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Non-self mappings in modular spaces and common fixed point theorems

Research Article

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Abstract: The aim of this paper, is to introduce the convex structure (specially, Takahashi convex structure) on modular spaces. Moreover, we are interested in proving some common fixed point theorems for non-self mappings in modular space.

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1. Introduction

Generalization of the classic function spaces L^p has been a problem that mathematician dealt with for almost fifty years. First attempts made by Orlicz and Birnbaum. This generalization found many applications in differential and integral equations with kernels of non-power types. Nakano [7] gave more abstract generalization which was based on replacing the particular integral form of the functional by an abstractly one that satisfy some good properties. This functional was called a modular. Totaly, theory of modular spaces is introduced by Nakano [7] in relation to ordered space theory and is generalized by Musielak and Orlicz [6]. The purpose of this paper is, introducing Takahashi convex modular space and generalizing fixed point theorems of Jungck [4], [5], Ćirić [1], Das and Naik [2] and Ume [9] on ρ-complete modular space. In order to do this, we recall the definition of modular spaces as follows:

Definition 1.1.
Let X be an arbitrary vector space over $K(= R$ or $C)$.

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(a) Functional $\rho : X \to [0, \infty)$ is called *modular* if:

(I) $\rho(x) = 0$ if and only if $x = 0$.

(II) $\rho(kx) = \rho(x)$ for all k with $|k| = 1$, for all $x \in X$.

(III) $\rho(kx + ly) \leq \rho(x) + \rho(y)$ for all $x, y \in X$ for all $k, l \geq 0$ where $k + l = 1$.

(b) If (III) is replaced by:

(III)′ $\rho(kx + ly) \leq k^s \rho(x) + l^s \rho(y)$ for all $k, l \geq 0$ where $k^s + l^s = 1$ with an $s \in (0, 1]$.

The modular ρ is called an *s-convex modular*; and if $s = 1$, ρ is called *convex modular*.

(c) Modular ρ defines a corresponding modular space, i.e. the space X_ρ given by:

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\}.$$

We recall the following definition.

Definition 1.2.

Let X be a vector space on R. A functional $\|.\| : X \to [0, \infty)$ is called a *F-norm* if for arbitrary $x, y \in X$:

(i) $\|x\| = 0$ if and only if $x = 0$.

(ii) $\|\alpha x\| = \|x\|$ if α is a scalar and $|\alpha| = 1$.

(iii) $\|x + y\| \leq \|x\| + \|y\|$.

(iv) $\|\alpha_k x_k - \alpha x\| \to 0$ if $\alpha_k \to \alpha$ and $\|x_k - x\| \to 0$, where $\{x_k\}$ is a sequence of elements from X.

It is clear, any F-norm $\|.\|$ can be regarded as a modular space provided $\alpha \to \|\alpha x\|$ is increasing for every $x \in X$. A modular space X_ρ can be equipped with an F-norm defined by

$$\|x\|_\rho = \inf\{\alpha > 0 : \rho(\alpha^{-1} x) \leq \alpha\}.$$

In the case of an s-convex modular space ρ, the formula

$$\|x\|_\rho = \inf\{\alpha^s > 0 : \rho(x/\alpha) \leq 1\}.$$

define a s-norm (a F-norm with the additional property $\|\alpha x\| = |\alpha|^s \|x\|$).

Example 1.1.

Let $(X, ||.||)$ be a norm space, then $||.||$ is a modular. But the converse is not true.

Example 1.2.

Let $(X, ||.||)$ be a norm space. For any $k \geq 1$, $||.||^k$ is a modular on X.

Example 1.3 (Musielak-Orlicz modular space).

Let

$$\rho(f) = \int_\Omega \varphi(w, f(w)) d\mu(w),$$

where μ is a σ-finite measure on Ω and $\varphi : \Omega \times R \to [0, \infty)$ satisfy the following conditions:

(i) $\varphi(w, u)$ is a continuous even function of u which is nondecreasing for $u > 0$, such that $\varphi(w, 0) = 0$, $\varphi(w, u) > 0$ for $u \neq 0$ and $\varphi(w, u) \to \infty$ as $u \to \infty$.

