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Altitude of wheels and wheel-like graphs

Research Article

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Abstract: An *edge-ordering* of a graph $G = (V, E)$ is a one-to-one mapping $f\colon E(G) \to \{1, 2, \dots, |E(G)|\}$. A path of length k in G is called a (k, f)-*ascent* if f increases along the successive edges forming the path. The *altitude* $\alpha(G)$ of G is the greatest integer k such that for all edge-orderings f, G has a (k, f)-ascent.
In our paper we give exact values of $\alpha(G)$ for all helms and wheels. Furthermore, we use our result to obtain altitude for graphs that are subgraphs of helms.

MSC: 05C78

Keywords: Altitude • Edge-ordering • Increasing paths

1. Introduction

A one-to-one mapping f from E to the set of positive integers is called an *edge-ordering* of a graph $G = (V, E)$. A path of G for which f increases along the edge-sequence, is called an f-*ascent* of G. An f-ascent of a length k will be called (k, f)-*ascent*. A (k, f)-ascent which cannot be extended to $(k+1, f)$-ascent we call *maximal* (k, f)-ascent. The *height* $h(f)$ of f is the maximum length of an f-ascent.
Denote the set of all edge-orderings of G by $\mathcal{F}$. The *altitude* $\alpha(G)$ of a graph G is the number

$$\alpha(G) = \min_{f \in \mathcal{F}} h(f).$$

This parameter was firstly studied by Chvátal and Komlós in [4]. Clearly, $\alpha(G) \geq 2$ for any graph G with a vertex of degree at least two. It is also obvious that if graph H is a subgraph of G then $\alpha(G) \geq \alpha(H)$. The altitude of some classes of graphs is easy to determine, for example the altitude of cycles.

Proposition 1.1.

$$\alpha(C_n) = \begin{cases} 2 & \text{if } n = 2k \text{ or } n = 3, \\ 3 & \text{otherwise.} \end{cases}$$

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Burger et al. [2] considered altitude for complete graphs.

Theorem 1.1 ([2]).

$$\alpha(K_n) = \begin{cases} 1 & \text{if } n = 2, \\ 2 & \text{if } n = 3 \text{ or } n = 4, \\ 3 & \text{if } n = 5, \\ 4 & \text{if } n = 6, \\ 5 & \text{if } n = 7 \text{ or } n = 8. \end{cases}$$

However, $\alpha(K_n)$ is still unknown for $n \geq 9$. Cockayne and Mynhardt [5] gave exact values for $\alpha(K_{3,n})$ and lower bound for $\alpha(K_{m,n})$. Some bounds for altitude of complete and complete bipartite graphs are given in [2]. A survey of some known results on $\alpha(G)$ can be found also e.g. in [3]. Recently, Katrenič and Semanišin [6] proved that the problem of determining the value of $h(f)$ for a given edge-ordering f is $\mathcal{NP}$-hard.
In the paper we use notation of $\mathbf{P}$-consistent ordering. Let $\mathbf{P} = (E_1, E_2, \ldots, E_t)$ be an ordered partition of the edge set E of G and let f be any edge-ordering of G satisfying

$$a \in E_i \text{ and } b \in E_j, \text{ where } i < j, \text{ implies } f(a) < f(b).$$

Such an edge-ordering is called $\mathbf{P}$-*consistent*. The appropriate sequence of labels $\mathbf{L} = (L_1, L_2, \ldots, L_t)$, $f(E_i) = L_i$ for $i = 1, 2, \ldots, t$, is called $\mathbf{L}$-*consistent labeling*. Sets L_i are formed by successive naturals.

Definition 1.1.
A wheel W_n, $n \geq 4$, *is a graph obtained from a cycle* C_{n-1} *on* $n-1$ *vertices* $\{v_1, \ldots, v_{n-1}\}$ *by adding a new vertex* w *and edges -* spokes $s_i = wv_i$, $i = 1, \ldots, n-1$. *The edges* $e_i = v_{i-1}v_i$ *(indices modulo* $n-1$*) form the* rim *of* W_n *(see Figure 1).*

Definition 1.2.
Let the fan F_n *[1] be a graph* $W_n - e_r$ *i.e. the wheel on* $n \geq 4$ *vertices without one edge on the rim (see Fig. 2(a)).*

Definition 1.3.
A gear graph G_n *[1], also sometimes known as a* bipartite wheel graph, *is a wheel graph* W_{n+1} *with a vertex added between each pair of adjacent vertices of the rim. The gear graph* G_n *has* $2n+1$ *vertices and* $3n$ *edges (see Fig. 2(b)).*

Definition 1.4.
A helm graph, *denoted by* H_n *[1],* $n \geq 4$, *is the graph obtained from a wheel graph* W_n *by adjoining a pendant edge at each vertex of the rim of the wheel (see Fig. 2(c)). We denote pendant edge adjacent to spoke* s_i *by* p_i.

