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Border bases and kernels of homomorphisms and of derivations

Research Article

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Abstract: Border bases are an alternative to Gröbner bases. The former have several more desirable properties. In this paper some constructions and operations on border bases are presented. Namely; the case of a restriction of an ideal to a polynomial ring (in a smaller number of variables), the case of the intersection of two ideals, and the case of the kernel of a homomorphism of polynomial rings. These constructions are applied to the ideal of relations and to factorizable derivations.

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1. Introduction

Let k be a field. We will denote by k^* the set $k \setminus \{0\}$. By $k[X]$ we mean $k[x_1, \ldots, x_n]$, the polynomial ring in n variables, and by $k(X)$ we mean $k(x_1, \ldots, x_n)$, the field of rational functions. Let $\mathbb{N}$ designate the set of nonnegative integers. For $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, we denote by X^α the monomial $x_1^{\alpha_1} \ldots x_n^{\alpha_n} \in k[X]$.

Let R be a commutative k-algebra. A k-linear map $d : R \to R$ is said to be a *k-derivation* (or simply a *derivation*) of R if $d(ab) = ad(b) + bd(a)$ for all $a, b \in R$. By R^d we denote the kernel of the derivation d. It forms a ring and we call it the *ring of constants* of d. It is well known that $k \subseteq R^d$. A *nontrivial constant* of the k-derivation d is the element of the set $R^d \setminus k$. For any given derivation $d : k[X] \to k[X]$ there exists exactly one derivation $\bar{d} : k(X) \to k(X)$ such that $\bar{d}_{|k[X]} = d$. By a *rational constant* of the derivation $d : k[X] \to k[X]$ we mean the constant of its corresponding derivation $\bar{d} : k(X) \to k(X)$.

In Section 2 we give the definition of a factorizable derivation. Important Lotka–Volterra derivations are examples of factorizable derivations. We show how to associate the factorizable derivation with any given derivation. The construction helps to establish new facts on our initial derivation, especially on its rational constants (see [7]). Furthermore we define

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an ideal of relations associated with a derivation and we estimate the number of generators of the ideal of relations. Finally, we apply these considerations to the rational constants.

A border basis is a set of generators of an ideal in a polynomial ring, with some additional properties (more details in Section 3). The notion is somewhat analogous to the Gröbner basis. There are some advantages of border bases over Gröbner bases:

1. Gröbner bases behave numerically unstable forming representation singularities. A small change in the coefficients of generators of an ideal leads to a big change in the reduced Gröbner basis (see 6.4.1 of [6]). Whereas border bases may change continuously into one another when we continuously vary the coefficients of generators (6.4.22 of [6]).

2. An ideal which is symmetric with respect to swapping indeterminates, may not have any symmetric reduced Gröbner basis, however has a symmetric border basis (6.4.2 and 6.4.22 of [6]).

3. Border bases are useful in statistics (design of experiments [5]). Order ideals appear in this field in a natural way and in simple shapes. Methods of border bases are very convenient to optimize many processes.

4. In some cases border bases algorithms are significantly faster than those of Gröbner bases (see 19 and 20 of [4]). The reason lies in the fact that the border basis computation requires polynomials of much lower degrees (although there are often more polynomials). Both approaches were tested using the CoCoA system and in some instances one theory is more effective while, in some, the other.

5. When we compute all the border bases of an ideal, then as a consequence we also easily obtain all the reduced Gröbner bases ([4]), whereas the reverse is not true.

The above examples indicate the merits of further investigation of border bases. In this paper we give some subsequent applications of border bases.

The main results of the paper, presented in Section 4, are Theorem 4.1 and the conclusions drawn from Propositions 4.1 and 4.2. They provide constructions of border bases in the following cases:

1. The restriction of an ideal to a polynomial ring in a smaller number of variables.

2. The intersection of two ideals.

3. The kernel of a homomorphism of polynomial rings (for example, an ideal of relations).

2. Factorizable derivations and ideals of relations

In this section k is a field of characteristic zero. A derivation $d : k[X] \to k[X]$ is called *factorizable* if $d(x_i) = x_i f_i$, where $f_i \in k[X]$ for $i = 1, \dots, n$. Now we show how to associate the factorizable derivation with a given derivation.