(ii) $\varphi(w,u)$ is measurable function of w for each $u \in R$.

The corresponding modular space is called a *Musielak-Orlicz (or a generalized Orlicz) modular function space*, and is denoted by L^φ. If φ does not depend on the first variable, then L^φ is called an *Orlicz space*. An example of functions which satisfies the above conditions, is given by

$$\varphi(u) = |u|^p, \quad \text{for} \quad p > 0.$$

Then L^φ is isomorphic to L^p.

Definition 1.3.

Let X_ρ be a modular space.

(a) Sequence x_n in X_ρ is called:

- (I) *ρ-Cauchy* if $\rho(x_n - x_m) \to 0$ as n and m go to ∞.
- (II) *ρ-convergent to $x \in X_\rho$* if $\rho(x_n - x) \to 0$ as $n \to \infty$.

(b) X_ρ is *ρ-complete* if any ρ-Cauchy sequence is ρ-convergent.

(c) A subset $B \subset X$ is called *ρ-closed*, if for any sequence $(x_n)_n \subset B$ where $x_n \to x$ then $x \in B$. $\overline{B}^\rho$ denotes the closure of B in the sense of ρ.

(d) A subset $B \subset X_\rho$ is called *ρ-bounded* if $\delta_\rho(B) = \sup \rho(x - y) < \infty$ for all $x, y \in B$. $\delta_\rho(B)$ is called the *ρ-diameter* of B.

(e) ρ is said to have *Fatou property* if:

$$\rho(x - y) \leq \liminf \rho(x_n - y_n),$$

whenever $x_n \to x$ and $y_n \to y$ as $n \to \infty$.

(f) ρ is said to satisfy the *Δ_2-condition* if:

$$\rho(2x_n) \to 0 \quad \text{as} \quad n \to \infty \quad \text{whenever} \quad \rho(x_n) \to 0 \quad \text{as} \quad n \to \infty.$$

Definition 1.4.

A function $f : X_\rho \to X_\rho$ is called *ρ-continuous*, if $\rho(x_n - x) \to 0$, then $\rho(f(x_n) - f(x)) \to 0$.

Definition 1.5.

Assume X_ρ is a *modular space*. For $x, y \in X_\rho$, define

$$seg[x,y] = \{z \in X_\rho : z = (1-t)x + ty, \quad 0 \leq t \leq 1\}.$$

Definition 1.6.

Let X_ρ be a modular space and I be the closed unit interval. A mapping $W : X_\rho \times X_\rho \times I \longrightarrow X_\rho$ (that $X_\rho \times X_\rho$ is a product modular space with product topology) is called a *convex structure on X_ρ*, if (III) is replaced by:

$$\rho(W(x,y,\lambda)) \leq \lambda\rho(x) + (1-\lambda)\rho(y),$$

for all $x, y \in X_\rho$ and $\lambda \in I$.
X_ρ together with a convex structure is called a *Takahashi convex modular space*.

Remark 1.1.

Clearly, any convex modular space is a Takahashi convex modular space with $W(x,y,\lambda) = \lambda x + (1-\lambda)y$.

Definition 1.7.
Let X_ρ be a Takahashi convex modular space. For $x, y \in X_\rho$, define

$$seg[x, y] = \{W(x, y, \lambda) | \lambda \in [0, 1]\}.$$

It is worth mentioning to state two propositions from [3], which will be used in the next section to prove Theorems 2.1 and 2.2.

