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On some properties of the functions from Sobolev-Morrey type spaces

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Abstract: In this paper the spaces of type Sobolev-Morrey- $W^l_{p,a,\Gamma,\tau}(Q,G)$-are constructed, the differential properties are studied and it is proved that the functions from these spaces satisfy Holder's condition, in the case, if the domain $G \subset R^n$ satisfies the flexible $\lambda-$horn condition.

Keywords: Sobolev-Morrey space, flexible λ-horn, mixed derivatives, the imbedding theorems, generalized derivatives, integral representation
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Let the domain $G \subset R^n$ satisfy the condition of flexible $\lambda-$horn introduced by O.V. Besov [1] and let $e_n = \{1, 2, \ldots, n\}$; $e^0_n = e_n \cup \{0\}$; Q be a fixed subset of the set e_n; $\varnothing \neq e \subset Q$; $p \in [1,\infty)$; $a \in [0,1]^n$; $\tau \in [1,\infty]$; $l \in N^n$; $\Gamma \in (0,\infty)^n$; $T = (T_1, T_2, \ldots, T_n)$, where $T_i = T_j$ for $i, j \in e_n \backslash Q$; $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, $\lambda_j = 1$ for $j \in Q$; $\lambda_j \in (0,\infty)$ for $j \in e_n\backslash Q$; $l^{e\vee i} = (l^{e\vee i}_1, l^{e\vee i}_2 \ldots l^{e\vee i}_n)$, $l^{e\vee i}_j = l_j$ for $j \in e \vee i$; $l^{e\vee i}_j = 0$ for $j \in e_n\backslash(e \vee i)$; $j \in e \vee i$ denotes, that, either $j \in e \subset Q$, or $j = i \in e_n\backslash Q$. Let also

$$\int_{a^{e\vee i}}^{b^{e\vee i}} f(x)dx^{e\vee i} = \left(\prod_{j\in e\vee i} \int_{a_j}^{b_j} dx_j\right) f(x),$$

that is, the integral involves only the variable x_j, and indices belong to set $e \vee i$.

Proceeding from the fact that some mixed derivatives $D^\nu f$ may not be estimated by derivative functions contained in the norm of the space $W^l_p(G)$ and on the other hand from undesireable higher order derivatives from the function f, there arose a necessity to consider other types of Sobolev spaces $W^l_p(Q,G)$, that are introduced and studied in [2]

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with the finite norm:

$$\|f\|_{W_p^l(Q,G)} = \sum_{e\subset Q} \sum_{i\in e_n^0\backslash Q} \left\| D^{l^{e\vee i}} f \right\|_{p,G},$$

where

$$\|f\|_{p,G} = \left\{ \int_R \left[\cdots \left\{ \int_R \left(\int_R |\chi_G(y)|f(y)|^{p_1} dy_1 \right)^{\frac{p_2}{p_1}} dy_2 \right\}^{\frac{p_3}{p_2}} \cdots \right]^{\frac{p_n}{p_{n-1}}} dy_n \right\}^{\frac{1}{p_n}},$$

with $\chi-$ the characteristic function of the set G.Note that the considered spaces also were defined by A.D.Djabrailov [3] but with the norm (1) replacing $D^{l^{e\vee i}}f$ by $D^{l^{e\cup i}}f$ (note, that in case [3] dominate mixed derivatives). Unlike in the paper [3] here either dominant unmixed derivatives, or mixed derivatives, or mixed derivatives and unmixed derivatives are equal.

At $|Q| = 1$ ($|Q|$ −the number of the set Q) the space $W_p^l(Q,G)$ coincides with the space of Sobolev-$W_p^l(G)$, at $Q \equiv e_n$ the space $W_p^l(Q,G)$ coincides with the space of Sobolev with dominant mixed derivatives $S_p^l W(G)$, introduced and studied by S.M.Nikolskii [4] with finite norm

$$\|f\|_{S_p^l W(G)} = \sum_{e\subseteq e_n} \left\| D^{l^e} f \right\|_{p,G},$$

where $l = (l_1, l_2, \cdots, l_n)$, $l_j \in N$ for $j \in e_n$; $l^e = ({l_1}^e, {l_2}^e, \cdots, {l_n}^e)$, ${l_j}^e = l_j$ for $j \in e$,${l_j}^e = 0$ for $j \in e_n\backslash e$.

For example, in equation

$$u + u'_x + u'_y + u''_{xy} + u'_z = 0$$

the norm u''_{xy} may not be estimated by the norm of the space $W^{(1,1,1)}$, but may be estimated by the norm of the Sobolev space with dominating mixed derivative $S^{(1,1,1)}W$, then we require additional derivatives from the function $u(x, y, z)$.

To study partial differential equations it is necessary to study the space of functions of many variables with parameters $W^l_{p,a,\Gamma}(G)$, for some partial values of indices studied in papers of Morrey [5-7], and later on developed in papers of Greco [8], Nirenberg [9], Campanato [10,11], Barossi [12], Il'yin [13], Netrusov [14], author [15-18] and others.

For all $x \in G$ and $t \in (0,\infty)^n$; $t_i = t_j$ for $i, j \in e_n\backslash Q$ assuming

$$I_{t^\Gamma}(x) = \left\{ y : |y_j - x_j| < \frac{1}{2} t_j^{\Gamma_j}, j \in e_n \right\},$$

$$G_{t^\Gamma}(x) = G \cap I_{t^\Gamma}(x).$$

Let's consider for $x \in G$ the vector-function

$$\rho(t^\lambda, x) = (\rho_1(t_1^{\lambda_1}, x), \rho_2(t_2^{\lambda_2}, x), \ldots, \rho_n(t_n^{\lambda_n}, x)),\ 0 \le t_j \le T_j,\ j \in e_n$$

where for all $j \in e_n$, $\rho_j(0,x) = 0$, the functions $\rho_j(u_j, x)$ are absolutely continuous with respect to u_j on $[0, T_j^{\lambda_j}]$ and $\left|\rho'_j(u_j,x)\right| \le 1$ for almost all $u_j \in [0, T_j^{\lambda_j}]$, where

