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A glimpse of deductive systems in algebra

Research Article

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Abstract: The concept of a deductive system has been intensively studied in algebraic logic, per se and in connection with various types of filters. In this paper we introduce an axiomatization which shows how several resembling theorems that had been separately proved for various algebras of logic can be given unique proofs within this axiomatic framework. We thus recapture theorems already known in the literature, as well as new ones. As a by-product we introduce the class of pre-BCK algebras.

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The concept of a *deductive system*, introduced by Diego [18, 19], is an algebraic counterpart of a theory closed under modus ponens. It has become an important tool in algebra of logic (Georgescu [21] emphasizes that the term *algebraic logic* should be reserved to the study of logic by algebraic means, while the study of algebras related to logic, mainly from an algebraic point of view, should be referred to as *algebra of logic*). The structures occurring in algebra of logic are usually endowed with an operation $\rightarrow$, called *implication*, and the concept of a deductive system makes sense for them. If the structure is also a (semi)lattice, then its deductive systems are usually (semi)lattice filters, and that is why they are sometimes called *implicative filters*, while the converse may or may not hold. It turns out that whenever deductive systems do not coincide with filters, another concept has been invented which is similar to or stronger than a filter and is equivalent to the concept of a deductive system. Besides, the lattice of deductive systems or equivalents of it has been studied in more detail; see e.g. (in chronological order) Nemitz [31], Bușneag [5, 6, 8], Jun [26], Chajda [11], Chajda and Halaš [12, 15], Bușneag and Piciu [9], Chajda, Halaš and Kühr [16] or Piciu [33].

The starting point of our paper is the existence of several repetitions in this area: certain results have identical formulations for various algebras such as BL algebras, MV algebras and others, while the proofs are very similar. We are going to point out the common background of these proofs, so that the corresponding theorems are proved here once

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and for all the involved classes of algebras[1]. We aim at algebraic direct proofs, without recourse to the logics described by the algebras under consideration.

Our aim described above amounts to a desideratum stated by a Romanian mathematician Dan Barbilian: prove a theorem under no more hypotheses than necessary. That is why we gradually introduce classes of algebras constructed ad hoc for "economically" proving the theorems that interest us and from which we immediately recapture the theorems known in the literature.

The paper is organized as follows. Firstly (Section 1) we deal only with deductive systems, then (Sections 2, 3) we relate them to other types of subsets, known as filters: product filters, filters (more exactly, semilattice filters), order filters and strong filters. Section 2 compares deductive filters to product filters, while comparison with other types of filters is done in Section 3. The criterion for this division was that product filters require the existence of an extra binary operation $\odot$, while in Section 3 the framework is the same as in Section 1, with the little exception of strong filters (dealt with in Proposition 3.3 and Corollary 3.4). The reason for which the comparison with product filters precedes the other comparisons is that the important Propositions 3.1a and 3.1b rely on Proposition 2.3.

The concept of a deductive system requires only the existence of a binary operation $\to$ and of a constant 1. As remarked in Section 1, this suffices to immediately recapture a result from the literature stating that the deductive systems can be made into an algebraic lattice. In the same section we prove that the internal structure of the deductive system generated by a subset of a Hilbert algebra or of a BCK algebra is in fact valid for *pre-BCK algebras*, a class of algebras which strictly includes BCK algebras. In Section 2 we introduce relational algebras $(A, \odot, \to, 1, \leq)$ equipped with two binary operations, a constant and a partial order. This context enables the introduction of a concept of $\odot$-*filter* and we provide conditions under which every $\odot$-filter is a deductive system, or every deductive system is a $\odot$-filter, or both. We thus recapture as particular cases results about pocrims, residuated lattices, BL and MV algebras. In Section 3 we specialize in the case when the reduct $(A, \odot, 1, \leq)$ is a meet semilattice with greatest element $(A, \wedge, 1)$, so that $\odot$-filters are the usual *filters*, well-known in lattice theory. Our study points out the role of the semilattice residuation condition $x \wedge y \leq z \iff x \leq y \to z$, which implies that $\leq$ is the *natural order* $x \leq y \iff x \to y = 1$. Several corollaries recapture results about relatively pseudocomplemented semilattices, also known as Hertz algebras, BL and MV algebras, Heyting algebras, Boolean algebras and implication algebras. We also deal with *strong filters*, suggested by Łukasiewicz–Moisil algebras, and with *order filters*.

The case of algebras endowed with two implications is left for a future paper.

It is understood that the variables of each formula are bound by universal quantifiers.

1. The lattice of deductive systems

Algebras of logic are endowed with two term operations $\to$ and 1, sometimes included in the signature, which represent logical implication and truth. So the concept of a *deductive system* makes sense in any algebra A which has a reduct $(A, \to, 1)$; it means a subset $S \subseteq A$ such that

$$1 \in S\,, \tag{s1}$$

$$x \in S \text{ and } x \to y \in S \Longrightarrow y \in S\,. \tag{s2}$$

It is clear that the family $\mathcal{DS}(A)$ of all deductive systems of A is a *closure system*, also called a *Moore family*. This means that $A \in \mathcal{DS}(A)$, and if $D_i \in \mathcal{DS}(A)$ $(\forall\, i \in I)$, then $\bigcap_{i\in I} D_i \in \mathcal{DS}(A)$. As is well known, this implies that the map Ded defined on $\mathcal{P}(A)$ by

$$Ded(X) = \bigcap\{D \in \mathcal{DS}(A) \mid X \subseteq D\} \;\; (\forall\, X \subseteq A)$$

is a *closure operator* (isotone, extensive and idempotent) and $Ded(X)$ is the smallest deductive system which includes X. Moreover,

$$(\mathcal{DS}(A), \bigvee, \bigcap, \{1\}, A)$$

[1] *One of us has undertaken a similar study for localizations and fractions in algebra of logic; cf. [37].*

is a complete lattice, where

$$\bigvee_{i\in I} D_i = Ded\left(\bigcup_{i\in I} D_i\right) .$$

Now recall the concept of algebraic lattice, which goes back to Birkhoff [3]. An element c of a complete lattice L is said to be *compact* if, whenever $c \leq \sup X$ for some $X \subseteq L$, there is a finite subset $X_0 \subseteq X$ such that $c \leq \sup X_0$. An *algebraic lattice* is a complete lattice in which every element is a join of compact elements. It is known that this concept is an abstract characterization of the lattice of subalgebras of an algebra.

It has been proved in the literature, for several algebras of logic, that $\mathcal{DS}(A)$ is an algebraic lattice, the proofs being based on properties of the algebras under consideration; cf. Nemitz [31], Bușneag [5, 8], Jun [26], Bușneag and Piciu [9], Chajda, Halaš and Kühr [16], Piciu [33]. In this section we first use a theorem in universal algebra to provide a single proof showing that for all algebras of logic A, $\mathcal{DS}(A)$ is an algebraic lattice (Theorem 1.1).