Proposition 1.1.
If (X_ρ, ρ) is a Takahashi convex modular space with convex structure W, then for every $x, y \in X_\rho$ and every $\lambda \in [0, 1]$

$$\rho(W(x, y, \lambda)) \leq \max\{\rho(x), \rho(y)\}.$$

Proof. Because (X_ρ, ρ) is a Takahashi convex modular space with convex structure W, then for every $x, y \in X_\rho$ and every $\lambda \in [0, 1]$ we have:

$$\begin{aligned} \rho(W(x, y, \lambda)) &\leq \lambda\rho(x) + (1 - \lambda)\rho(y) \\ &\leq (\lambda + (1 - \lambda)) \max\{\rho(x), \rho(y)\} \\ &= \max\{\rho(x), \rho(y)\}. \end{aligned}$$

□

Proposition 1.2 ([3]).
Let K be a nonempty closed subset of the Takahashi convex modular space with convex structure W that is continuous on it's third variable. Let $x \in K$ and $y \notin K$. Then there exists $\lambda \in [0, 1]$ such that

$$W(x, y, \lambda) \in seg[x, y] \cap \partial K.$$

L. B. Ćirić [1] has proved fixed point theorem for quasi contraction non-self mappings on Banach spaces. Afterward, common fixed point theorem for some of mappings that satisfy special property by K. M. Das and K. V. Naik are stated. Indeed, our main results are generalization of Theorems [8].

2. Main result

In this section, we prove some common fixed point theorems in modular space.

Theorem 2.1.
Let X_ρ be a ρ-complete modular space where ρ satisfies the Δ_2-condition, C a nonempty ρ-closed subset of X_ρ and $\partial C \neq \emptyset$(boundary of C). Suppose $g : C \to X_\rho$, $f : X_\rho \to X_\rho$, $f(C) \subset C$ and f is ρ-continuous. Moreover, assume f and g satisfy the following conditions.

1) There exists a constant $\lambda \in (0, 1)$ such that for every $x, y \in C$:

$$\rho(l(g(x) - g(y))) \leq \lambda M(x, y) \text{ for all } l \geq 1,$$

where

$$M(x, y) = \max\{\rho(f(x) - f(y)), \rho(f(x) - g(x)), \rho(f(y) - g(y)), \rho(f(y) - g(x)), \rho(f(x) - g(y))\}.$$

2) f and g are ρ-commutative on their coincidence points.

3) $g(C) \cap C \subset f(C)$.

4) $g(\partial C) \subset C$.

5) $f(\partial C) \supset \partial C$.

6) $f(C)$ *is* ρ*-closed.*

Then f *and* g *have a unique common fixed point in* C.

Proof. Assume $x_0 \in \partial C$, we construct a sequence x_n of points in C as follows: Assumption (4) shows $g(x_0) \in C$. Then by (3), there is $x_1 \in C$ such that $f(x_1) = g(x_0)$. Now, $g(x_1)$. If $g(x_1) \in C$ again by (3) there is $x_2 \in C$ such that $f(x_2) = g(x_1)$. If $g(x_1) \notin C$, by (5), there is $x_2 \in \partial C$ such that

$$f(x_2) \in \partial C \cap [f(x_1), g(x_1)].$$

Hence, by induction one can construct a sequence $\{x_n\}$ of points in C as follows:
If $g(x_n) \in C$, then by (3)

$$f(x_{n+1}) = g(x_n) \text{ for some } x_{n+1} \in C.$$

If $g(x_n) \notin C$, then by (5) pick $x_{n+1} \in \partial C$ such that

$$f(x_{n+1}) \in \partial C \cap seg[f(x_n), g(x_n)].$$

Now, it is enough to show, $\{f(x_n)\}$ and $\{g(x_n)\}$ are ρ-Cauchy sequences. Due to this, we need to prove if $f(x_{n+1}) \neq g(x_n)$, then $f(x_n) = g(x_{n-1})$. Assume to the contrary that $f(x_n) \neq g(x_{n-1})$, then $x_n \in \partial C$, and by (3), $g(x_n) \in C$. Hence $f(x_{n+1}) = g(x_n)$, and this contradicting the assumption. In order to prove ρ-Cauchy of the sequences, one can apply the following notations:

$$\begin{aligned} &B(n,k) = \{f(x_j), g(x_j) : n \leq j \leq n+k\},\\ &B(n) = \{f(x_j), g(x_j) : n \leq j\},\\ &b(n,k) = \sup\{\rho(x-y) : x, y \in B(n,k)\},\\ &b(n) = \sup\{\rho(x-y) : x, y \in B(n)\}. \end{aligned}$$