In the paper we give the exact value of altitude for all helms, wheels and some related classes of graphs like gears, fans. The appropriate algorithm of finding „the best" edge-ordering is also given.

2. Altitude of helms

In this section we consider the altitude of helms. We have the following

Theorem 2.1.
Let H_n *be a helm graph and* $n \geq 4$. *Then*

$$\alpha(H_n) = 3$$

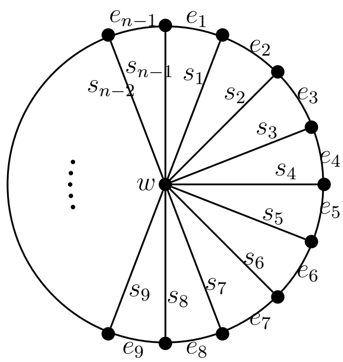

Figure 1. Wheel W_n.

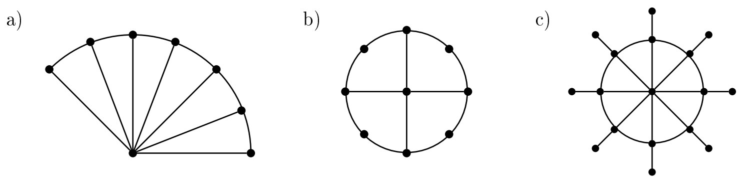

Figure 2. (a) Fan F_8. (b) Gear G_4. (c) Helm H_9.

Proof. Firstly, we prove that $\alpha(H_4) = 3$. Suppose that there is an edge-ordering f of H_4 of height two. Let us consider the path formed by the following sequence of edges of H_4: $(p_1, e_2, s_2, s_3, p_3)$. To avoid a $(3, f)$-ascent on this path, without loss of generality we can assume that $f(p_1) < f(e_2)$, $f(s_2) < f(e_2)$, $f(s_2) < f(s_3)$ and $f(p_3) < f(s_3)$. To avoid a $(3, f)$-ascent on (s_2, s_3, e_1) we obtain that $f(e_1) < f(s_3)$. Moreover, we have that $f(e_1) < f(p_1)$ (consider path (p_1, e_1, s_3)), $f(e_1) < f(e_3)$ $((e_3, e_1, p_1))$, $f(s_2) < f(e_3)$ $((e_1, e_3, s_2))$, $f(p_3) < f(e_3)$ $((p_3, e_3, s_2))$ and $f(e_2) < f(e_3)$ $((e_2, e_3, p_3))$. But in this case we have the following $(3, f)$-ascent (p_1, e_2, e_3), a contradiction.
Now we have that $n \geq 5$. Since C_5 is a subgraph of H_n, $n \geq 5$, $\alpha(H_n) \geq 3$. All we need is to fix an edge-ordering f of H_n of height three. The proof is divided into two steps. First step describes labelling for $W_n \subset H_n$ while in Step II we label pendant edges of H_n.

Step I.

In this step the edges of the rim and spokes are labelled. This part of the proof is divided into four cases depending on the residue class of n mod 4. In every case we define $\mathbf{L}$-consistent labeling with $\mathbf{P}$-consistent ordering for wheel W_n.

Case 1.
$n \equiv 1 \mod 4$

Let $\mathbf{L} = (L_1, \ldots, L_7)$, where $|L_i| = (n-1)/2$ for $1 \leq i \leq 4$ and $i = 7$, $|L_5| = |L_6| = (n-1)/4$. $(E_1, \ldots, E_7)$-consistent ordering is formed as follows.

1. We label the edges of the rim $(e_1, \ldots, e_{n-1})$ periodically with the labels from the classes L_1, L_3, L_6, L_4.

   One can easily observe that the cycle C_{n-1} is covered by maximal $(3, f)$-ascents (a_1, a_2, a_3), where $a_1 \in E_1, a_2 \in E_3 \cup E_4$ and $a_3 \in E_6$, as given in Fig. 3(a).