$\rho'_j(u_j,x) = \frac{\partial}{\partial u_j}\rho_j(u_j,x)$. For $\theta \in (0,1]^n$; $\theta_i = \theta_j$ for $i,j \in e_n \backslash Q$. We'll call each of the sets $V(\lambda,x,\theta) = \bigcup 0 \le t_j \le T_j, j \in e_n \left[\rho(t^\lambda,x) + t^\lambda\theta^\lambda I\right]$ and $x+ V(\lambda,x,\theta)$−as flexible horn−λ and point x as a top $x + V(\lambda,x,\theta)$. We'll suppose that $x+ V(\lambda,x,\theta) \subset G$. In the case of, $t_1 = t_2 = \ldots = t_n = t$, $\theta_1 = \theta_2 = \ldots = \theta_n = \theta \in (0,1]$, $V(\lambda,x,\theta)$−is flexible $\lambda - horn$, found in O.V. Besov in [1].

Definition. The Sobolev-Morrey-$W^l_{p,a,\Gamma,\tau}(Q,G)$ type space is called the Banach space of locally summable on G functions f with finite norm:

$$\|f\|_{W^l_{p,a,\Gamma,\tau}(Q,G)} = \sum_{e\subset Q}\sum_{i\in e_n^0\backslash Q}\left\|D^{l^{e\vee i}}f\right\|_{p,a,\Gamma,\tau;G} \tag{1}$$

here

$$\|f\|_{p,a,\Gamma,\tau;G} = \|f\|_{L_{p,a,\Gamma,\tau}(G)} = \sup_{x\in G}\left\{\int_0^{t_0}\left[\prod_{j\in e_n}[t_j]_1^{-\frac{\Gamma_j a_j}{p}}\|f\|_{p,G_{t^\Gamma}(x)}\right]^\tau\prod_{j\in Q\vee i}\frac{dt_j}{t_j}\right\}^{\frac{1}{\tau}}, \tag{2}$$

$$\|f\|_{\mathbf{p},G_{t^\Gamma}(x)} = \left\{\int_{G_{t^{\Gamma_n}}(x_n)}\left[\cdots\left\{\int_{G_{t^{\Gamma_2}}(x_2)}\left(\int_{G_{t^{\Gamma_1}}(x_1)}|f(y)|^{p_1}dy_1\right)^{\frac{p_2}{p_1}}dy_2\right\}^{\frac{p_3}{p_2}}\cdots\right]^{\frac{p_n}{p_{n-1}}}dy_n\right\}^{\frac{1}{p_n}}, \tag{3}$$

where $[t_j]_1 = \min\{1,t_j\}$, $j \in e_n$; $t_0 = (t_{01},t_{02},\ldots,t_{0n})$−a fixed positive vector, $t_{0i} = t_{0j}$ for $i,j \in e_n\backslash Q$, $l_0 = 0$. At $\tau = \infty$ the $W^l_{p,a,\Gamma,\infty}(Q,G) = W^l_{p,a,\Gamma}(Q,G)$−space studied in [18] . At $\tau = \infty$ and $a = 0$ the space $W^l_{p,a,\Gamma,\tau}(Q,G)$ coincides with the space of type Sobolev $W^l_p(Q,G)$.

In the case $\tau = \infty$, $a = (a,\cdots a)$, $p = (p,\cdots,p)$, $t = (t,\cdots,t)$ Sobolev-Morrey space $W^l_{p,a,\Gamma,\infty}(G) \equiv W^l_{p,a,\Gamma}(G)$ were introduced and studied by V.P.Il'yin [13], more precisely,

$$\|f\|_{W^l_{p,a,\Gamma}(G)} = \|f\|_{p,a,\Gamma;G} + \sum_{i=1}^n\left\|D_i^{l_i}f\right\|_{p,a,\Gamma;G},$$

where

$$\|f\|_{p,a,\Gamma;G} = \sup_{x\in G,t>0}\left([t]_1^{-\sum_{j=1}^n\frac{\Gamma_j a_j}{p_j}}\|f\|_{\mathbf{p},G_{t^\Gamma}(x)}\right),$$

Besov-Morrey spaces $B^l_{p,\theta,a,\Gamma,\infty}(G) \equiv B^l_{p,\theta,a,\Gamma}(G)$ with finite norm:

$$\|f\|_{B^l_{p,\theta,a,\Gamma}(G)} = \|f\|_{p,a,\Gamma,G} + \sum_{i=1}^n\left\{\int_0^{h_0}\left[\frac{\left\|\triangle_i^{m_i}(h,G)D_i^{k_i}f\right\|_{p,a,\Gamma}}{h^{l_i-k_i}}\right]^\theta\frac{dh}{h}\right\}^{1/\theta},$$

$$\Delta_i^{m_i}(h)f(x) = \sum_{i=0}^{m_i}(-1)^{m_i-j}C^j_{m_i}f(x+jhe_i),$$

were introdused and studied by Yu.V.Netrusov [14]. In the papers [15-17] introdused and studied Besov-Morrey type spaces $B^l_{p,\theta,a,\Gamma,\tau}(G,\lambda)$ with finite norm ($m_i > l_i - k_i > 0$):

$$\|f\|_{B^l_{p,\theta,a,\Gamma,\tau}(G,\lambda)} = \|f\|_{p,a,\Gamma,\tau;G} + \sum_{i=1}^{n} \left\{ \int_0^{h_0} \left[\frac{\left\| \triangle_i^{m_i}(h,G,\lambda) D_i^{k_i} f \right\|_{p,a,\Gamma,\tau}}{h^{l_i-k_i}} \right]^\theta \frac{dh}{h} \right\}^{1/\theta}$$

where $\triangle_i^{m_i}(h,G,\lambda) = \triangle_i^{m_i}(h,G_{h^\lambda})$, Lizorkin-Triebel-Morrey type spaces $F^l_{p,\theta,a,\Gamma,\tau}(G)$ with finite norm ($m_i > l_i > k_i \geq 0$):