Recall that for any field F and a derivation $d : F \to F$, by a *logarithmic derivative* of the derivation d we mean the mapping $L : F^* \to F$ defined by $L(a) = d(a)/a$ for all $a \in F^*$. If $\varphi \in k(X)^*$ is of the form $\varphi = a x_1^{\alpha_1} \dots x_n^{\alpha_n}$, where $a \in k^*$ and $\alpha_1, \dots, \alpha_n \in \mathbb{Z}$, then

$$L(\varphi) = \alpha_1 L(x_1) + \dots + \alpha_n L(x_n).$$

Let $d : k[X] \to k[X]$ be a derivation defined on the variables as follows:

$$d(x_i) = m_{i1} + m_{i2} + \dots + m_{ir_i}$$

for $i = 1, \dots, n$, where m_{ij} for $j = 1, \dots, r_i$, are monomials of the form $a x_1^{\beta_1} \dots x_n^{\beta_n}$, where $a \in k^*$ and $\beta_1, \dots, \beta_n \in \mathbb{N}$. Define the set

$$S = \left\{ \frac{m_{ij}}{x_i};\ 1 \le i \le n,\ 1 \le j \le r_i \right\} \subseteq k(X).$$

Let R be the smallest subring in $k(X)$ containing k and S, that is, $R = k[S]$. Then $d(x_i)/x_i \in R$ for $i = 1, \dots, n$ and the inclusion $\bar{d}(R) \subseteq R$ holds.

Now define the set S', obtained from the set S by normalization of the elements of S, that is, all coefficients are now equal to 1 (note that from all associated elements of S we obtain only one element of S') and then by crossing out, if it exists, the element from the field k. Obviously, $R = k[S']$. Let $S' = \{\varphi_1, \dots, \varphi_p\}$. For $i = 1, \dots, n$ we have

$$\frac{d(x_i)}{x_i} \in ke_d + k\varphi_1 + \dots + k\varphi_p,$$

where $e_d = 0$ if no element from k^* belongs to S and $e_d = 1$ in the opposite case. The mapping $\bar{d}_{|R} : R \to R$ is well-defined and

$$\bar{d}(\varphi_i) = \varphi_i(b_i e_d + a_{i1}\varphi_1 + \dots + a_{ip}\varphi_p)$$

for $i = 1, \dots, p$ and some $b_1, \dots, b_p, a_{ij} \in k$, where $1 \le j \le p$.

We introduce new variables $y_1, \dots, y_p$. We will denote by $k[Y]$ the polynomial ring $k[y_1, \dots, y_p]$. Let $\delta : k[Y] \to k[Y]$ be a derivation defined by

$$\delta(y_i) = y_i(b_i e_d + a_{i1}y_1 + \dots + a_{ip}y_p)$$

for $i = 1, \dots, p$. With the above notations we call δ the *factorizable derivation associated with derivation d*. For more details we refer the reader to [7].

Let $d : k[X] \to k[X]$ be a derivation and let $S' = \{\varphi_1, \dots, \varphi_p\}$ be its corresponding set defined above. Let $\varphi_i = x_1^{\alpha_{i1}} \dots x_n^{\alpha_{in}} = X^{\alpha_i}$ for $i = 1, \dots, p$. Then we have the matrix $\alpha = (\alpha_{ij})_{1 \le i \le p, 1 \le j \le n} \in M_{p \times n}(\mathbb{Z})$.

Let $R = k[\varphi_1, \dots, \varphi_p] \subseteq k(X)$. Define a homomorphism $\eta : k[y_1, \dots, y_p] \to R$ by $\eta(y_i) = \varphi_i$ for $i = 1, \dots, p$. It is an epimorphism. Therefore, $R \approx k[y_1, \dots, y_p]/\ker(\eta)$. By the *ideal of relations* associated with the derivation d we mean the kernel of the homomorphism η. We denote it by $A_{(\alpha)}$. Hence, $A_{(\alpha)} = \left\{h \in k[y_1, \dots, y_p];\ h(\varphi_1, \dots, \varphi_p) = 0\right\}$. The ideal $A_{(\alpha)}$ is prime and differential. It contains neither a linear form nor a nonzero monomial ([8]).

