{26}$$

Now, the formulas (13), (26) and (4) with $f = \widehat{\varphi}(\widehat{U})$, $W = B(X,Y)Z$ enable us to find

$$\widehat{\varphi}(\widehat{U})\widehat{C}(\widehat{X},\widehat{Y})\widehat{Z} = \widehat{\varphi}(\widehat{U})(B(X,Y)Z)\widehat{\ } = (\varphi(U)R(X,Y)Z - \varphi(JU)JR(X,Y)Z)\widehat{\ }.$$

Finally, using the above expression and (23), we claim that the condition (25) is equivalent to (24). To have the proof complete, one should also note that at a point of the manifold M, $\widehat{C} = 0$ if and only if $B = 0$.

(ii) This is an obvious consequence of (18). □

5 Examples

The main idea of obtaining examples of Kähler-Norden manifolds of parallel or H-recurrent H-Weyl conformal curvature is to make the complexification of some real (pseudo-)Riemannian metrics. There are many examples of (pseudo-)Riemannian metrics which have parallel or recurrent Weyl conformal curvature; cf.[4, 5, 14, 15, 16], etc. Below, basing on the paper [15], we present only one of the known classes of such manifolds.

Fix $m \in \mathbb{N}$, $m > 3$ and assume that the Greek indices run through the range $\{2, 3, \ldots, m-1\}$.

Let p, q be non-constant functions of an one complex variable only, which are holomorphic on an open connected subset $U_1 \subset \mathbb{C}$ such that q is non-zero on U_1. Let f be a holomorphic function given on the open connected subset $U = U_1 \times \mathbb{C}^{m-1} \subset \mathbb{C}^m$ by

$$f(z^1, \ldots, z^m) = \sum \big(p(z^1)\, k_{\alpha\beta} + q(z^1)\, c_{\alpha\beta}\big)\, z^\alpha z^\beta,$$

where $c_{\alpha\beta}$, $k_{\alpha\beta}$ are complex constants such that the $(m-2)$-by-$(m-2)$ matrices $[c_{\alpha\beta}]$, $[k_{\alpha\beta}]$ are symmetric, Rank $[c_{\alpha\beta}] > 1$, Rank $[k_{\alpha\beta}] = m-2$ and $\sum k^{\alpha\beta} c_{\alpha\beta} = 0$, $k^{\alpha\beta}$ being the entries of the inverse matrix $[k_{\alpha\beta}]^{-1}$.

Let $\widehat{g}$ be the holomorphic Riemannian metric defined on U by

$$\widehat{g} = f\, dz^1 \otimes dz^1 + \sum k_{\alpha\beta}\, dz^\alpha \otimes dz^\beta + dz^1 \otimes dz^m + dz^m \otimes dz^1.$$

Then it is a straightforward verification that the metric $\widehat{g}$ is of H-recurrent H-Weyl conformal curvature with $\widehat{\psi} = (q'/q)dz^1$ as its recurrence form. In the case when q is a non-zero constant, this metric is of parallel H-Weyl conformal curvature. And it can be also verified that the holomorphic Riemann curvature is not holomorphically recurrent.

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