Note that $b(n,k)_{k\to\infty} \uparrow b(n)$ and $b(n) \downarrow$. Hence $b = \lim_{n\to\infty} b(n) \geq 0$ exists. We claim that

$$b(n,k) \leq \lambda b(n-2, k+2) \quad n, k \geq 2. \tag{1}$$

In order to show the above inequality, we have to consider three cases as follows:
I) $b(n,k) = \rho(f(x_i) - g(x_j))$, for $n \leq i, j \leq n+k$. Then according to the definition of $b(n,k)$, we have two cases:
a) $f(x_i) = g(x_{i-1})$, then

$$\begin{aligned} b(n,k) = \;& \rho(g(x_{i-1}) - g(x_j))\\ \leq\;& \rho(l(g(x_{i-1}) - g(x_j)))\\ \leq\;& \lambda M(x_{i-1}, x_j)\\ \leq\;& b(n-2, k+2). \end{aligned}$$

b) If $f(x_i) \neq g(x_{i-1})$, then by the pervious argument, $f(x_{i-1}) = g(x_{i-2})$. Since

$$f(x_i) \in seg[f(x_{i-1}), g(x_{i-1})] = seg[g(x_{i-2}), g(x_{i-1})],$$

then by Proposition 1.1, we have

$$\begin{aligned} b(n,k) = \rho(f(x_i) - g(x_j)) \;&\leq\; \max\{\rho(g(x_{i-2}) - g(x_j)), \rho(g(x_{i-1}) - g(x_j))\}\\ &\leq\; \max\{\rho(l(g(x_{i-2}) - g(x_j))), \rho(l(g(x_{i-1}) - g(x_j)))\}\\ &\leq\; \lambda \max\{M(x_{i-2}, x_j), M(x_{i-1}, x_j)\}\\ &\leq\; \lambda b(n-2, k+2). \end{aligned}$$

II) $b(n,k) = \rho(f(x_i) - f(x_j))$, for $n \leq i, j \leq n+k$. If $f(x_j) = g(x_{j-1})$ then we go back to the case (I). Otherwise, $f(x_j) \neq g(x_{j-1})$ for $j \geq 2$ then $f(x_{j-1}) = g(x_{j-2})$. Since $f(x_j) \in \partial C \cap seg[g(x_{j-2}), g(x_{j-1})]$. Then

$$\begin{aligned} b(n,k) = & \rho(f(x_i) - f(x_j)) \\ \leq & \max\{\rho(f(x_i) - g(x_{j-2})), \rho(f(x_i) - g(x_{j-1}))\} \end{aligned}$$

and this shows that case II reduces to case I and inequality (1) is hold.
III) $b(n,k) = \rho(g(x_i) - g(x_j))$, for $n \leq i, j \leq n+k$. The inequality (1) is hold.
By taking limit of inequality (1) when $k \to \infty$, we obtain $b(n) \leq \lambda b(n-2)$, and by taking limit when $n \to \infty$, we have $b \leq \lambda b$. But this shows $b = 0$.
By the definition of $b(n)$ and $b(n,k)$, one can conclude $\{f(x_n)\}$ and $\{g(x_n)\}$ are ρ-Cauchy sequences.
Since $f(x_n) \in C$ and C is ρ-closed , then $\lim_{n\to\infty} f(x_n) = y \in C$. Moreover, $\rho(f(x_n) - g(x_n)) \leq b(n)$ and $b(n) \to 0$ when $n \to \infty$, then $\rho(f(x_n) - g(x_n)) \to 0$ when $n \to \infty$. Thus

$$\begin{aligned} \rho(g(x_n) - y) = & \rho(g(x_n) - f(x_n) + f(x_n) - y) \\ = & \rho(1/2(2(g(x_n) - f(x_n))) + 1/2(2(f(x_n) - y))) \\ \leq & \rho(2(g(x_n) - f(x_n))) + \rho(2(f(x_n) - y)). \end{aligned}$$