The internal structure of $Ded(X)$ was determined in the case of Hilbert algebras by Diego [18, 19] for finite sets X and by Rasiowa [34] for arbitrary subsets. The latter result says that $Ded(X)$ consists of the elements $x \in A$ such that

$$x_1 \to (x_2 \to (\ldots \to (x_n \to x)\ldots)) = 1$$

for some $n \in \mathbb{N}\backslash\{0\}$ and $x_1,\ldots,x_n \in X$. This result was then extended to the class of BCK algebras; see e.g. Iséki and Tanaka [25], Nemitz [31]. We prove below that this representation of $Ded(X)$ is valid for a class of algebras which strictly includes BCK algebras (Theorem 1.2).

To obtain the first announced result we refer the reader to Grätzer [22], Ch.0, §6, Theorem 4 and Lemmas 3 and 4, for the following theorem. Suppose $\mathcal{C}$ is a closure system such that $\bigcup\mathcal{F} \in \mathcal{C}$ for any directed family $\mathcal{F} \subseteq \mathcal{C}$. Then the lattice $(\mathcal{C}, \subseteq)$ is algebraic and its compact elements are the *finitely generated* $\mathcal{C}$−sets, that is, the sets of the form $\mathcal{C}(X)$ where X runs over all the finite subsets of A. Moreover, the closure of an arbitrary subset $X \subseteq A$ is given by

$$\mathcal{C}(X) = \bigcup\{\mathcal{C}(Y) \mid Y \subseteq X, Y \text{ finite}\} .$$

See also Roman [35], Theorems 7.7 and 7.8.

Now we can prove

Theorem 1.1.

Suppose A is an algebra whose signature includes a binary operation $\to$ and a constant 1. Then the family $\mathcal{DS}(A)$ of deductive systems of A is an algebraic lattice whose compact elements are the finitely generated deductive systems.

Proof. Clearly $\mathcal{DS}(A)$ is a closure system. In view of the result mentioned above, it suffices to prove that the union of any directed family of deductive systems is a deductive system. Let $\mathcal{F}$ be such a family and take $x, x \to y \in \bigcup\mathcal{F}$. Then $x \in A$ and $x \to y \in B$ for some $A, B \in \mathcal{F}$. Let C be an upper bound of A and B in $\mathcal{F}$. Then $x, x \to y \in C$, hence $y \in C$, therefore $y \in \bigcup\mathcal{F}$. Clearly $1 \in \bigcup\mathcal{F}$. □

Before tackling the problem of the internal structure of $Ded(X)$, it seems convenient to briefly recall the classes of Hilbert algebras and BCK algebras. Like many axiomatic definitions, the definitions of Hilbert algebras and BCK algebras are economical, meaning that there are few axioms and each of them does not require too much. Such an axiom system is advantageous for proving that a given mathematical object belongs to the class defined by the axioms. However, sometimes there is a cost to be paid for this facility, namely that the most important properties of the defined concept remain hidden, and this is precisely the case of the original definitions of Hilbert algebras and BCK algebras. That is why we prefer the following unorthodox equivalent definitions. The two classes have a common genus proximus: a structure $(A, \to, 1, \leq)$, where $(A, \to, 1)$ is an algebra of type (2,0), $(A, \leq, 1)$ is a poset with greatest element, and $\leq$ is the *natural order*, that is,

$$x \leq y \Longleftrightarrow x \to y = 1 \,. \tag{nat}$$

Hilbert algebras are then characterized by the identities

$$x \leq y \to x \,, \tag{H1}$$

$$x \to (y \to z) \leq (x \to y) \to (x \to z) \,, \tag{H2}$$

while identities

$$1 \to x = x \,, \tag{BCK1}$$

$$x \to y \leq (y \to z) \to (x \to z) \,, \tag{BCK2}$$

characterize BCK *algebras*. Indeed, these definitions collect well-known properties of Hilbert algebras and BCK algebras and imply the properties which occur in the original definitions.

The monograph by Iorgulescu [24] emphasizes that many algebras of logic have a reduct $(A, \to, 1, \leq)$ which is a BCK algebra. In particular Remark 2.1.32(1) shows that Hilbert algebras are exactly those BCK algebras which satisfy

$$x \to (y \to z) = (x \to y) \to (x \to z) \,. \tag{$\overline{\text{H2}}$}$$

Alternatively, since it can be proved that (H1) holds in every BCK algebra, Hilbert algebras are also characterized by (H2) within the class of BCK algebras. Another characterization is mentioned by Iorgulescu [24], Proposition 2.1.31:

$$x \to (x \to y) = x \to y \,.$$

Now we are going to prove that the internal description of $Ded(X)$ recalled above is in fact valid for a class of algebras $(A, \to, 1)$ which strictly includes the class of BCK algebras.

A convenient technical device is the compact notation

$$x_1 \to (x_2 \to (\ldots \to (x_n \to x) \ldots)) = (x_1, \ldots, x_n; x) \,,$$

which can be defined by induction:

$$(x_1; x) = x_1 \to x \,,$$

$$(x_1, \ldots, x_{n+1}; x) = (x_1, \ldots, x_n; x_{n+1} \to x) \,.$$

This inductive definition immediately implies the identity

$$x_1 \to (x_2, \ldots, x_{n+1}; x) = (x_1, x_2, \ldots, x_{n+1}; x) \,,$$

which will be used in the sequel.

Theorem 1.2.

Suppose an algebra $(A, \to, 1)$ of type (2,0) satisfies the axioms

$$x \to x = 1 \,, \tag{a1}$$

$$1 \to x = x \,, \tag{a2}$$

$$x \to 1 = 1 \,, \tag{a3}$$

$$x \to (y \to z) = y \to (x \to z) \,, \tag{a4}$$

$$x \to y = 1 \Longrightarrow (z \to x) \to (z \to y) = 1 \,. \tag{a5}$$

Then if $\varnothing \neq X \subseteq A$ we have

$$Ded(X) = \{x \in A \mid (x_1, \ldots, x_n; x) = 1;\; x_1, \ldots, x_n \in X,\; n \in \mathbb{N} \backslash \{0\}\} \,.$$

Proof. Note first that (a4) can be generalized, to the effect that

$$(x_{\sigma(1)}, \ldots, x_{\sigma(n)}; x) = (x_1, \ldots, x_n; x) \tag{$\overline{\text{a4}}$}$$

for every permutation σ of $(1, \ldots, n)$.

Now we are going to show that the right-hand side X' of the equality to be proved is indeed the smallest deductive system which includes X.