2. We label spokes $(s_1, \ldots, s_{n-1})$ periodically with the labels from the classes L_7, L_1, L_3, L_7.

   We can easily observe that bold spokes in Fig. 3(a) are assigned value smaller than values assigned to adjacent edges from the cycle, ie. $f(s_i) < f(e_i)$ and $f(s_i) < f(e_{i+1})$. Moreover, every unbold spoke is labelled with number from class L_7.

Following properties hold:

- none of maximal f-ascents from cycle C_{n-1} can be extended by bold spokes
- only paths of length 1 on the rim can be extended by spokes (maximal two), labelled with numbers from L_7; we have maximal $(3, f)$-ascents (a_1, a_2, a_3) such that $a_1 \in E_1 \cup E_3$ and $a_2, a_3 \in E_7$
- every $(2, f)$-ascent formed by two bold spokes is extended by at most one edge from the rim.

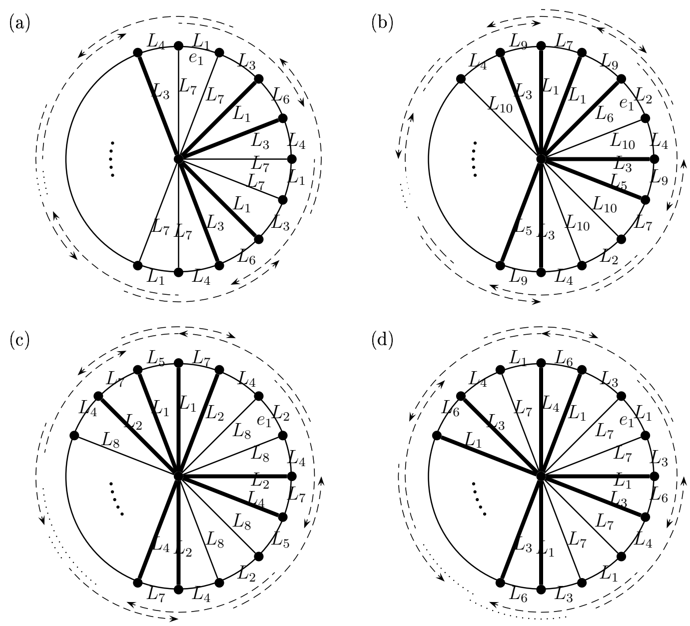

Figure 3. The subgraph of H_n: wheel W_n (Step I of the algorithm), (a) $n \equiv 1 \mod 4$, (b) $n \equiv 2 \mod 4$, (c) $n \equiv 3 \mod 4$, (d) $n \equiv 0 \mod 4$ with fixed, by dashed arrows, maximal $(3, f)$-ascents ((a)) or with fixed maximal $(3, f)$- and $(2, f)$-ascents ((b), (c), (d)) on the cycle C_{n-1}.

Case 2.

$n \equiv 2 \mod 4$

Let $\mathbf{L} = (L_1, \ldots, L_{10})$, where $|L_1| = 2$, $|L_i| = (n-2)/4$ for $i = 2, 4, 7, 8$, $|L_3| = (3n-2)/4$, $|L_5| = |L_{10}| = (n-4)/2$, $|L_6| = 1$ and $|L_9| = (n+2)/4$. $(E_1, \ldots, E_{10})$-consistent ordering is formed as follows.

1. We label the edges of the rim $(e_1, \ldots, e_{n-2})$ periodically with the labels from the classes L_2, L_4, L_9, L_7 and we set $f(e_{n-1}) \in L_9$.

   One can easily observe that each edge of the cycle C_{n-1} belongs to at least one maximal $(3, f)$-ascent (a_1, a_2, a_3), where $a_1 \in E_2, a_2 \in E_4 \cup E_7$ and $a_3 \in E_9$, or maximal $(2, f)$-ascent (a_1, a_2), where $a_1 \in E_2 \cup E_7$ and $a_2 \in E_9$ (see Fig. 3(b)).

2. We label spokes $(s_1, \ldots, s_{n-6})$ periodically with the labels from the classes L_{10}, L_3, L_5, L_{10} and spokes $(s_{n-5}, \ldots, s_{n-1})$ with the labels from classes $L_{10}, L_3, L_1, L_1, L_6$.