By applying the Δ_2-condition, we have $g(x_n) \to y$ when $n \to \infty$. Since $f(C)$ is ρ-closed, there exists $u \in C$ such that $y = f(u)$. Now, there is a subsequence $\{u_n\}$ of $\{x_n\}$ such that

$$\begin{aligned} \rho(g(u) - g(u_{n-1})) \leq & \rho(l(g(u) - g(u_{n-1}))) \\ \leq & M(u, u_{n-1}). \end{aligned}$$

By the definition of $M(u, u_{n-1})$, one can obtain

$$g(u) = f(u) = y.$$

Moreover, the ρ-commutatively implies that

$$fg(u) = gf(u) = gg(u),$$

and

$$\begin{aligned} \rho(gg(u) - g(u)) \leq & \rho(l(gg(u) - g(u))) \\ \leq & \lambda M(g(u), u) \\ = & \lambda \rho(gg(u) - g(u)). \end{aligned}$$

Thus $\rho(gg(u) - g(u)) = 0$. Therefore $gg(u) = g(u)$. Since $fg(u) = gg(u)$, then $g(u)$ is a fixed point of f. But $g(u)$ is a fixed point of g. Therefore, $g(u)$ is a common fixed point of f and g.
In order to prove the uniqueness, assume $z \in C$ is another common fixed point of f and g, i.e. $f(z) = g(z) = z$. Thus

$$\begin{aligned} \rho(z - g(u)) = & \rho(g(z) - gg(u)) \\ \leq & \rho(l(g(z) - gg(u))) \\ \leq & \lambda \rho(z - g(u)). \end{aligned}$$

Then $\rho(z - g(u)) = 0$. This means that $z = g(u)$ and the common fixed point $g(u)$ is unique. □

Theorem 2.2.

Let X_ρ be a ρ-complete modular space with Takahashi convex structure (i.e. $W : X_\rho \times X_\rho \times I \longrightarrow X_\rho$) such that W is continuous in third variable and $W(x,y,\lambda) - z = W(x-z, y-z, \lambda)$ for all $x, y, z \in X_\rho$ and $\lambda \in [0,1]$. Also, ρ has Fatou property. Assume C is a nonempty ρ-closed subset of X_ρ and $\partial C \neq \emptyset$(boundary of C). Suppose $g : C \to X_\rho$, $f : X_\rho \to X_\rho$, $f(C) \subset C$ and f is ρ-continuous. Moreover, let f and g satisfy the following conditions:

1) There exists a constant $\lambda \in (0,1)$ such that for every $x, y \in C$

$$\rho(g(x) - g(y)) \leq \lambda M(x,y)$$

where

$$M(x,y) = \max\{\rho(f(x) - f(y)), \rho(f(x) - g(x)), \rho(f(y) - g(y)), \rho(f(y) - g(x)), \rho(f(x) - g(y))\}.$$

2) f and g are ρ-Commutative on their coincidence points.

3) $g(C) \cap C \subset f(C)$.

4) $g(\partial C) \subset C$.

5) $f(C)$ is ρ-closed.

Then f and g have a unique common fixed point in C.

Proof. Assume $x_0 \in \partial C$, we construct a sequence x_n of points in C as follows: Assumption (4) shows $g(x_0) \in C$. Then (3) implies that there is $x_1 \in C$ such that $f(x_1) = g(x_0)$. Now consider $g(x_1)$. If $g(x_1) \in C$ again by (3) there is $x_2 \in C$ such that $f(x_2) = g(x_1)$. If $g(x_1) \notin C$, because W is continuous in third variable, there is $\lambda_{11} \in [0,1]$ such that

$$W(f(x_1), g(x_1), \lambda_{11}) \in \partial C \cap seg[f(x_1), g(x_1)].$$

By (2) there is $x_2 \in \partial C$ such that $f(x_2) = W(f(x_1), g(x_1), \lambda_{11})$. Hence, by induction we construct a sequence x_n of points in C as follows: If $g(x_n) \in C$, then by (3)