$$\|f\|_{F^l_{p,\theta,a,\Gamma,\tau}(G)} = \|f\|_{p,a,\Gamma,\tau;G} + \sum_{i=1}^{n} \left\| \left\{ \int_0^{t_0} \left[t^{(k_i-l_i)\lambda_i} \delta_i^{m_i-k_i}(t^\lambda) D_i^{k_i} f(\cdot) \right]^\theta \frac{dt}{t} \right\}^{\frac{1}{\theta}} \right\|_{p,a,\Gamma,\tau},$$

where

$$\delta_i^{m_i}(t^\lambda) f(x) = \int_{-1}^{1} \left| \triangle_i^{m_i}(t^{\lambda_i}u, G_{t^\lambda}) f(x) \right| du,$$

h_0 and t_0 fixed positive numbers and Besov-Morrey type spaces with dominating mixed derivative $S^l_{p,\theta,a,\Gamma,\tau}B(G_h)$ with finite norm ($m_j > l_j - k_j > 0$):

$$\|f\|_{S^l_{p,\theta,a,\Gamma,\tau}B(G_h)} = \sum_{e \subseteq e_n} \left\{ \int_{0^e}^{h_0^e} \left[\frac{\left\| \Delta^{m^e}(h,G_h) D^{k^e} f \right\|_{p,a,\Gamma,\tau}}{\prod_{j\in e} h_j^{l_j-k_j}} \right]^\theta \prod_{j\in e} \frac{dh_j}{h_j} \right\}^{\frac{1}{\theta}}$$

where

$$\Delta^{m^e}(h) f(x) = \left(\prod_{j\in e} \Delta_j^{m_j}(h_j) \right) f(x),$$

$$\|f\|_{p,a,\Gamma,\tau;G} = \sup_{x\in G} \left\{ \int_0^{t_1^0} \cdots \int_0^{t_n^0} \left[\prod_{j=1}^{n} [t_j]_1^{-\Gamma_j \frac{a_j}{p_j}} \left\{ \int_{G_{t_n^{\Gamma_n}}(x_n)} \left[\cdots \left\{ \int_{G_{t_2^{\Gamma_2}}(x_2)} \left(\int_{G_{t_1^{\Gamma_1}}(x_1)} |f|^{p_1} dy_1 \right)^{p_2/p_1} dy_2 \right\}^{p_3/p_2} \cdots \right]^{p_n/p_{n-1}} dy_n \right\}^{1/p_n} \right]^\tau \prod_{j=1}^{n} \frac{dt_j}{t_j} \right\}^{1/\tau}.$$

Note some properties of the spaces $L_{p,a,\Gamma,\tau}(G)$ and $W^l_{p,a,\Gamma,\tau}(Q,G)$.

1) The embedding theorems are valid:

$L_{p,a,\Gamma,\tau}(G) \hookrightarrow L_{p,a,\Gamma}(G)$, $W^l_{p,a,\Gamma,\tau}(Q,G) \hookrightarrow W^l_{p,a,\Gamma}(Q,G)$ i.e.

$$\|f\|_{p,a,\Gamma;G} \leq C \|f\|_{p,a,\Gamma,\tau;G} \tag{4}$$

and

$$\|f\|_{W^l_{p,a,\Gamma}(Q,G)} \le C\,\|f\|_{W^l_{p,a,\Gamma,\tau}(Q,G)}\,. \tag{5}$$

2) For all the real $c>0$ it

$$\|f\|_{p,a,c\Gamma,\tau;G} = \frac{1}{c^{\frac{1}{\tau}}}\,\|f\|_{p,a,\Gamma,\tau;G} \tag{6}$$

and

$$\|f\|_{W^l_{p,a,c\Gamma,\tau}(G)} = \frac{1}{c^{\frac{1}{\tau}}}\,\|f\|_{W^l_{p,a,\Gamma,\tau}(G)}\,; \tag{7}$$

are valid. 3) a) $\|f\|_{p,0,\Gamma,\infty;G} = \|f\|_{p;G}$ and $\|f\|_{W^l_{p,0,\Gamma,\infty}(Q,G)} = \|f\|_{W^l_p(Q,G)}$;

b) $\|f\|_{\infty;G} \le \|f\|_{p,1,\Gamma,\tau;G}$ and $\|f\|_{W^l_p(Q,G)} \le \|f\|_{W^l_{p,1,\Gamma,\tau}(Q,G)}$.

Let $M_{e,i}(\cdot,y,z)\in C_0^\infty(R^n)$ and support of the function $M_{e,i}(\cdot,y,z)$ be such that

$$S(M_{e,i}) = \operatorname{supp} M_{e,i} \subset I_1 = \left\{x : |x_j| < \frac{1}{2},\, j\in e_n\right\}$$

Denote

$$V = \bigcup_{0<t_j\le T_j,\, j\in e_n} \left\{y : \left(\frac{y}{t^\lambda}\right)\in S(M_{e,i})\right\}.$$

Obviously $V\subset I_{T^\lambda}$, U is a open set, contained in G. Further, suppose that $U+V\subset G$. Let

$$G_{T^\Gamma}(U) = \bigcup_{x\in G} G_{T^\Gamma}(x) = (U + I_{T^\Gamma}(x))\cap G,$$

if $0<T_j\le 1$, $0<\Gamma_j\le\lambda_j$, $j\in e_n$, then $I_{T^\lambda}\subset I_{T^\Gamma}$, it follows that $U+V\subset G_{T^\Gamma}(U) = Z$.