We have $X \subseteq X'$ by (a1). Then, as $X \neq \varnothing$, it follows that $1 \in X'$ by (a3). Further, suppose $x, x \to y \in X'$, say $(x_1, \ldots, x_n; x) = 1$ and $(z_1, \ldots, z_m; x \to y) = 1$, where $x_1, \ldots, x_n, z_1, \ldots, z_m \in X$. This implies

$$x \to (z_1, \ldots, z_m; y) = (x, z_1, \ldots, z_m; y) = (z_1, \ldots, z_m, x; y) = (z_1, \ldots, z_m; x \to y) = 1 ,$$

hence it follows from (a5) that

$$(x_n; x) \to (x_n, z_1, \ldots, z_m; y) = (x_n \to x) \to (x_n \to (z_1, \ldots, z_m; y)) = 1 .$$

In a similar manner we obtain

$$(x_{n-1}, x_n; x) \to (x_{n-1}, x_n, z_1, \ldots, z_m; y) = (x_{n-1} \to (x_n; x)) \to (x_{n-1} \to (x_n, z_1, \ldots, z_m; y)) = 1 .$$

Clearly this process continues until we get

$$(x_1, \ldots, x_n; x) \to (x_1, \ldots, x_n, z_1, \ldots, z_m; y) = 1$$

and since $(x_1, \ldots, x_n; x) = 1$, this implies $(x_1, \ldots, x_n, z_1, \ldots, z_m; y) = 1$ by (a2), showing that $y \in X'$. So we have proved that $X' \in \mathcal{DS}(A)$.

Finally suppose $X \subseteq D \in \mathcal{DS}(A)$. Take $x \in X'$, say $(x_1, \ldots, x_n; x) = 1 \in D$, where $x_1, \ldots, x_n \in X$. Then $x_1, \ldots, x_n \in D$ and $x_1 \to (x_2, \ldots, x_n; x) = (x_1, \ldots, x_n; x) \in D$, hence $(x_2, \ldots, x_n; x) \in D$. We continue in the same way and obtain $(x_3, \ldots, x_n; x) \in D, \ldots, (x_n; x) \in D$. So $x_n \to x \in D$, therefore $x \in D$. This proves that $X' \subseteq D$. □

Now we are going to establish the relationship between BCK algebras and the algebras defined by (a1)–(a5).

Lemma 1.1 (Iséki and Tanaka [25]).

BCK algebras satisfy (a1)–(a5).

Proposition 1.1.

The class of algebras $(A, \to, 1)$ *defined by axioms* (a1)–(a5) *strictly includes the class of* BCK *algebras.*

Proof. In view of Lemma 1.1, it remains to provide an algebra $(A, \to, 1)$ which satisfies (a1)–(a5) but is not BCK.

Take a three-element set $\{a, b, 1\}$ and define $1 \to a = a, 1 \to b = b, x \to y = 1$ otherwise. Then $a \to b = 1 = b \to a$ and $a \neq b$, therefore this algebra is not BCK. It remains to prove (a1)–(a5).

It is readily seen that (a1)–(a3) are satisfied.

To prove (a4) we note that $1 \to x = x$ and discuss several cases.

Case 1: $z = 1$. We have $x \to (y \to 1) = x \to 1 = 1 = y \to 1 = y \to (x \to 1)$.

Case 2: $z = a$. There are two subcases.

2.1: $y = 1$. We have $x \to (1 \to a) = x \to a = 1 \to (x \to a)$.

2.2: $y \neq 1$. Then $x \to (y \to a) = x \to 1 = 1 = y \to (x \to a)$.

Case 3: $z = b$. Similar to case 2.

Finally we prove (a5). Take x, y such that $x \to y = 1$. Suppose there is z such that $(z \to x) \to (z \to y) \neq 1$, say $(z \to x) \to (z \to y) = a$. In our algebra this is possible only if $z \to x = 1$ and $z \to y = a$. Again, the latter equality is possible only if $z = 1$ and $y = a$. Then from $z \to x = 1$ we obtain $x = 1$ by (a2), therefore $x \to y = 1 \to a = a$. Contradiction. □

Lemma 1.2 (Iorgulescu).

In every algebra which verifies the axioms (a1)–(a5)*, the following identities hold:*

$$x \to (y \to x) = 1\ , \tag{a6}$$

$$y \to ((y \to x) \to x) = 1\ , \tag{a7}$$

$$(x \to y) \to ((y \to z) \to (x \to z)) = 1\ . \tag{a8}$$

Proof. By applying (a4) and (a1) we obtain

$$x \to (y \to x) = y \to (x \to x) = y \to 1 = 1 \text{ by (a3)}\ ,$$

$$y \to ((y \to x) \to x) = (y \to x) \to (y \to x) = 1\ ,$$

and since $y \to ((y \to z) \to z) = 1$ by (a7), we get

$$(x \to y) \to (x \to ((y \to z) \to z)) = 1$$

by (a5), whence (a8) follows by (a4). □

Corollary 1.1 (Iorgulescu).

Let $(A, \to, 1)$ *be an algebra satisfying* (a1)–(a5)*. Define*

$$x \le y \iff x \to y = 1\ . \tag{nat}$$

Then $\le$ *is a pre-order which satisfies* $x \le 1$, (BCK1) *and* (BCK2).

Proof. The last three properties are translations of (a3), (a2) and (a8), respectively, while (a1) expresses reflexivity. To prove transitivity, suppose $x \to y = y \to z = 1$. Then (a8) and (a2) imply $(y \to z) \to (x \to z) = 1$, that is, $1 \to (x \to z) = 1$, hence $x \to z = 1$. □

Proposition 1.2 (Iorgulescu).

BCK algebras coincide with the algebras satisfying (a1)–(a5) *and*

$$x \to y = y \to x = 1 \text{ imply } x = y\ . \tag{a9}$$

Proof. Every BCK algebra satisfies (a1)–(a5) by Lemma 1.1, and also (a9), which expresses the antisymmetry of $\le$. Conversely, it follows from Corollary 1.1 that if the algebra $(A, \to, 1)$ satisfies (a1)–(a5) and (a9), then $\le$ is a partial order, therefore A is a BCK algebra. □

In view of the above we adopt the suggestion of our colleague Afrodita Iorgulescu and refer to the algebras satisfying (a1)–(a5) as *pre-BCK algebras*. The point is that, roughly speaking, pre-BCK algebras are obtained by relaxing the partial order to a pre-order.

Now Proposition 1.2 can be paraphrased as follows:

Proposition 1.2' (Iorgulescu).

BCK algebras are the same as the pre-BCK algebras that satisfy (a9).

Another characterization of pre-BCK algebras is the following:

Proposition 1.3 (Iorgulescu).

The axiom system {(a2), (a3), (a4), (a8)} *defines pre-BCK algebras.*

Proof. It remains to infer (a1) and (a5) from the above conditions.

Taking $x := y := 1$ in (a8) and using (a3), we obtain $1 \to ((1 \to z) \to (1 \to z)) = 1$, whence (a2) implies first $(1 \to z) \to (1 \to z) = 1$, then $z \to z = 1$.