$$f(x_{n+1}) = g(x_n) \text{ for some } x_{n+1} \in C.$$

If $g(x_n) \notin C$, then there is $\lambda_{nn} \in [0,1]$ such that

$$W(f(x_n), g(x_n), \lambda_{nn}) \in \partial C \cap seg[f(x_n), g(x_n)]$$

and there is $x_{n+1} \in \partial C$ such that $f(x_{n+1}) = W(f(x_n), g(x_n), \lambda_{nn})$. Now, it is enough to show $\{f(x_n)\}$ and $\{g(x_n)\}$ are ρ-Cauchy sequences. Due to this, we need to prove $f(x_{n+1}) \neq g(x_n)$ then $f(x_n) = g(x_{n-1})$. By contrary $f(x_n) \neq g(x_{n-1})$, then $x_n \in \partial C$ and by (3), $g(x_n) \in C$. Hence $f(x_{n+1}) = g(x_n)$, and this is a contradiction. In order to prove ρ-Cauchy of the sequences, one can apply the following notations:

$$\begin{aligned}
&B(n,k) = \{f(x_j), g(x_j) : n \leq j \leq n+k\},\\
&B(n) = \{f(x_j), g(x_j) : n \leq j\},\\
&b(n,k) = \sup\{\rho(x-y) : x, y \in B(n,k)\},\\
&b(n) = \sup\{\rho(x-y) : x, y \in B(n)\}.
\end{aligned}$$

Note that $b(n,k)_{k\to\infty} \uparrow b(n)$ and $b(n) \downarrow$. Hence $b = \lim_{n\to\infty} b(n) \geq 0$ exists. To see that $f(x_n)$ and $g(x_n)$ are ρ-Cauchy sequences, it is enough to show that $b = 0$. We claim that

$$b(n,k) \leq \lambda b(n-2, k+2) \quad n, k \geq 2. \tag{2}$$

In order to show the above inequality, we have to consider three cases as follows:
I) $b(n,k) = \rho(f(x_i) - g(x_j))$ for, $n \leq i, j \leq n+k$ we have two cases:
a) $f(x_i) = g(x_{i-1})$

$$\begin{aligned}
b(n,k) = &\ \rho(g(x_{i-1}) - g(x_j))\\
\leq &\ \lambda M(x_{i-1}, x_j)\\
\leq &\ b(n-2, k+2).
\end{aligned}$$

b) $f(x_i) \neq g(x_{i-1})$, then by the previous argument $f(x_{i-1}) = g(x_{i-2})$. Then, there exists $\lambda_{i-1i-1} \in [0,1]$ such that $f(x_i) = W(f(x_{i-1}), g(x_{i-1}), \lambda_{i-1i-1})$. Thus by Proposition 1.1, we have

$$\begin{aligned} b(n,k) =& \ \rho(f(x_i) - g(x_j)) \\ =& \ \rho(W(f(x_{i-1}), g(x_{i-1}), \lambda_{i-1i-1}) - g(x_j)) \\ =& \ \rho(W(g(x_{i-2}) - g(x_j), g(x_{i-1}) - g(x_j), \lambda_{i-1i-1})) \\ \leq& \ \max\{\rho(g(x_{i-2}) - g(x_j)), \rho(g(x_{i-1}) - g(x_j))\} \\ \leq& \ \lambda \max\{M(x_{i-2}, x_j), M(x_{i-1}, x_j)\} \\ \leq& \ \lambda b(n-2, k+2). \end{aligned}$$

II) $b(n,k) = \rho(f(x_i) - f(x_j))$, for $n \leq i, j \leq n+k$. If $f(x_j) = g(x_{j-1})$ then we go back to the case I. Otherwise,

$$f(x_j) \neq g(x_{j-1}) \quad j \geq 2 \text{ then } f(x_{j-1}) = g(x_{j-2}),$$

and we have

$$f(x_j) = W(g(x_{j-2}), g(x_{j-1}), \lambda_{j-1j-1}) \in \partial C \cap seg[g(x_{j-2}), g(x_{j-1})].$$