Lemma 1. Let $1\le p\le q\le r\le\infty$, $0<\Gamma_j\le\lambda_j$, $0<t_j\le T_j\le 1$, $j\in e_n$, $1\le\tau\le\infty$, $\eta=(\eta_1,\eta_2,\ldots,\eta_n)$, $0<\eta_j\le T_j$, $j\in e_n$, $\eta_i=\eta_j$, for $i,j\in e_n\backslash Q$; $\nu=(\nu_1,\nu_2,\ldots,\nu_n)$, $\nu_j\ge 0-$ be integer, $j\in e_n$, $\phi\in L_{p,a,\Gamma,\tau}(G)$ and let $k\in Q$

$$\varepsilon_j = \lambda_j l_j - \sum_{j=k\vee j=i\in e_n\backslash Q}\left[\lambda_j\nu_j + (\lambda_j - \Gamma_j a_j)\left(\frac{1}{p}-\frac{1}{q}\right)\right],$$

$$A^{e,i}_\eta(x) = \prod_{j\in Q\backslash e} T^{-1-\nu_j}\int_{0^{e\vee i}}^{\eta^{e\vee i}} \frac{\varphi_{e,i}(x,t)\,dt^{e\vee i}}{\prod_{j=k\in e\vee j=i} t_j^{1-\lambda_j l_j+\sum_{j=k\vee j=i\in e_n\backslash Q}(\lambda_j+\lambda_j\nu_j)}}, \tag{8}$$

$$A^{e,i}_{\eta T}(x) = \prod_{j\in Q\backslash e} T^{-1-\nu_j}\int_{\eta^{e\vee i}}^{T^{e\vee i}} \frac{\varphi_{e,i}(x,t)\,dt^{e\vee i}}{\prod_{j=k\in e\vee j=i} t_j^{1-\lambda_j l_j+\sum_{j=k\vee j=i\in e_n\backslash Q}(\lambda_j+\lambda_j\nu_j)}}, \tag{9}$$

where

$$\varphi_{e,i}(x,t) = \int_{R^n}\phi(x+y)M_{e,i}\left(\frac{y}{t^{\lambda^{e_n\backslash(Q\backslash e)}} + T^{\lambda^{Q\backslash e}}},\right.$$

$$\frac{\rho(t^{\lambda^{e_n\backslash(Q\backslash e)}}+T^{\lambda^{Q\backslash e}},x)}{t^{\lambda^{e_n\backslash(Q\backslash e)}}+T^{\lambda^{Q\backslash e}}},\rho'(t^{\lambda^{e_n\backslash(Q\backslash e)}}+T^{\lambda^{Q\backslash e}},x)\Bigg)\,dy. \tag{10}$$

Then the following inequalities hold:

$$\sup_{x\in G}\left\|A_{\eta}^{e,i}\right\|_{q,U_{\gamma^\Gamma}(\overline{x})}\le C_1\left\|\phi\right\|_{p,a,\Gamma,\tau;Z}\times$$
$$\times\prod_{j\in e_n}[\gamma_j]_1^{\frac{\Gamma_j a_j}{q}}\prod_{j\in Q\backslash e}T_j^{-\nu_j-(1-\Gamma_j a_j)\left(\frac{1}{p}-\frac{1}{q}\right)}\prod_{j\in e\vee i}\eta_j^{\varepsilon_j}\ (\varepsilon_j>0), \tag{11}$$

$$\sup_{x\in G}\left\|A_{\eta T}^{e,i}\right\|_{q,U_{\gamma^\Gamma}(\overline{x})}\le C_2\left\|\phi\right\|_{p,a,\Gamma,\tau;Z}\prod_{j\in e_n}[\gamma_j]_1^{\frac{\Gamma_j a_j}{q}}\times$$
$$\times\prod_{j\in Q\backslash e}T_j^{-\nu_j-(1-\Gamma_j a_j)\left(\frac{1}{p}-\frac{1}{q}\right)}\begin{cases}\prod\limits_{j=k\in e\vee j=i}T_j^{\varepsilon_j}, & \varepsilon_j>0,\\ \prod\limits_{j=k\in e\vee j=i}\ln\frac{T_j}{\eta_j}, & \varepsilon_j=0,\\ \prod\limits_{j=k\in e\vee j=i}\eta_j^{\varepsilon_j}, & \varepsilon_j<0,\end{cases} \tag{12}$$

where $\gamma\in(0,\infty)^n$, $\gamma_i=\gamma_j$ for $i,j\in e_n\backslash Q$, C_1 and $C_2$are constants independent of ϕ,γ,η and T.

Proof of Lemma 1. Applying the generalized Minkowskii inequalities for all $x\in U$, we have

$$\sup_{x\in G}\left\|A_{\eta}^{e,i}\right\|_{q,U_{\gamma^\Gamma}(\overline{x})}\le\prod_{j\in Q\backslash e}T_j^{-1-\nu_j}\times$$
$$\times\int\limits_{0^{e\vee i}}^{\eta^{e\vee i}}\frac{1}{\prod\limits_{j=k\in e\vee j=i}t_j^{1-\lambda_j l_j+\sum\limits_{j=k\vee j=i\in e_n\backslash Q}(\lambda_j+\lambda_j\nu_j)}}\left\|\varphi_{e,i}(\cdot,t)\right\|_{q,U_{\gamma^\Gamma}(\overline{x})}dt^{e\vee i} \tag{13}$$

We estimate the norm $\left\|\varphi_{e,i}(\cdot,t)\right\|_{q,U_{\gamma^\Gamma}(\overline{x})}$. By virtue of Holder's inequalities $(q\le r)$ we have

$$\left\|\varphi_{e,i}(\cdot,t)\right\|_{q,U_{\gamma^\Gamma}(\overline{x})}\le\left\|\varphi_{e,i}(\cdot,t)\right\|_{r,U_{\gamma^\Gamma}(\overline{x})}\prod_{j\in e_n}\gamma_j^{\Gamma_j\left(\frac{1}{q}-\frac{1}{r}\right)} \tag{14}$$