By a *monic binomial* we mean a polynomial of the form $Y^a - Y^b \in k[Y]$, where $a, b \in \mathbb{N}^p$.

Theorem 2.1 ([8], 5.7).
Let $\varphi_1 = X^{\alpha_1}, \dots, \varphi_p = X^{\alpha_p} \in k(X)$, where $\alpha_1, \dots, \alpha_p$ are pairwise distinct elements of the set $\mathbb{Z}^n \setminus \{(0, \dots, 0)\}$. If the ideal $A_{(\alpha)}$ is nonzero, then it has $(p - \operatorname{rank}_{\mathbb{Z}}\alpha)$ algebraically independent α-homogeneous irreducible monic binomials.

It is shown in [8] that if $A_{(\alpha)} \neq 0$, then the derivation δ has a nontrivial rational constant, and, an estimate can be given.

Theorem 2.2 ([8], 6.5).
If $A_{(\alpha)} \neq 0$, then the derivation δ has $(p - \operatorname{rank}_{\mathbb{Z}}\alpha)$ rational constants which are algebraically independent over k.

The main application of factorizable derivations is determining whether the given derivations have nontrivial rational constants (in general, still an open problem). For instance, in [7], this tool gives a full and extensive description of all monomial derivations of $k[x, y, z]$ with nontrivial rational constants, as well as other applications.

3. Border bases

A non-empty set of terms $\mathcal{O}$ in $k[X]$ is called an *order ideal* if $t \in \mathcal{O}$ implies $t' \in \mathcal{O}$ for every term t' dividing t. This ideal is a monoid ideal, not a ring ideal. The *border* of $\mathcal{O}$ is the set of terms $\partial\mathcal{O} = (x_1\mathcal{O} \cup \dots \cup x_n\mathcal{O}) \setminus \mathcal{O}$. Let $\mathcal{O}$ be finite and $\partial\mathcal{O} = \{b_1, \dots, b_r\}$. A set of polynomials $G = \{g_1, \dots, g_r\} \in k[X]$ is called an *$\mathcal{O}$-border prebasis* if $g_i = b_i + h_i$, where $h_i \in k[X]$ satisfies $\operatorname{Supp}(h_i) \subseteq \mathcal{O}$ for $1 \le i \le r$.

Let I be a zero-dimensional ideal of $k[X]$ containing $\mathcal{O}$-border prebasis G. We say that G is an *$\mathcal{O}$-border basis* of I if the residue classes of the elements of $\mathcal{O}$ form a k-vector space basis of $k[X]/I$.

There are many equivalent conditions to the above definition; eighteen of them are given in [3]. We will use the condition of Proposition 3.1, called the Buchberger criterion for border bases. The *S-polynomial* of two distinct elements $g_i, g_j \in G$ is defined by

$$S(g_i, g_j) = (\mathrm{lcm}(b_i, b_j)/b_i)g_i - (\mathrm{lcm}(b_i, b_j)/b_j)g_j.$$

Two prebasis polynomials, g_i and g_j are *neighbors* if their border terms are related according to $x_p b_i = x_s b_j$ or $x_p b_i = b_j$ for some indeterminates x_p, x_s.

Proposition 3.1 ([4], 4).

An $\mathcal{O}$-border prebasis $G = \{g_1, \dots, g_r\}$ is an $\mathcal{O}$-border basis of an ideal I if and only if $G \subseteq I$ and, for each pair of neighbors in G, there are constant coefficients $c_l \in k$ such that $S(g_i, g_j) = c_1 g_1 + \dots + c_r g_r$.

For every zero-dimensional ideal I there exists an order ideal $\mathcal{O}$ supporting a border basis. An example is $\mathcal{O}_\sigma\{I\} := \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{I\}$, where, here and throughout, $\mathbb{T}^n$ denotes the monoid of terms and $\mathrm{LT}_\sigma\{I\}$ denotes the set of leading terms of I with respect to a term ordering σ. If for a given $\mathcal{O}$ a border basis G of I exists, then G is uniquely determined and generates I.