Further we note that $(y \to z) \to ((x \to y) \to (x \to z)) = 1$ by (a8) and (a4). If $y \to z = 1$, the latter equality implies $(x \to y) \to (x \to z) = 1$ by (a2), and this implication is (a5). □

Corollary 1.2.

The class of pre-BCK algebras is equational.

Pre-BCK algebras can also be nicely related to BCC algebras, discussed by Chajda, Halaš and Kühr [16], pp. 194–195. A *BCC algebra* is an algebra $(A, \to, 1)$ which satisfies (a1), (a2), (a3), (a9) and

$$(x \to y) \to ((z \to x) \to (z \to y)) = 1\,. \tag{a10}$$

It is proved that the class of BCC algebras strictly includes the class of BCK algebras, the latter being characterized as those BCC algebras which satisfy the *exchange axiom* (a4).

Proposition 1.4.

An algebra $(A, \to, 1)$ is a BCK algebra if and only if it is both a pre-BCK algebra and a BCC algebra.

Proof. The "only if" part is already established. Conversely, if a BCC algebra is also a pre-BCK algebra, then it satisfies (a4), hence it is a BCK algebra. □

In the case of Hilbert algebras Theorem 1.2 can be strengthened.

Theorem 1.3.

The deductive system generated by a non-empty subset X of a Hilbert algebra $(A, \to, 1, \leq)$ is given by

$$Ded(X) = \big\{x \in A \mid (x_1, \dots, x_n; x) = 1;\; x_1, \dots, x_n \in X;\; x_1, \dots, x_n \text{ pairwise distinct},\; n \in \mathbb{N}\backslash\{0\}\big\}\,.$$

Proof. In view of Theorem 1.2, it suffices to prove that if $(x_1, ..., x_n; x) = 1$ and $x_k = x_h$ for some k, h with $k \neq h$, then $(x_k, y_1, \dots, y_{n-2}; x) = 1$, where $y_1, \dots, y_{n-2}$ are the elements x_i with $i \in \{1, \dots, n\}\backslash\{k, h\}$.

To prove this, set $(y_1, \dots, y_{n-2}; x) = z$. Taking into account $(\overline{H2})$ and $(\overline{a4})$, we have

$$(x_k, y_1, \dots, y_{n-2}; x) = x_k \to z = 1 \to (x_k \to z) = (x_k \to x_h) \to (x_k \to z)$$

$$= x_k \to (x_h \to z) = (x_k, x_h, y_1, \dots, y_{n-2}; x) = (x_1, \dots, x_n; x) = 1\,. \qquad \square$$

In particular Theorem 1.3 yields a characterization of finitely generated deductive systems.

Corollary 1.3 (Diego [18, 19]).

Suppose A is a Hilbert algebra and $X = \{x_1, \dots, x_n\} \subseteq A$. Then

$$Ded(X) = \{x \in A \mid (x_1, \dots, x_n; x) = 1\}\,.$$

Proof. In view of Theorem 1.3, an element $x \in A$ is in $Ded(X)$ if and only if $(z_1, \dots, z_m; x) = 1$ for some pairwise distinct elements $z_1, \dots, z_m \in X$, $m \leq n$. It suffices to note that if $x_i \notin \{z_1, \dots, z_m\}$, then

$$(x_i, z_1, \dots, z_m; x) = x_i \to (z_1, \dots, z_m; x) = x_i \to 1 = 1\,. \qquad \square$$

To realize the extent of Theorem 1.3 and Corollary 1.3, let us recall a few important subclasses of Hilbert algebras. *Implication algebras* were introduced by Abbott [1, 2]. They can be characterized as Hilbert algebras satisfying the identity

$$(x \to y) \to y = (y \to x) \to x\ ; \tag{IA}$$

cf. [1], Theorem 26, or [2], Theorem 8. A *Hertz algebra* is a Hilbert algebra which is also a meet semilattice with respect to the natural order (nat) and satisfies

$$x \to (y \to x \wedge y) = 1\ . \tag{Hz}$$

The following definition is due to Katriňák [27]: a *poset* $(P, \leq)$ is said to be *relatively pseudocomplemented* if there is a binary operation $\to$ defined on P such that

$$\forall z\,[z \leq a \text{ and } z \leq x \Longrightarrow z \leq b] \iff x \leq a \to b\ ;$$

using the notation $(c] = \{z \in P \mid z \leq c\}$, the latter property can be written in the compact form

$$(a] \cap (x] \subseteq (b] \iff (x] \subseteq (a \to b]\ .$$

Rudeanu [36] proved that any such poset is a Hilbert algebra with respect to the operation $\to$. If P is a lattice (a meet semilattice), this definition can be written in the form

$$a \wedge x \leq b \iff x \leq a \to b \tag{mres}$$

(*meet residuation*), P is said to be a *relatively pseudocomplemented lattice* (*semilattice*) and it has greatest element $1 = a \to a$. The Hertz algebras mentioned above are the same as relatively pseudocomplemented semilattices; cf., e.g., Figallo Jr. and Ziliani [20]. The *bounded relatively pseudocomplemented lattices*, also known as *Heyting algebras*, are those relatively pseudocomplemented lattices which have least element 0. *Boolean algebras* become Heyting algebras with respect to the relative pseudocomplementation $a \to b = a' \vee b$.

2. Deductive systems versus product filters

In the sequel we deal with algebras A (in the sense of universal algebra) which are also endowed with a binary relation $\leq$, and we refer to these structures as *relational algebras*.

Let A be a relational algebra which has a reduct $(A, \odot, 1, \leq)$, where $(A, \odot, \leq)$ is an algebra of type (2,0). By a $\odot$-*filter* or *product filter*, we mean a subset $S \subseteq A$ which satisfies

$$1 \in S\ , \tag{s1}$$

$$x \in S \text{ and } x \leq y \Longrightarrow y \in S\ , \tag{s3}$$

$$x, y \in S \Longrightarrow x \odot y \in S\ . \tag{s4}$$

Proposition 2.1.

Let A be a relational algebra having a reduct $(A, \odot, 1, \leq)$. The family $\mathcal{PF}$ of $\odot$-filters of A is an algebraic lattice whose compact elements are the finitely generated $\odot$-filters.

Proof. Quite similar to the proof of Theorem 1.1. □

It is easy to determine, under mild conditions, the structure of the $\odot$-filter $Fil(X)$ generated by a subset $X \subseteq A$.

Recall that an *ordered semigroup* is a relational algebra $(A, \odot, \leq)$, where $(A, \odot)$ is a semigroup, i.e., the operation $\odot$ is associative, $(A, \leq)$ is a poset and the operation $\odot$ is *isotone*, that is,

$$x \leq y \Longrightarrow z \odot x \leq z \odot y \text{ and } x \odot z \leq y \odot z\ .$$

Proposition 2.2.