Let $\chi-$ be the characteristic function of the set $S(M_{e,i})$. Observing that $1\le p\le r\le\infty$, $s\le r$ $\left(\frac{1}{s}=1-\frac{1}{p}+\frac{1}{r}\right)$ we represent the right hand side of equality (10) as

$$|\phi M_{e,i}|=\left(|\phi|^p|M_{e,i}|^s\right)^{\frac{1}{r}}\left(|\phi|^p\chi\right)^{\frac{1}{p}-\frac{1}{r}}\left(|M_{e,i}|^s\right)^{\frac{1}{s}-\frac{1}{r}} \tag{15}$$

and again applying to $|\varphi_{e,i}(x,t)|$ Holder's inequality with the exponent $\left(\frac{1}{r}+\left(\frac{1}{p}-\frac{1}{r}\right)+\left(\frac{1}{s}-\frac{1}{r}\right)=1\right)$, we have

$$\|\varphi_{e,i}(\cdot,t)\|_{r,U_{\gamma^\Gamma}(\overline{x})} \leq \sup_{x\in U_{\gamma^\Gamma}(\overline{x})}\left(\int_{R^n}|\phi(x+y)|^p\,\chi\left(\frac{y}{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}}\right)dy\right)^{\frac{1}{p}-\frac{1}{r}}\times$$
$$\times\sup_{y\in V}\left(\int_{U_{\gamma^\Gamma}(\overline{x})}|\phi(x+y)|^p\,dx\right)^{\frac{1}{r}}\left(\int_{R^n}\left|M_{e,i}\left(\frac{y}{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}},\right.\right.\right.$$
$$\left.\left.\left.\frac{\rho(t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}},x)}{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}},\rho'(t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}},x)\right)\right|^s dy\right)^{\frac{1}{s}}. \qquad (16)$$

Obviously $Z_{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}}(x)\subset Z_{t^{\Gamma^{e_n\setminus(Q\setminus e)}}+T^{\Gamma^{Q\setminus e}}}(x)$ for $0<t_j\leq T_j\leq 1$, $\Gamma_j\leq\lambda_j$, $j\in e_n$ and for any $x\in U$

$$\int_{R^n}|\phi(x+y)|^p\,\chi\left(\frac{y}{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}}\right)dy\leq$$
$$\leq\int_{Z_{t^{\Gamma^{e_n\setminus(Q\setminus e)}}+T^{\Gamma^{Q\setminus e}}}(x)}|\phi(x+y)|^p\,\chi\left(\frac{y}{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}}\right)dy\leq$$
$$\leq\|\phi\|^p_{p,a,\Gamma;Z}\prod_{j\in e_n\setminus(Q\setminus e)}t_j^{\Gamma_j a_j}\prod_{j\in Q\setminus e}T_j^{\Gamma_j a_j}. \qquad (17)$$

For $y\in V$

$$\int_{U_{\gamma^\Gamma}(\overline{x})}|\phi(x+y)|^p\,dx\leq\int_{Z_{\gamma^\Gamma}(\overline{x+y})}|\phi(x)|^p\,dx\leq\|\phi\|^p_{p,a,\Gamma;Z}\prod_{j\in e_n}[\gamma_j]_1^{\Gamma_j a_j}, \qquad (18)$$

$$\int_{R^n}\left|M_{e,i}\left(\frac{y}{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}},\frac{\rho(t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}},x)}{t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}}},\right.\right.$$
$$\left.\left.\rho'(t^{\lambda^{e_n\setminus(Q\setminus e)}}+T^{\lambda^{Q\setminus e}},x)\right)\right|^s dy=\|M_{e,i}\|_s^s\prod_{j\in e_n\setminus(Q\setminus e)}t_j^{\lambda_j}\prod_{j\in Q\setminus e}T_j^{\lambda_j}. \qquad (19)$$

From the inequalities (14) and (16)-(19) it follows that

$$\|\varphi_{e,i}(\cdot,t)\|_{q,U_{\gamma^\Gamma}(\overline{x})}\leq\|\phi\|_{p,a,\Gamma;Z}\prod_{j\in Q\setminus e}T_j^{1-(1-\Gamma_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)}\prod_{j\in e_n}\gamma_j^{\Gamma_j\left(\frac{1}{q}-\frac{1}{r}\right)}\times$$
$$\times\prod_{j\in e_n}[\gamma_j]_1^{\Gamma_j a_j}\prod_{j=k\in e\vee j=i}t_j^{\lambda_j l_j+\sum\limits_{j=k\vee j=i\in e_n\setminus Q}\left[\lambda_j-(\lambda_j-\Gamma_j a_j)\left(\frac{1}{p}-\frac{1}{r}\right)\right]}. \qquad (20)$$

Taking into account inequality (4) and substituting the inequality (19) in (13) for $r=q$ we obtain the inequality (11). Analogously the inequality (12) is proved. □

Lemma 2. Let $1 \le p \le q < \infty$, $0 < \Gamma_j \le \lambda_j$, $0 < T_j \le 1$, $j \in e_n$, $1 \le \tau_1 \le \tau_2 \le \infty$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \ge 0$ be integer, $j \in e_n$, and $\varepsilon_j > 0$,

$$\varepsilon_j^0 = \lambda_j l_j - \sum_{j=k \vee j=i \in e_n \setminus Q} \left[\lambda_j \nu_j + (\lambda_j - \Gamma_j a_j) \frac{1}{p} \right], k \in Q,$$

then for the function $A_T^{e,i}(x)$, defined by the equality (8) the following estimate is valid:

$$\left\| A_\eta^{e,i} \right\|_{q,b,\Gamma,\tau_2;U} \le C \left\| \phi \right\|_{p,a,\Gamma,\tau_1;Z}, \tag{21}$$

where $b = (b_1, b_2, \ldots b_n)$, b_j is any number, satisfying the inequalities:

$$\begin{aligned} &0 \le b_j \le 1, \quad \text{if} \quad \varepsilon_j^0 > 0, \quad j \in e_n \setminus (Q \setminus e), \\ &0 \le b_j < 1, \quad \text{if} \quad \varepsilon_j^0 = 0, \quad j \in e_n \setminus (Q \setminus e); \quad 0 \le b_j \le a_j, \quad j \in Q \setminus e \\ &0 \le b_j \le 1 + \frac{\varepsilon_j^0 q (1 - a_j)}{|\lambda|_{(e_n \setminus Q) \vee k} - |\Gamma, a|_{(e_n \setminus Q) \vee k}}, \quad \text{if} \quad \varepsilon_j^0 < 0, \quad j \in e_n \setminus (Q \setminus e), \end{aligned}$$

$k \in Q$ and C_1 is a constant independent of ϕ.