   One can easily observe that labels assigned to bold spokes in Fig. 3(b), with exception s_{n-1}, satisfy $f(s_i) < f(e_i)$ and $f(s_i) < f(e_{i+1})$. Let us notice that $f(e_{n-1}) > f(s_{n-1}) > f(e_1)$ and

$$f(s_{n-1}) > \max_{i \in S} f(s_i),$$

   where $S = \{s_2, s_3, s_6, s_7, \ldots, s_{n-4}, s_{n-3}, s_{n-2}\}$. Moreover, every unbold spoke is labelled with number from class E_{10}.

Following properties hold:

- only $(1, f)$-ascent (e_1) from cycle C_{n-1} can be extended by bold spoke s_{n-1}; then we have maximal $(3, f)$-ascent (e_1, s_{n-1}, a_3) (for $n = 6$ maximal $(2, f)$-ascent (e_1, s_{n-1})), where $e_1 \in E_2$, $s_{n-1} \in E_6$ and $a_3 \in E_{10}$,
- only paths of length 1 on the rim can be extended by spokes (maximal two), labelled with numbers from L_{10}; we have maximal $(3, f)$-ascents (a_1, a_2, a_3) (for $n = 6$ maximal $(2, f)$-ascent (a_1, a_2)) such that $a_1 \in E_2 \cup E_4$ and $a_2, a_3 \in E_{10}$,
- every $(2, f)$-ascent formed by two bold spokes is extended by at most one edge from the rim.

Case 3.

$n \equiv 3 \mod 4$

Let $\mathbf{L} = (L_1, \ldots, L_8)$, where $|L_1| = 2$, $|L_2| = |L_5| = (n-1)/2$, $|L_3| = |L_7| = (n+1)/4$, $|L_4| = (3n-9)/4$, $|L_6| = (n-3)/4$ and $|L_8| = (n-3)/2$. $(E_1, \ldots, E_8)$-consistent ordering is formed as follows.

1. We label the edges of the rim $(e_1, \ldots, e_{n-3})$ periodically with the labels from the classes L_2, L_4, L_7, L_5 and $f(e_{n-2}) \in L_7$, $f(e_{n-1}) \in L_4$.

   One can easily observe that each edge of the cycle C_{n-1} forms at least one maximal $(3, f)$-ascent (a_1, a_2, a_3), where $a_1 \in E_2, a_2 \in E_4 \cup E_5$ and $a_3 \in E_7$, or $(2, f)$-ascent (a_1, a_2), where $a_1 \in E_5$ and $a_2 \in E_7$ (see Fig. 3(c)).

2. We label spokes $(s_1, \ldots, s_{n-7})$ periodically with the labels from the classes L_8, L_2, L_4, L_8 and spokes $(s_{n-6}, s_{n-5}, \ldots, s_{n-1})$ with labels from sets $(L_8, L_2, L_1, L_1, L_2, L_8)$.

   We can easily observe that bold spokes in Fig. 3(c) are labelled in such a way that $f(s_i) < f(e_i)$ and $f(s_i) < f(e_{i+1})$. Every unbold spoke is assigned number from class L_8.

Following properties hold:

- none of maximal f-ascents from cycle C_{n-1} can be extended by bold spokes,
- only paths of length 1 on the rim can be extended by spokes (maximal two), labelled with numbers from L_8; we have maximal $(3, f)$-ascents (a_1, a_2, a_3) such that $a_1 \in E_2 \cup E_4 \cup E_5$ and $a_2, a_3 \in E_8$,
- every $(2, f)$-ascent formed by two bold spokes is extended by at most one edge from the rim,
- maximal $(2, f)$-ascents on the rim can be preceded by one spoke labelled with number from class L_1; then we have maximal $(3, f)$-ascent (a_1, a_2, a_3), where $a_1 \in E_1$, $a_2 \in E_5$ and $a_3 \in E_7$.

Case 4.

$n \equiv 0 \mod 4$

Let $\mathbf{L} = (L_1, \ldots, L_7)$, where $|L_1| = |L_4| = n/2$, $|L_2| = |L_6| = n/4$, $|L_3| = (3n-4)/4$, $|L_5| = (n-4)/4$ and $|L_7| = (n-2)/2$. $(E_1, \ldots, E_7)$-consistent ordering is formed as follows.