Suppose the relational algebra $(A, \odot, \to, 1, \leq)$ satisfies

(i) $(A, \leq, 1)$ *is a poset with greatest element;*

(ii) $(A, \odot, \leq)$ *is an ordered semigroup.*

Then the $\odot$-filter $Fil(X)$ generated by a subset $X \subseteq A$ is determined as follows: $Fil(\varnothing) = \{1\}$, while if $X \neq \varnothing$ then

$$Fil(X) = \left\{x \in A \mid x_1 \odot \ldots \odot x_n \leq x;\ x_1, \ldots, x_n \in X, n \in \mathbb{N} \setminus \{0\}\right\} .$$

Proof. Routine. □

In Propositions 2.1 and 2.2 the binary operation $\odot$ is alone. If the relational algebra has a reduct $(A, \odot, \to, 1, \leq)$, then both deductive systems and $\odot$-filters exist, therefore it is natural to investigate the relationship between these concepts.

Proposition 2.3.

If the relational algebra $(A, \odot, \to, 1, \leq)$ satisfies

$$x \odot (x \to y) \leq y \text{ or } (x \to y) \odot x \leq y , \tag{1}$$

then every $\odot$-filter is a deductive system, while if

$$x \leq y \Longrightarrow x \to y = 1 , \tag{2}$$

$$x \to (y \to x \odot y) = 1 \text{ or } y \to (x \to x \odot y) = 1 , \tag{3}$$

then every deductive system is a $\odot$-filter.

Caution: In axioms (1) and (3) it is meant that some pairs x, y may fulfil the first alternative and some pairs may satisfy the second alternative. Moreover, these sets of pairs need not be the same for the two axioms.

Proof. Suppose (1) holds and let S be a $\odot$-filter. Then (s1) holds. To prove (s2) take $x, x \to y \in S$. Then $x \odot (x \to y), (x \to y) \odot x \in S$ by (s4), hence $y \in S$ by (1) and (s3).

Now suppose (2) and (3) hold and let S be a deductive system. Then (s1) holds.

To prove (s3) let $x \in S$ and $x \leq y$. Then $x \to y = 1$ by (2), hence $x \to y \in S$ by (s1), therefore $y \in S$ by (s2).

To prove (s4) take $x, y \in S$. In view of (3) there are two cases. (i) If $x \to (y \to x \odot y) = 1$, then $x \to (y \to x \odot y) \in S$ by (s1), whence it follows by applying (s2) twice that $y \to x \odot y \in S$, then $x \odot y \in S$. (ii) If $y \to (x \to x \odot y) = 1$, the proof is similar. □

Corollary 2.1.

If the relational algebra $(A, \odot, \to, 1, \leq)$ satisfies

$$z \leq x \to y \Longrightarrow x \odot z \leq y , \tag{res$\leftarrow$}$$

then every $\odot$-filter is a deductive system, while if (2) and

$$x \odot z \leq y \Longrightarrow z \leq x \to y \tag{res$\rightarrow$}$$

hold, then every deductive system is a $\odot$-filter.

Proof. In view of Proposition 2.3 it suffices to prove that (res$\leftarrow$) $\Longrightarrow$ (1) and (res$\rightarrow$) $\Longrightarrow$ (3). Indeed, taking $z := x \to y$ in (res$\leftarrow$) we obtain $x \odot (x \to y) \leq y$, and taking $y := x \odot z$ in (res$\rightarrow$) we get $z \leq x \to x \odot z$, hence $z \to (x \to x \odot z) = 1$ by (2). □

A similar corollary working with $z \odot x$ is left to the reader.

In numerous algebras of logic studied in the literature $\odot$-filters coincide with deductive systems and $\leq$ is the natural order (nat). The next proposition points out that the common cause of these properties is the *residuation condition* (res) below.

Proposition 2.4.

Let $(A, \odot, \rightarrow, 1, \leq)$ be a relational algebra in which the relation $\leq$ is antisymmetric and reflexive, while the identities $x \leq 1$ and $1 \odot x = x$ hold. If the equivalence

$$x \odot y \leq z \Longleftrightarrow x \leq y \rightarrow z \tag{res}$$

is fulfilled, then $\odot$-filters coincide with deductive systems and

$$x \leq y \Longleftrightarrow x \rightarrow y = 1\,. \tag{nat}$$

Proof. In view of Proposition 2.3, the sameness of $\odot$-filters and deductive systems will follow from (1), (2) and (3).

Taking $x := y \rightarrow z$ in (res) it follows that $(y \rightarrow z) \odot y \leq z$, which implies (1). Taking $x := 1$ in (res) we obtain $y \leq z \Longleftrightarrow 1 \leq y \rightarrow z$. But $y \rightarrow z \leq 1$, hence $1 \leq y \rightarrow z \Longleftrightarrow 1 = y \rightarrow z$ by antisymmetry and reflexivity, therefore $y \leq z \Longleftrightarrow 1 = y \rightarrow z$. We have thus proved (nat), which implies (2). Taking $z := x \odot y$ in (res) and using reflexivity, we obtain $x \leq y \rightarrow x \odot y$, which is equivalent to $x \rightarrow (y \rightarrow x \odot y) = 1$ by (nat). Therefore (3) holds as well. □

Corollary 2.2.

Let $(A, \odot, \rightarrow, 1, \leq)$ be a relational algebra such that $(A, \leq, 1)$ is a poset with greatest element and identity $1 \odot x = x$ holds. If (res) *is satisfied, then $\odot$-filters coincide with deductive systems and $\leq$ is the natural order* (nat).

Proposition 2.4 and Corollary 2.2 can be applied in order to obtain the coincidence of deductive systems with $\odot$-filters in certain algebras of logic that have been studied in the literature.

Thus, according to the book by Chajda, Halaš and Kühr [16], a *pocrim* is a relational algebra $(A, \odot, \rightarrow, 1, \leq)$ such that $(A, \leq, 1)$ is a poset with greatest element, $(A, \odot, 1)$ is an ordered commutative monoid, and the residuation property (res) holds.

Corollary 2.3.

In every pocrim deductive systems coincide with $\odot$-filters.