Theorem 1. Let the open set $G \subset R^n$ satisfy the condition of flexible $\lambda-$horn, $1 \le p \le q \le \infty$, $0 < \Gamma_j \le \lambda_j$, $0 < T_j \le 1$, $j \in e_n$, $1 \le \tau_1 \le \tau_2 \le \infty$, $\nu = (\nu_1, \nu_2, \ldots, \nu_n)$, $\nu_j \ge 0$ be integer, $j \in e_n$, $f \in W^l_{p,a,\Gamma,\tau_1}(Q, G)$ and let $\varepsilon_j > 0$, $j \in e_n$ $(k \in Q)$ then

$$D^\nu : W^l_{p,a,\Gamma,\tau_1}(Q, G) \hookrightarrow L_{q,b,\Gamma,\tau_2}(G),$$

i.e., for the function f the generalized deriatives $D^\nu f$ exist and the inequalities are valid:

$$\|D^\nu f\|_{q,G} \le C_1 \sum_{e \subset Q} \sum_{i \in e_n^0 \setminus Q} \prod_{j \in Q \vee i} T_j^{s_j} \left\| D^{l^{e \vee i}} f \right\|_{p,a,\Gamma,\tau_1;G} \tag{22}$$

$$\|D^\nu f\|_{q,b,\Gamma,\tau_2;G} \le C_2 \|f\|_{W^l_{p,a,\Gamma,\tau_1}(Q,G)}, (p \le q < \infty) \tag{23}$$

where

$$s_j = \begin{cases} \varepsilon_j, & j \in e \vee i, \\ -\nu_j - (1 - \Gamma_j a_j) \left(\frac{1}{p} - \frac{1}{q} \right), & j \in Q \setminus e. \end{cases}$$

In particular, if $\varepsilon_j^0 > 0$, $j \in e_n$ $(k \in Q)$, then $D^\nu f$ is continous on G and the inequalitiy is valid:

$$\sup_{x \in G} |D^\nu f| \le C_1 \sum_{e \subset Q} \sum_{i \in e_n^0 \setminus Q} \prod_{j \in Q \vee i} T_j^{s_j^0} \left\| D^{l^{e \vee i}} f \right\|_{p,a,\Gamma,\tau_1;G}, \tag{24}$$

where

$$s_j^0 = \begin{cases} \varepsilon_j^0, & j \in e \vee i, \\ -\nu_j - (1 - \Gamma_j a_j) \frac{1}{p}, & j \in Q \setminus e, \end{cases}$$

here $T_j \in (0, \min(1, t_{0j})]$, $j \in e_n$, C_1 and C_2 are constants independent of f, and a constant C_1 doesn't depend on T also.

Proof of Theorem 1. First of all note, that as far as $\overline{\Gamma} = c\Gamma$, $c > 0$, we can consider $f \in W^l_{p,a,\overline{\Gamma},\tau_1}(Q,G)$ and we can replace everywhere in inequalities (22) -(24), ε_j and ε_j^0, Γ_j into $\overline{\Gamma}_j$, $j \in e_n$. We'll prove exactly such inequalities (the more Γ_j, the more ε_j). From the conditions of our theorem it follows from theorem 2 of [2] that on G there exist generalized derivatives $D^\nu f$ and $D^\nu f \in L_p(G)$. Really, if $\varepsilon_j > 0$, $j \in e_n$ $(k \in Q)$, then $\lambda_j l_j - \sum\limits_{j=k \vee j=i \in e_n \setminus Q} \lambda_j \nu_j > 0$, $j \in e_n$ $(k \in Q)$, since $1 \le p \le q \le \infty$, $0 < a_j \le 1$, $0 < \Gamma_j \le \lambda_j$, $j \in e_n$. Since $f \in W^l_{p,a,\Gamma,\tau_1}(Q,G) \hookrightarrow W^l_{p,a,\Gamma}(Q,G) \hookrightarrow W^l_p(Q,G)$, then by virtue of theorem 2 of [2] on G the generalized derivatives exist $D^\nu f \in L_p(G)$. Then for almost every point $x \in G$ the integral identity holds:

$$D^\nu f(x) = \sum_{e \subset Q} \sum_{i \in e_n^0 \setminus Q} (-1)^{\left|\nu^{e \vee i}\right|} \prod_{j \in Q \setminus e} T_j^{-1-\nu_j} \times$$
$$\times \int\limits_{0^{e\vee i}}^{T^{e\vee i}} \int\limits_{R^n} \frac{dt^{e \vee i}}{\prod\limits_{j=k \in e \vee j=i} t_j^{1-\lambda_j l_j + \sum\limits_{j=k \vee j=i \in e_n \setminus Q} (\lambda_j + \lambda_j \nu_j)}} L_{e,i}^{(\nu)} \left(\frac{y}{t^{\lambda^{e_n \setminus (Q \setminus e)}} + T^{\lambda^{Q \setminus e}}}, \right.$$
$$\left. \frac{\rho(t^{\lambda^{e_n \setminus (Q \setminus e)}} + T^{\lambda^{Q \setminus e}}, x)}{t^{\lambda^{e_n \setminus (Q \setminus e)}} + T^{\lambda^{Q \setminus e}}}, \rho'(t^{\lambda^{e_n \setminus (Q \setminus e)}} + T^{\lambda^{Q \setminus e}}, x) \right) D^{l^{e \vee i}} f(x+y) dy \tag{25}$$

the functions $L_{e,i} \in C_0^\infty(R^n)$, have support contained in in I_1, because the support of representation (25) is contained in flexible $\lambda-$horn $x+ +V(\lambda, x, \theta) \subset G$. Here by virtue of Minkowskii inequalities, we have

$$\|D^\nu f(x)\|_{q,G} \le \sum_{e \subset Q} \sum_{i \in e_n^0 \setminus Q} \left\|A_T^{e,i}\right\|_{q,G}. \tag{26}$$

With the help of inequality (10) for $U = G$, $D^{l^{e \vee i}} f = \phi$, $M_{e,i} = L_{e,i}^{(\nu)}$, $\gamma \to \infty$ we have

$$\left\|A_T^{e,i}\right\|_{q,G} \le C \left\|D^{l^{e \vee i}} f\right\|_{p,a,\overline{\Gamma},\tau;G} \prod_{j \in Q \vee i} T_j^{s_j}, \tag{27}$$

and, here the inequality (22) follows. Analogously by virtue of inequality (21) the inequality (23) follows.