In [4] there are several algorithms for computing border bases presented. We will employ the following Basis Transformation Algorithm ([4], Proposition 5).

Proposition 3.2.

Let $I \subseteq k[X]$ be a zero-dimensional ideal and let $\mathcal{O} = \{t_1, \dots, t_m\}$ be an order ideal. The following algorithm checks whether $\mathcal{O}$ supports a border basis of I and, in the affirmative, computes the $\mathcal{O}$-border basis $\{g_1, \dots, g_r\}$ of I.

(T1) Choose a term ordering σ and compute $\mathcal{O}_\sigma\{I\} := \mathbb{T}^n \setminus \mathrm{LT}_\sigma\{I\}$.

(T2) If $\#(\mathcal{O}_\sigma\{I\}) \neq m$, then return "$\mathcal{O}$ has the wrong cardinality to support a border basis of I," and stop.

(T3) Let $\mathcal{O}_\sigma\{I\} = \{s_1, \dots, s_m\}$. For $1 \le p \le m$, compute the coefficients $\tau_{ip} \in k$ of the normal form $\mathrm{NF}_{\sigma,I}(t_p) = \sum_{i=1}^m \tau_{ip} s_i$. Let T be the matrix $(\tau_{ip})_{1\le i,p\le m}$.

(T4) If $\det T = 0$, then return "$\mathcal{O}$ has the wrong form to support a border basis of I," and stop.

(T5) Let $\partial\mathcal{O} = \{b_1, \dots, b_r\}$. For $1 \le j \le r$, compute the coefficients $\beta_{ij} \in k$ of $\mathrm{NF}_{\sigma,I}(b_j) = \sum_{i=1}^m \beta_{ij} s_i$. Let B be the matrix $(\beta_{ij})_{1\le i\le m, 1\le j\le r}$.

(T6) Compute $(\alpha_{ij}) = T^{-1}B$. Return $g_j := b_j - \sum_{i=1}^m \alpha_{ij} t_i$ for $1 \le j \le r$.

4. Operations on border bases

Let $k[X,Y] = k[x_1, \dots, x_n, y_1, \dots, y_m]$. If $\mathcal{O}$ is an order ideal in $k[X,Y]$, then $\mathcal{O} \cap k[Y]$ is an order ideal in $k[Y]$. If $I \lhd k[X,Y]$ is a zero-dimensional ideal, then obviously the same is true for $I \cap k[Y]$. A term ordering σ such that $t >_\sigma t'$ for terms $t \in k[X]$, $t' \in k[Y]$ is, among many others, $\sigma = \mathrm{Lex}$. The following reduction requires only some set-theoretical condition on cardinality. Note also that we can just take $\mathcal{O} = \mathcal{O}_\sigma\{I\}$.

Theorem 4.1.

Let σ be a term ordering on $k[X,Y]$ such that, $t >_\sigma t'$ for $t \in k[X]$, $t' \in k[Y]$. Let I be a zero-dimensional ideal in $k[X,Y]$. Consider an order ideal $\mathcal{O}_\sigma\{I\} = \mathbb{T}^{n+m} \setminus \mathrm{LT}_\sigma\{I\}$. Let $\mathcal{O}$ be an order ideal in $k[X,Y]$ supporting a border basis of I, such that $\#(\mathcal{O} \cap k[Y]) = \#(\mathcal{O}_\sigma\{I\} \cap k[Y])$. If G is $\mathcal{O}$-border basis of I, then $G \cap k[Y]$ is $(\mathcal{O} \cap k[Y])$-border basis of $I \cap k[Y]$.