Proof. By Corollary 2.2. □

A *bounded residuated lattice* is an algebra $(L, \wedge, \vee, \odot, \rightarrow, 0, 1)$ where $(L, \wedge, \vee, 0, 1)$ is a bounded lattice, $(L, \odot, 1)$ is a commutative ordered monoid and the residuation property (res) holds; the concept of a *residuated lattice* is obtained by dropping the existence of 0. In particular BL *algebras*, introduced by Hájek, are bounded residuated lattices (see e.g. Piciu [33]), while MV *algebras* are a particular case of BL algebras, e.g. by Proposition 54 in Turunen [38]. So, according to Corollary 2.2, *in bounded residuated lattices and in particular in* BL *algebras and* MV *algebras deductive systems coincide with $\odot$-filters.*

The coincidence of deductive systems with $\odot$-filters had already been proved by Turunen [39] for BL algebras and by Piciu [33], Remark 1.11, for bounded residuated lattices. As a matter of fact, Corollary 2.2 shows that the existence of 0 is not necessary: *in residuated lattices deductive systems coincide with $\odot$-filters.*

So, deductive systems coincide with $\odot$-filters in certain relational algebras. Since on the other hand we have internal descriptions of deductive systems and of $\odot$-filters (Theorem 1.2 and Proposition 2.2), it is natural to look for cases in which the two descriptions coincide. As shown by the next proposition, the basis for this identification is a property which we metaphorically call the *deduction theorem* (DT).

Proposition 2.5.

Suppose the structure $(A, \odot, \to, \leq)$ *satisfies*

(i) $(A, \leq)$ *is a poset;*

(ii) $(A, \odot, \leq)$ *is an ordered semigroup;*

(res) $x \odot y \leq z \Longleftrightarrow x \leq y \to z$.

Then

$$x \to (y \to z) = x \odot y \to z \,, \tag{DT}$$

$$(x_1, \dots, x_n; x) = x_1 \odot \dots \odot x_n \to x \,. \tag{nDT}$$

Proof. Taking $x := y \to z$ in (res), we get

$$(y \to z) \odot y \leq z \,, \tag{4}$$

which for $y := x$ and $z := y \to z$ yields

$$(x \to (y \to z)) \odot x \leq y \to z \,,$$

hence

$$(x \to (y \to z)) \odot x \odot y \leq (y \to z) \odot y \leq z \,,$$

therefore (res) implies

$$x \to (y \to z) \leq x \odot y \to z \,. \tag{5}$$

On the other hand, it also follows from (4) that

$$(x \odot y \to z) \odot x \odot y \leq z \,,$$

whence (res) implies in turn

$$(x \odot y \to z) \odot x \leq y \to z, \qquad x \odot y \to z \leq x \to (y \to z).$$

The latter inequality and (5) yield (DT).

The inductive step in proving (nDT) is immediate. □

Corollary 2.4.

If the operation $\odot$ *is commutative, then*

$$x \to (y \to z) = y \to (x \to z) \,. \tag{a4}$$

Proposition 2.6.

Suppose $(A, \odot, \to, 1, \leq)$ *is a relational algebra such that* $(A, \leq, 1)$ *is a poset with greatest element,* $(A, \odot, 1, \leq)$ *is an ordered semigroup with left unit, and the residuation property* (res) *holds. If* $\varnothing \neq X \subseteq A$ *then*

$$\begin{aligned} Fil(X) &= \left\{x \in A \mid x_1 \odot \dots \odot x_n \leq x;\ x_1, \dots, x_n \in X, n \in \mathbb{N}\backslash\{0\}\right\} \\ &= \left\{x \in A \mid (x_1, ..., x_n; x) = 1;\ x_1, \dots, x_n \in X, n \in \mathbb{N}\backslash\{0\}\right\} = Ded(X) \,. \end{aligned} \tag{6}$$

Proof. $Fil(X)$ equals the first set in (6) by Proposition 2.2. It follows from Corollary 2.2 and Proposition 2.5 that

$$x_1 \odot \dots \odot x_n \leq x \Longleftrightarrow x_1 \odot \dots \odot x_n \to x = 1 \Longleftrightarrow (x_1, \dots, x_n; x) = 1 \,,$$

therefore the two sets in (6) are equal. Finally, deductive systems coincide with $\odot$-filters again by Corollary 2.2, therefore $Fil(X) = Ded(X)$. □

Corollary 2.5.

Relations (6) *hold in pocrims and residuated lattices; in particular they are valid in* BL *algebras and* MV *algebras.*

3. Deductive systems versus semilattice and order filters

Recall that a *filter* of a lattice $(A, \wedge, \vee, 1)$ with greatest element 1 is a subset $S \subseteq A$ which satisfies

$$1 \in S\,, \tag{s1}$$

$$x \in S \text{ and } x \leq y \Longrightarrow y \in S\,, \tag{s3}$$

$$x, y \in S \Longrightarrow x \wedge y \in S\,. \tag{s5}$$

As a matter of fact, this definition does not require the entire structure of the lattice: it makes sense even in the meet reduct $(A, \wedge, 1)$.

On the other hand, we have noted in Section 1 that Diego's concept of a *deductive system*, also called *implicative filter*, can be formally defined in any algebra $(A, \rightarrow, 1)$ of type (2,0) as a subset $S \subseteq A$ which satisfies (s1) and

$$x \in S \text{ and } x \rightarrow y \in S \Longrightarrow y \in S\,. \tag{s2}$$

However in the literature this concept has been rendered valuable and connected with logic only in lattice-theoretical context.

Therefore it seems plausible that the following definition introduces the most general algebras for which the concepts of filter and deductive system not only make sense, but keep the original meaning behind them.

Definition 3.1.
A *meet-arrow algebra* (MA) is an algebra $(A, \wedge, \rightarrow, 1)$ of type (2,2,0) where $(A, \wedge, 1)$ is a meet semilattice with greatest element.

Definition 3.2.
A subset S of an MA A is called a *filter* if it satisfies (s1), (s3), (s5), and a *deductive system* if it satisfies (s1), (s2). If the MA is also endowed with a unary operation φ, then by a *strong filter relative to* φ, or simply a *strong filter* when there is no danger of confusion, we mean a filter S which satisfies

$$x \in S \Longrightarrow \varphi(x) \in S\,. \tag{s6}$$

As to the relationship between $\odot-$filters, (semilattice or $\wedge$-) filters and strong filters, the following result is obvious.

Remark 3.1.
1) Suppose $(A, \wedge, \odot, \rightarrow, 1, \leq)$ is a relational algebra such that $(A, \wedge, \rightarrow, 1)$ is an MA and $\leq$ is the semilattice order. If A satisfies $x \odot y \leq x \wedge y$, then every $\odot-$filter is a filter.

2) If A is an MA endowed with a unary operation φ, set $x \odot_\varphi y = x \wedge y \wedge \varphi(x \wedge y)$. Then $\odot_\varphi-$filters coincide with strong filters. We see that every strong filter satisfies (s4) by applying in turn (s5), (s6) and again (s6), while if F is a $\odot_\varphi-$filter, then by applying (s6) and (s3) we see that F satisfies (s5) and (s6).

Corollary 3.1.
In residuated lattices and in particular in BL algebras and MV algebras, every $\odot$-filter is a filter.

Proof. By Remark 3.1 and Turunen [38], Proposition 13, (1.37). □

Proposition 3.1a below is the only property in this paper which establishes a necessary and sufficient condition.