Let now $\varepsilon_j^0 > 0$, $j \in e_n$ $(k \in Q)$. Show that $D^\nu f$ is continous on G. By virtue of identity (25) and inequality (22) for $q = \infty$, $\varepsilon_j = \varepsilon_j^0 > 0$, $j \in e_n$ $(k \in Q)$, we have

$$\|D^\nu f - D^\nu f_{T^\lambda}\|_{\infty,G} \le C_1 \sum_{e \subset Q} \sum_{i \in e_n^0 \setminus Q} \prod_{j \in Q \vee i} T_j^{s_j^0} \left\|D^{l^{e \vee i}} f\right\|_{p,a,\overline{\Gamma},\tau;G}. \tag{28}$$

Here it follows that the left hand side of inequality (28) converges to zero for $T_j \to 0$, $j \in e_n$. Since $D^\nu f_{T^\lambda}$ is continuous on G then convergence on $L_\infty(G)$ in this case coincides with the uniform and consequently $D^\nu f$ is continuous in G. Theorem 1 is proved. □

Theorem 2. Let the domain G, the parameters p, q, τ_1, τ_2 and vectors $\overline{\Gamma}$, Γ, ν satisfiy the condition of theorem 1. If $\varepsilon_j > 0$, $j \in e_n$, $k \in Q$, then the derivatives $D^\nu f$ satisfiy the Holder's condition in metric L_q with exponent β^1, more exactly,

$$\|\Delta(\xi, G)D^\nu f\|_{q,G} \leq C\,\|f\|_{W^l_{p,a,\Gamma,\tau}(Q,G)} \prod_{j\in Q} |\xi_j|^{\beta_j^1}\,|\xi|^{\beta^1}_{e_n\backslash Q}\,, \tag{29}$$

where $\beta^1 = (\beta_1^1, \beta_2^1, \ldots, \beta_n^1)$, $\beta_i^1 = \beta_j^1$ for $i, j \in e_n\backslash Q$ and β_j^1- for any number satisfying the following inequalities:

$$\begin{array}{llll} 0 \leq \beta_j^1 \leq 1, & \text{if} & \varepsilon_j > 1 & \text{for} \quad j \in e, \\ 0 \leq \beta_j^1 < 1, & \text{if} & \varepsilon_j = 1 & \text{for} \quad j \in e; \\ 0 \leq \beta_j^1 \leq 1 & \text{for} & j \in Q\backslash e; & \\ 0 \leq \beta_j^1 \leq \varepsilon_j, & \text{if} & \varepsilon_j < 1 & \text{for} \quad j \in e, \\ 0 \leq \beta^1 \leq 1, & \text{if} & \frac{\varepsilon_0}{\lambda_0} > 1, & \\ 0 \leq \beta^1 < 1, & \text{if} & \frac{\varepsilon_0}{\lambda_0} = 1, & \\ 0 \leq \beta^1 \leq \frac{\varepsilon_0}{\lambda_0}, & \text{if} & \frac{\varepsilon_0}{\lambda_0} < 1, & \end{array}$$

where $\lambda_0 = \max\limits_{j\in e_n\backslash Q} \lambda_j$, $\varepsilon_0 = \min \varepsilon_{Q,i}$, $\varepsilon_{Q,i} = \lambda_i l_i - \sum\limits_{j\in e_n\backslash Q} \left[\lambda_j \nu_j + (\lambda_j - \Gamma_j a_j)\left(\frac{1}{p} - \frac{1}{q}\right)\right]$, $i \in e_n\backslash Q$.

If $\varepsilon_j^0 > 0$, $j \in e_n, k \in Q$, then

$$\sup_{x\in G} |\Delta(\xi, G)D^\nu f| \leq C\,\|f\|_{W^l_{p,a,\Gamma,\tau}(Q,G)} \prod_{j\in Q} |\xi_j|^{\beta_j^{1,0}}\,|\xi|^{\beta^{1,0}}_{e_n\backslash Q}\,, \tag{30}$$

where $\beta_j^{1,0}(j \in e_n)$ satisfies the same conditions, that β_j^1 with the substitution of ε_j by ε_j^0.

Proof of Theorem 2. As in proof of theorem 1, we can replace vector Γ by $\overline{\Gamma}$. Let ξ be an $n-$ dimensional vector. According to lemma 8.6 in [1] there exists domain $G_\sigma \subset G$, $\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n)$, $\sigma_j = \varsigma_j r_\lambda^\lambda$ for $j \in e_n$; $\sigma_i = \sigma_j$ for $i, j \in e_n\backslash Q$; $r_\lambda = dist_\lambda(x, \partial G)$, $x \in G$. We'll suppose that $\xi_j < \sigma_j$ for $j \in Q$, $|\xi|_{e_n\backslash Q} < \sigma$, then for every $x \in G_\sigma$, segment $[x, x+\xi]$ is contained in G. For any point of the segment $[x, x+\xi]$ integral representation (25) is valid with the same kernel. After some transformation we have