Therefore

$$\begin{aligned} b(n,k) =& \ \rho(f(x_i) - f(x_j)) \\ \leq& \ \max\{\rho(f(x_i) - g(x_{j-2})), \rho(f(x_i) - g(x_{j-1}))\}, \end{aligned}$$

and case II reduces to I and inequality (2) is hold.
III) $b(n,k) = \rho(g(x_i) - g(x_j))$ for $n \leq i, j \leq n+k$. The inequality 2 is hold.
By taking limit of inequality (2) when $k \to \infty$, $b(n) \leq \lambda b(n-2)$, and when $n \to \infty$, $b \leq \lambda b$. This means that $b = 0$. By the definition of $b(n)$ and $b(n,k)$, one can conclude $\{f(x_n)\}$, $\{g(x_n)\}$ are ρ-Cauchy sequences.
Since $f(x_n) \in C$ and C is ρ-closed subset X_ρ, then $\lim_{n\to\infty} f(x_n) = y \in C$. Moreover, $\rho(f(x_n) - g(x_n)) \leq b(n)$ and $b(n) \to 0$ when $n \to \infty$, then $\rho(f(x_n) - g(x_n)) \to 0$ when $n \to \infty$. Therefore, since ρ has Fatou property, we conclude $g(x_n) \to y$.
Moreover, since $f(C)$ is ρ-closed, there exists $u \in C$ such that $y = f(u)$. Now, there is a subsequence u_n of x_n such that

$$\rho(g(u) - g(u_{n-1})) \leq M(u, u_{n-1}).$$

Hence,

$$g(u) = f(u) = y.$$

Moreover, because f and g are ρ-commutative in coincidence points, we conclude

$$gf(u) = fg(u) = gg(u),$$

and

$$\rho(g(u) - gg(u)) \leq \lambda M(u, g(u)).$$

Then

$$\rho(g(u), gg(u)) = 0 \to g(u) = gg(u).$$

This means that $gf(u) = fg(u) = gg(u) = g(u)$, so $g(u)$ is a common fixed point of f and g. Uniqueness is trivial to see. □

We end this section by the next example:

Example 2.1.

Let $X = (0, \infty)$, define modular $\rho : Y \to [0, \infty)$ as follows:

$$\rho(f) = \int_0^\infty |f(x)|^{x+1} dx,$$

where Y is the set of measurable functions $f : X \to \mathbb{R}$. Suppose B be the set of ρ-continuous function f on X such that $0 \leq f(x) \leq 1/2$. It is clear that B is a ρ-closed subset of Y_ρ. Now, define two operators $T^* : B \to Y_\rho$ and $T : Y_\rho \to Y_\rho$ by the following formulas:

$$T^*(f)x = \begin{cases} f(x-1), & x \geq 1, \\ 0, & x \in [0, 1), \end{cases}$$

and

$$T(f)x = \begin{cases} f(x), & x \geq 1, \\ 0, & x \in [0, 1). \end{cases}$$

We claim that for all $f, g \in B$,

$$\rho(T^*(f) - T^*(g)) \leq 1/2 \rho(T(f) - T(g)).$$

Indeed,

$$\begin{aligned} \rho(T^*(f) - T^*(g)) &= \int_0^\infty |T^*(f)x - T^*(g)x|^{x+1} dx \\ &= \int_1^\infty |f(x-1) - g(x-1)|^{x+1} dx \\ &= \int_0^\infty |f(x) - g(x)|^{x+1} |f(x) - g(x)| dx \\ &\leq 1/2 \int_0^\infty |f(x) - g(x)|^{x+1} dx \\ &= 1/2 \rho(T(f) - T(g)). \end{aligned}$$

Also, it is easily to see, this modular satisfies in Δ_2- condition and the corresponding modular space is ρ-complete. Moreover, it is straightforward to show which all of conditions of Theorem 2.1 hold in this example. Thus according to Theorem 2.1, T and T^* have a unique common fixed point in B, and it is simple to verify that zero function is a common fixed point of these functions.

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