Proposition 3.1a.
In any MA $(A, \wedge, \rightarrow, 1)$, *every filter is a deductive system if and only if*

$$x \wedge (x \rightarrow y) \leq y\,. \tag{7}$$

Proof. The "if" part follows by Proposition 2.3 with $\odot := \wedge$. To prove the "only if" part, suppose (7) fails. Then there are $a, b \in A$ such that $a \wedge (a \to b) \nleq b$. Let S be the principal filter generated by $a \wedge (a \to b)$. Since the principal filter generated by an element is the set of all upper bounds of that element, it follows that $a \in S$ and $a \to b \in S$ but $b \notin S$, therefore S is not a deductive system. □

Corollary 3.2.

In any MA $(A, \wedge, \to, 1)$, *if*

$$x \leq y \to z \implies x \wedge y \leq z \, , \tag{8}$$

then every filter is a deductive system.

Proof. Taking $x := y \to z$ we obtain condition (7). □

Proposition 3.1b.

In any MA $(A, \wedge, \to, 1)$, *conditions*

$$x \leq y \implies x \to y = 1 \, , \tag{2}$$

$$x \to (y \to x \wedge y) = 1 \, , \tag{Hz}$$

imply that every deductive system is a filter.

Proof. By Proposition 2.3 with $\odot := \wedge$. □

Corollary 3.3.

In any MA $(A, \wedge, \to, 1)$, *the condition*

$$x \wedge y \leq z \implies x \leq y \to z \tag{9}$$

implies that every deductive system is a filter.

Proof. Taking $x := 1$ in (9) we obtain (2), and taking $z := x \wedge y$ we obtain $x \leq y \to x \wedge y$, which implies (Hz) in view of (2). □

Proposition 3.2.

In Hertz algebras filters coincide with deductive systems.

Proof. By Corollaries 3.2 and 3.3, since Hertz algebras satisfy (mres). □

In particular *filters coincide with deductive systems in relatively pseudocomplemented lattices, Heyting algebras, Boolean algebras with $x \to y = x' \vee y$, and implication algebras with* 0. This sameness is folklore for Boolean algebras, was established by Monteiro [29] in the case of Heyting algebras, and was proved by Buşneag [7], Lemma 2.5, for Hertz algebras. Recall that every implication algebra with 0 is a Boolean algebra; cf. Abbott [1], Theorem 19, or [2], Theorem 8. A referee has called our attention to the fact that the coincidence of deductive systems with filters in relatively pseudocomplemented semilattices was widely known already in the eighties.

A special situation occurs in connection with Łukasiewicz–Moisil algebras. Recall that an *n-valued Łukasiewicz–Moisil algebra* (without negation), or *LMn algebra* for short, is an algebra $(L, \vee, \wedge, \varphi_1, \dots, \varphi_{n-1}, 0, 1)$ such that $(L, \vee, \wedge, 0, 1)$ is a bounded distributive lattice, $\varphi_1, \dots, \varphi_{n-1}$ are bounded-lattice endomorphisms with values in the Boolean algebra $(B(L), \vee, \wedge, ', 0, 1)$ of complemented elements of L, and the following conditions are satisfied:

$$\varphi_1 \leq \dots \leq \varphi_{n-1} \, ,$$

$$\varphi_i \varphi_j = \varphi_j \; (i, j = 1, \dots, n-1) \, ,$$

$$\varphi_i(x) = \varphi_i(y) \; (i = 1, \dots, n-1) \implies x = y \, .$$

Corollary 3.4.

In every LMn algebra L endowed with the implication

$$x \to y = y \vee \bigwedge_{i=1}^{n-1}(\varphi_i'(x) \vee \varphi_i(y)) \ ,$$

filters coincide with deductive systems.

Proof. L is a Heyting algebra (cf. Moisil and Cignoli; see e.g. Theorem 4.3.3 in Boicescu et al. [4]). □

The concept of a strong filter defined by (s1), (s3), (s5) and (s6), was introduced by Moisil in the case of *LMn* algebras, with $\varphi := \varphi_1$. The next proposition relates deductive systems to strong filters in a very general case.

Proposition 3.3.

Suppose the MA $(L, \wedge, \to, 1)$ *is endowed with a unary operation* φ*. If the condition*

$$\varphi(x) \wedge (x \to y) \leq y \tag{10}$$

holds, then every strong filter is a deductive system. If conditions

$$x \leq y \Longrightarrow x \to y = 1 \ , \tag{2}$$

$$y \to (x \to y) = 1 \ , \tag{11}$$

$$x \to y \leq x \to x \wedge y \ , \tag{12}$$

$$x \to \varphi(x) = 1 \tag{13}$$

hold, then every deductive system is a strong filter.

Proof. Suppose (10) holds and let S be a strong filter. Then (s1) holds. Now take $x, x \to y \in S$. Then $\varphi(x) \in S$ by (s6), hence (s5) implies $\varphi(x) \wedge (x \to y) \in S$, therefore $y \in S$ by (10) and (s3).

Now suppose conditions (2) and (11)–(13) hold and let S be a deductive system. Then (s1) holds.

To prove (s3) take $x \in S$ and $x \leq y$. Then $x \to y \in S$ by (2) and (s1), therefore $y \in S$ by (s2).

To prove (s5) note first that $y \to (x \to y) = 1 \in S$ by (11) and (s1), therefore if $y \in S$ then $x \to y \in S$ by (s2). Now take $x, y \in S$. Then $x \to y \in S$, hence (12) and (s3) imply $x \to x \wedge y \in S$, therefore $x \wedge y \in S$ by (s2).

To prove (s6) suppose $x \in S$. Since $x \to \varphi(x) = 1 \in S$ by (13) and (s1), it follows that $\varphi(x) \in S$. □

In the next corollary the role of $\to$ is played by the *weak implication*

$$x \overset{M}{\to} y = \varphi_1'(x) \vee y \ ,$$

introduced by Monteiro and Cignoli. Unlike $\to$, the weak implication $\overset{M}{\to}$ does not make L a Heyting algebra.

Corollary 3.5 (Monteiro [30] and Cignoli [17]; cf. Proposition 5.1.39 in Boicescu et al. [4]).