$$\begin{aligned} |\Delta(\xi, G)D^\nu f| \leq C_1 \sum_{e\subset Q} \sum_{i\in e_n\backslash Q} \prod_{j\in Q\backslash e} T_j^{-1-\nu_j} \times \\ \times \int\limits_{0^{e\vee i}}^{|\xi^e|\vee|\xi|^{\frac{1}{\lambda_0}}_{e_n\backslash Q}} \frac{dt^{e\vee i}}{\prod\limits_{j=k\in e\vee j=i} t_j^{1-\lambda_j l_j + \sum\limits_{j=k\vee j=i\in e_n\backslash Q}(\lambda_j+\lambda_j\nu_j)}} \times \\ \times \int\limits_{R^n} \left|L_{e,i}^{(\nu)}\right| \left|\Delta(\xi, G)D^{l^{e\vee i}} f(x+y)\right| dy + C_2 \sum_{e\subset Q} \sum_{i\in e_n\backslash Q} \prod_{j\in Q\backslash e} T_j^{-1-\nu_j} \times \end{aligned} \tag{31}$$

$$\times \prod_{j\in Q} |\xi_j|\, |\xi|_{e_n\setminus Q} \int\limits_{|\xi^e| \vee |\xi|^{\frac{1}{\lambda_0}}_{e_n\setminus Q}}^{T^{e\vee i}} \frac{dt^{e\vee i}}{\prod\limits_{j=k\in e\vee j=i} t_j^{1-\lambda_j l_j+\lambda_j+\sum\limits_{j=k\vee j=i\in e_n\setminus Q}(\lambda_j+\lambda_j\nu_j)}} \times$$

$$\times \int\limits_{R^n} \left|L^{(\nu+1)}_{e,i}\right| \int\limits_0^1 \left|D^{l^{e\vee i}} f(x+y+\xi_1\varsigma_1+\cdots+\xi_n\varsigma_n)\right| d\varsigma dy =$$

$$= C_1 \sum_{e\subset Q}\sum_{i\in e_n\setminus Q} A_1^{e,i}(x,\xi) + C_2 \sum_{e\subset Q}\sum_{i\in e_n\setminus Q} A_2^{e,i}(x,\xi), \tag{32}$$

Here $|\xi^e| = (|\xi_1^e|, |\xi_2^e|, \ldots, |\xi_n^e|)$, $\left|\xi_j^e\right| = |\xi_j|$ for $j\in e$; $\left|\xi_j^e\right| = 0$ for $j \in Q\setminus e$ and we can consider, that $|\xi_j| < T_j$ for $j\in Q$; $|\xi|_{e_n\setminus Q} < T$, $0 < T_j \le \min(1, t_{j,0})$ for $j\in e_n$, consequently $|\xi_j| < \min(\sigma_j, T_j)$ for $j \in Q$; $|\xi|_{e_n\setminus Q} < \min(\sigma^{\lambda_0}, T^{\lambda_0})$. If $x\in G\setminus G_\sigma$, then by definition

$$\Delta(\xi, G) D^\nu f = 0.$$

Then

$$\begin{aligned}\|\Delta(\xi,G)D^\nu f\|_{q,G} &= \|\Delta(\xi,G)D^\nu f\|_{q,G_\sigma}\\ &\le C_1 \sum_{e\subset Q}\sum_{i\in e_n\setminus Q}\left\|A_1^{e,i}(\cdot,\xi)\right\|_{q,G_\sigma} + C_2 \sum_{e\subset Q}\sum_{i\in e_n\setminus Q}\left\|A_2^{e,i}(\cdot,\xi)\right\|_{q,G_\sigma}. \end{aligned}\tag{33}$$

With the help of inequality (11) for $U = G$, $D^{l^{e\vee i}} f = \phi, M_{e,i} = L^{(\nu)}_{e,i}$, $\gamma\to\infty$ we have

$$\left\|A_1^{e,i}(\cdot,\xi)\right\|_{q,G_\sigma} \le C_3 \prod_{j\in e} |\xi_j|^{\varepsilon_j} |\xi|^{\frac{\varepsilon_{Q,i}}{\lambda_0}}_{e_n\setminus Q} \left\|D^{l^{e\vee i}} f\right\|_{p,a,\overline{\Gamma},\tau;G}, \tag{34}$$

and, with the help of inequality (12) for $U = G$, $D^{l^{e\vee i}} f = \phi$, $M_{e,i} = L^{(\nu)}_{e,i}$, $\gamma\to\infty$ we have

$$\begin{aligned}\left\|A_2^{e,i}(\cdot,\xi)\right\|_{q,G_\sigma} &\le C_4 \prod_{j\in e} |\xi_j|^{\varepsilon_j} \prod_{j\in Q\setminus e} |\xi_j|\, |\xi|^{\frac{\varepsilon_{Q,i}}{\lambda_0}}_{e_n\setminus Q} \left\|D^{l^{e\vee i}} f\right\|_{p,a,\overline{\Gamma},\infty;G}\\ &\le C_5 \prod_{j\in Q} |\xi_j|^{\beta_j} |\xi|^{\beta}_{e_n\setminus Q} \left\|D^{l^{e\vee i}} f\right\|_{p,a,\overline{\Gamma},\infty;G}\\ &= C_6 \prod_{j\in Q} |\xi_j|^{\beta_j^1} |\xi|^{\beta^1}_{e_n\setminus Q} \left\|D^{l^{e\vee i}} f\right\|_{p,a,\overline{\Gamma},\tau;G}, \end{aligned}\tag{35}$$

From here with the help of inequality (32) we have (29), here $\beta_j^1 > \beta_j$, $j\in e_n$.

Assume now, that $|\xi_j| \ge \min(\sigma_j, T_j)$ for $j \in Q$; $|\xi|_{e_n\setminus Q} \ge \min(\sigma^{\lambda_0}, T^{\lambda_0})$. Then

$$\|\Delta(\xi,G)D^\nu f\|_{q,G} \le 2\,\|D^\nu f\|_{q,G} \le C(\sigma,T)\,\|D^\nu f\|_{q,G} \prod_{j\in Q} |\xi_j|^{\beta_j^1} |\xi|^{\beta^1}_{e_n\setminus Q}. \tag{36}$$

We estimate $\|D^\nu f\|_{q,G}$ with the help of inequality (22) again we have inequality (29). Theorem 2 is proved.

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