1. We label the edges of the rim $(e_1, \ldots, e_{n-4})$ periodically with the labels from the classes L_1, L_3, L_6, L_4 and we set $f(e_{n-3}) \in L_1$, $f(e_{n-2}) \in L_6$ and $f(e_{n-1}) \in L_3$.

   One can easily observe that each edge of the cycle C_{n-1} is in at least one maximal $(3, f)$-ascent (a_1, a_2, a_3), where $a_1 \in L_1$, $a_2 \in L_3 \cup L_4$ and $a_3 \in L_6$ (see Fig. 3(d)).

2. We label spokes $(s_1, \ldots, s_{n-4})$ periodically with the labels from the classes L_7, L_1, L_3, L_7 and spokes $(s_{n-3}, s_{n-2}, s_{n-1})$ with the labels from classes L_4, L_1, L_7, respectively.

   One can easily observe that labels assigned to bold spokes in Fig. 3(d), with exception s_{n-3}, satisfy $f(s_i) < f(e_i)$ and $f(s_i) < f(e_{i+1})$. Let us notice that $f(e_{n-2}) > f(s_{n-3}) > f(e_{n-3})$ and

   $$f(s_{n-3}) > \max_{i \in S} f(s_i),$$

   where $S = \{s_2, s_3, s_6, s_7, \ldots, s_{n-6}, s_{n-5}, s_{n-2}\}$. Moreover, every unbold spoke is assigned number from L_7.

The following properties hold:

- only $(1, f)$-ascent (e_{n-3}) from the cycle C_{n-1} can be extended by bold spoke s_{n-3}; then we have maximal $(3, f)$-ascent (e_{n-3}, s_{n-3}, a_3), where $e_{n-3} \in E_1$, $s_{n-3} \in E_4$ and $a_3 \in E_7$,
- only paths of length 1 on the rim can be extended by spokes (maximal two), labelled with numbers from L_7; we have maximal $(3, f)$-ascents (a_1, a_2, a_3) such that $a_1 \in E_1 \cup E_3 \cup E_4$ and $a_2, a_3 \in E_7$,
- every $(2, f)$-ascent formed by two bold spokes is extended by at most one edge from the rim.

In all cases we obtained maximal (k, f)-ascents, $k \leq 3$.

Step II.

In this step we label pendant edges in H_n. Firstly, let us notice that each pendant edge of H_n is adjacent to exactly one spoke - the bold or unbold one. One can easily observe that each pendant edge p_i adjacent to bold spoke s_i must receive a larger value than $f(s_i)$. Otherwise, in most cases, we immediately have a $(4, f)$-ascent which can be formed by the pendant edge, adjacent bold spoke, another bold spoke of W_n and one edge from the rim. By using similar argument we obtain that each pendant edge p_i adjacent to unbold spoke s_i must obtain smaller value than $f(s_i)$. These situations are presented by straight dotted arrows in Figure 4.
The rest of the dotted arrows in Figure 4 are forced by avoiding $(4, f)$-ascents which could be formed from $(3, f)$-ascents in the edge-ordering of W_n (Step I of the algorithm). There are no dotted arrows in some cases of pendant edges. It means that the labelling of the corresponding edges does not depend on values of adjacent edges on the rim.
Below we give edge-ordering fulfilling above conditions.

Case 1.
$n \equiv 1 \mod 4$

We label pendant edges $(p_1, \ldots, p_{n-1})$ periodically with the labels from classes L_2, L_4, L_5, L_2.

Case 2.
$n \equiv 2 \mod 4$

We label pendant edges $(p_1, \ldots, p_{n-2})$ periodically with the labels from classes L_3, L_5, L_8, L_3 and we set $f(p_{n-1}) \in L_3$.