In every LMn algebra endowed with the weak implication, strong filters relative to φ_1*, or* LMn strong filters*, coincide with deductive systems.*

Proof. We check the hypotheses of Proposition 3.3 with $\varphi := \varphi_1$. For (2), (11) and (13) we refer to Proposition 5.1.36 in [4]. We check (10) and (12) by using the identity $\varphi_1(x) \le x$:

$$\varphi_1(x) \wedge (x \overset{M}{\to} y) = \varphi_1(x) \wedge (\varphi_1'(x) \vee y) = \varphi_1(x) \wedge y \le y \ ,$$

$$x \overset{M}{\to} x \wedge y = \varphi_1'(x) \vee (x \wedge y) \ge \varphi_1'(x) \vee (\varphi_1(x) \wedge y) = (\varphi_1'(x) \vee \varphi_1(x)) \wedge (\varphi_1'(x) \vee y) = 1 \wedge (x \overset{M}{\to} y) = x \overset{M}{\to} y \ .$$

□

The concept of an *LMn* strong filter is actually stronger than that of an *LMn* (lattice) filter. For instance, unlike the principal strong filter generated by an element x, the principal filter generated by x does not contain $\varphi_1(x)$. The converse of the implication $x \le y \Longrightarrow x \overset{M}{\to} y = 1$ does not hold. For instance, $x \overset{M}{\to} \varphi_1(x) = 1$, but in general $x \not\le \varphi_1(x)$ because $\varphi_i\varphi_1(x) = \varphi_1(x) \le \varphi_i(x)$ for all i, which implies $\varphi_1(x) \le x$.

There is one more concept which can be related to deductive systems. An *order filter* of a poset $(A, \le)$ is a non-empty subset $S \subseteq A$ satisfying (s3) and

$$x, y \in S \Longrightarrow \exists z \in S \ \ z \le x, y \ . \tag{s5'}$$

As usual, if the poset has greatest element 1, then condition $S \ne \varnothing$ can be replaced by (s1). The dual concept of *order ideal* was recently used by Celani, Cabrer and Montangie [10] in the context of Hilbert algebras.

Note that in a meet semilattice (join semilattice) order filters (order ideals) coincide with filters (ideals).

Proposition 3.4.
In a Hilbert algebra every order filter is a deductive system.

Proof. Let S be an order filter of a Hilbert algebra A. Take $x, x \to y \in S$. If $z \in S$ is a lower bound of x and $x \to y$, then by applying in turn (BCK1), (nat), ($\overline{\text{H2}}$) and again (nat), we obtain

$$z \to y = 1 \to (z \to y) = (z \to x) \to (z \to y) = z \to (x \to y) = 1 \ ,$$

hence $z \le y$, therefore $y \in S$. □

Proposition 3.5.
The following conditions are equivalent for a Hilbert algebra A:

(i) *every deductive system is an order filter,*

(ii) *A is a Hertz algebra.*

Proof. (ii) $\Longrightarrow$ (i) : Since order filters are just filters, we can apply Corollary 3.6.

(i) $\Longrightarrow$ (ii) : Take $a, b \in A$. Then $D = Ded(\{a, b\})$ is an order filter, hence there is $c \in D$ such that $c \le a$ and $c \le b$. Note that

$$D = \{x \in A \mid a \to (b \to x) = 1\} \tag{14}$$

by Corollary 1.1, hence $a \to (b \to c) = 1$. We will prove that $c = a \wedge b$, which will imply that A is a meet semilattice.

Take $z \in A$ such that $z \le a$ and $z \le b$. Then by applying (nat) and (BCK1) we get

$$z \to c = 1 \to (1 \to (z \to c)) = (z \to a) \to ((z \to b) \to (z \to c)) = z \to (a \to (b \to c)) = z \to 1 = 1 \ ,$$

hence $z \le c$. This proves that $c = a \wedge b$.

So $a \wedge b = c \in D$, hence (14) implies $a \to (b \to a \wedge b) = 1$. This is property (Hz), therefore A is a Hertz algebra. □

Corollary 3.6.

In Hertz algebras deductive systems coincide with filters and with order filters. In addition, the finitely generated deductive systems are principal filters:

$$Ded(\{x_1, \dots, x_n\}) = [x_1 \wedge \dots \wedge x_n) \ .$$

Proof. The first statement follows from Propositions 3.4 and 3.5.

We have noted in Section 1 that Corollary 1.1 applies to Hertz algebras. Besides, a Hertz algebra $(A, \wedge, \rightarrow, 1, \leq)$ satisfies the hypotheses of Proposition 2.5. Therefore

$$Ded(\{x_1, \dots, x_n\}) = \{x \in A \mid (x_1, \dots, x_n; x) = 1\} = \{x \in A \mid x_1 \wedge \dots \wedge x_n \rightarrow x = 1\} = \{x \in A \mid x_1 \wedge \dots \wedge x_n \leq x\} \ .$$

□

We conclude with an extension of property 1) in Remark 3.1.

Proposition 3.6.

Suppose $(A, \wedge, \odot, \rightarrow, 1, \leq)$ *is a relational algebra such that* $(A, \wedge, \rightarrow, 1)$ *is an MA and* $\leq$ *is the semilattice order.*

α) *If every filter is a* $\odot$*-filter, then* $x \wedge y \leq x \odot y$ *identically.*

β) *If* (Hz), (nat) *and* (res) *are satisfied, then* $x \odot y \leq x \wedge y$ *identically.*

γ) *If* (mres) *and* (res) *are satified, then* $x \odot y \leq x \wedge y$ *identically.*

δ) *Every filter is a* $\odot$*-filter if and only if* $x \wedge y \leq x \odot y$ *identically.*

Proof. Take $x, y \in A$.

α) Since $x, y \in [x \wedge y)$ and $[x \wedge y)$ is a $\odot$-filter, it follows that $x \odot y \in [x \wedge y)$, that is, $x \wedge y \leq x \odot y$.

β) The identity (Hz) is $x \rightarrow (y \rightarrow x \wedge y) = 1$ and it implies $x \leq y \rightarrow x \wedge y$ by (nat), therefore $x \odot y \leq x \wedge y$ by (res).

γ) A satisfies (nat) and (Hz) by Proposition 3.2, therefore we can apply β).

δ) If $x \wedge y \leq x \odot y$ then (s5)$\Longrightarrow$(s4). The converse holds by α). □

4. Conclusions

Our axiomatization provides a uniform approach to the problem of determining the relationship between filters and deductive systems in various contexts. We have found several simple conditions which ensure either an implication between these concepts or their equivalence. The results imply (most of) the theorems already known in the literature in this respect and a few new ones.

The next step of our research would be the extension of our results to the non-commutative case. This means algebras in which the product $\odot$ is not commutative, so that there are two implications, $\rightarrow$ and $\rightsquigarrow$, instead of just one. It is actually not difficult to duplicate the Propositions of the present paper, but we are looking for more compact results specific to the non-commutative case.

As a matter of fact, there are more variants of the concepts of filter than those dealt with in this paper; see e.g. Liu and Li [28] and the literature quoted therein. See also Chajda and Halaš [13–15] and Halaš [23]. It would be interesting to extend the present paper to other types of filters and deductive systems.

Most of the results in Sections 2 and 3 establish only sufficient conditions. An open problem which seems difficult is to obtain necessary and sufficient conditions, like in Proposition 3.1a.

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