Case 3.
$n \equiv 3 \mod 4$

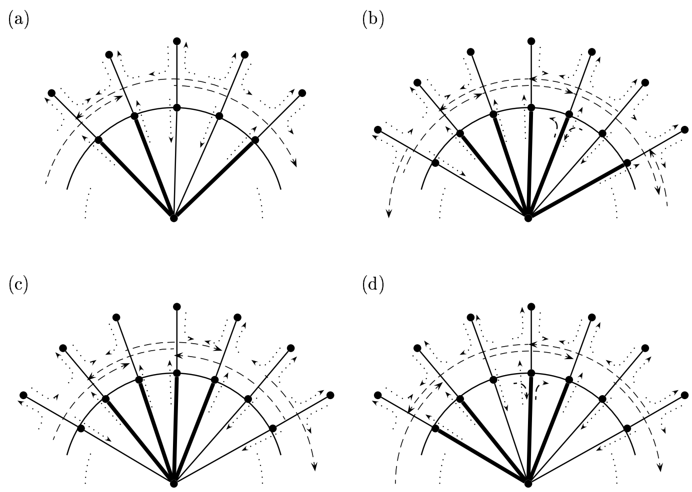

Figure 4. A part of helm H_n (Step II of the algorithm), (a) $n \equiv 1 \mod 4$, (b) $n \equiv 2 \mod 4$, (c) $n \equiv 3 \mod 4$, (d) $n \equiv 0 \mod 4$. Dashed arrows denote maximal $(3,f)$- or maximal $(2,f)$-ascents on the rim of wheel W_n. Dotted arrows denote $(2,f)$-ascents.

We label pendant edges $(p_1, \ldots, p_{n-3})$ periodically with the labels from classes L_3, L_5, L_6, L_4 and we set $f(p_{n-2}) \in L_5$, $f(p_{n-1}) \in L_3$.

Case 4.

$n \equiv 0 \mod 4$

We label pendant edges $(p_1, \ldots, p_{n-4})$ periodically with the labels from classes L_2, L_4, L_5, L_3 and we set $f(p_{n-3}) \in L_3$, $f(p_{n-2}) \in L_4$, $f(p_{n-1}) \in L_2$.

After two steps of the proof we have obtained an edge-ordering of H_n of height three, so $\alpha(H_n) = 3$. □

3. Altitude of wheels, fans and gears

In this section we consider the altitude of wheel-like graphs.

Theorem 3.1.

Let W_n be a wheel and $n \geq 4$. Then

$$\alpha(W_n) = \begin{cases} 2 & \text{if } n = 4, \\ 3 & \text{otherwise.} \end{cases} \tag{1}$$

Proof. Since $W_4 = K_4$, the first part of our result follows from Theorem 1.1.
Since C_5 is a subgraph of W_n, $n \geq 5$, then $\alpha(W_n) \geq 3$. On the other hand W_n is a subgraph of H_n, then by Theorem 2.1 our proposition holds. Step I of the proof of Theorem 2.1 gives proper ordering for wheels. □

Since fans are subgraphs of wheels then we obtain the following

Corollary 3.1.

Let F_n be a fan and $n \geq 4$. Then

$$\alpha(F_n) = \begin{cases} 2 & \text{if } n = 4, \\ 3 & \text{otherwise.} \end{cases}$$

Proof. The case of $n = 4$ is obvious. For $n \geq 5$ we can observe that F_n which is a subgraph of W_n contains a cycle C_5. By using Proposition 1.1 and Theorem 3.1 we obtain the result. □

We obtain the following result for gears.

Proposition 3.1.

Let G_n be a gear graph and $n \geq 3$. Then

$$\alpha(G_n) = 3$$

Proof. It is evident that G_n is a subgraph of W_{2n+1}, so by Theorem 3.1 we have that $\alpha(G_n) \leq 3$. It remains to show that there is no edge-ordering of G_n of height two. Suppose to the contrary that f is an edge-ordering of G_n with $h(f) = 2$. Let us denote the edges of outer cycle of G_n by $e_1, e_2, \ldots, e_{2n}$ and the spokes by $s_2, s_4, \ldots s_{2n}$, where s_i is adjacent to e_i. Without loss of generality we may assume that $f(e_1) < f(e_2)$. To avoid the existence of $(3, f)$-ascent (e_1, e_2, s_2), we have $f(s_2) < f(e_2)$ and, next, to avoid $(3, f)$-ascent (s_4, s_2, e_2), we have $f(s_4) > f(s_2)$. Therefore, one can easily observe that inequlities $f(s_2) < f(e_3) < f(e_2)$ hold. Up to now we have the following $(2, f)$-ascents: $(e_1, e_2), (s_2, e_2), (e_3, e_2), (s_2, e_3), (s_2, s_4)$. If $f(e_4) > f(e_3)$, then (s_2, e_3, e_4) is a $(3, f)$-ascent, in the opposite case if $f(e_4) < f(e_3)$, then (e_4, e_3, e_2) is a $(3, f)$-ascent – a contradiction with $h(f) = 2$. Hence $\alpha(G_n) = 3$. □

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