Proof. Observe that the border of $\mathcal{O} \cap k[Y]$ is $\partial\mathcal{O} \cap k[Y]$. Let $\partial\mathcal{O} \cap k[Y] = \{b_1, \dots, b_r\}$ and $\mathcal{O} \cap k[Y] = \{t_1, \dots, t_p\}$. By the assumption on cardinality, $\mathcal{O}_\sigma\{I\} \cap k[Y] = \{s_1, \dots, s_p\}$ for some terms $s_1, \dots, s_p \in k[Y]$. Let t be a term in $k[Y]$. If $t \in \{s_1, \dots, s_p\}$, then $\mathrm{NF}_{\sigma,I}(t) = t \in \langle s_1, \dots, s_p \rangle_k$.

Suppose $t \notin \{s_1, \dots, s_p\}$. Then $t = \mathrm{LT}_\sigma(f)$ for some $f \in I$. If a term $t' \in \mathrm{Supp}(f)$ depends on x_i for $1 \le i \le n$, then $t' >_\sigma t$, which is a contradiction to $t = \mathrm{LT}_\sigma(f)$. Hence, $f \in k[Y]$. Let $f = t + h$ for $h \in k[Y]$ and $\mathrm{LT}_\sigma(h) <_\sigma t$. Let $f_1, \dots, f_l$

be the reduced Gröbner basis of the ideal $I \cap k[Y]$ with respect to the term ordering $\sigma_{|_{k[Y]}}$. Then $f = h_1 f_1 + \cdots + h_l f_l$ for $h_1, \ldots, h_l \in k[Y]$. We apply the Division Algorithm for Gröbner bases to obtain $\mathrm{NF}_{\sigma,I}(t)$. In the above situation we subtract a polynomial in $k[Y]$ and the difference also belongs to $k[Y]$. At each step a reduction has the same properties, because every subsequent term is either in $\{s_1, \ldots, s_p\}$ or is a leading term of a polynomial in $I \cap k[Y]$. Finally, the rest has support in $\mathcal{O}_\sigma\{I\} \cap k[Y]$. Therefore $\mathrm{NF}_{\sigma,I}(t) \in \langle s_1, \ldots, s_p \rangle_k$.

Let $\mathcal{O} \setminus \{t_1, \ldots, t_p\} = \{t_{p+1}, \ldots, t_u\}$. We compute the $\mathcal{O}$-border basis G of I according to the algorithm in Proposition 3.2. We calculate the matrix T of step (T3). $\mathrm{NF}_{\sigma,I}(t_i) = \sum_{j=1}^{p} \tau_{ji} s_j$ for $1 \le i \le p$. Since $\mathcal{O}$ and $\mathcal{O}_\sigma\{I\}$ are order ideals both supporting border bases of I, then $\#\mathcal{O}_\sigma\{I\} = u$. Thus,

$$T = \left[\begin{array}{ccc|c} \tau_{11} & \cdots & \tau_{1p} & \\ \vdots & & \vdots & * \\ \tau_{p1} & \cdots & \tau_{pp} & \\ \hline 0 & \cdots & 0 & \\ \vdots & & \vdots & * \\ 0 & \cdots & 0 & \end{array}\right] \in \mathrm{Mat}_{u \times u}(k).$$

By Proposition 3.2 the matrix T is invertible. It follows from Gauss–Jordan algorithm that T^{-1} is of the form $\left[\begin{array}{c|c} A & * \\ \hline \mathbf{0} & C \end{array}\right]$, where $A \in \mathrm{Mat}_{p\times p}(k)$ and $C \in \mathrm{Mat}_{(u-p)\times(u-p)}(k)$. Let $\partial\mathcal{O} \setminus \{b_1, \ldots, b_r\} = \{b_{r+1}, \ldots, b_w\}$. Analogously to step (T3) we obtain in step (T5) that $B = \left[\begin{array}{c|c} A' & * \\ \hline \mathbf{0} & C' \end{array}\right]$, where $A' \in \mathrm{Mat}_{p\times r}(k)$ and $C' \in \mathrm{Mat}_{(u-p)\times(w-r)}(k)$. Therefore $T^{-1}B = (\alpha_{ij})_{1\le i\le u, 1\le j\le w}$, where $\alpha_{ij} = 0$ for $p+1 \le i \le u$, $1 \le j \le r$. By (T6), we have $g_j = b_j - \sum_{i=1}^{u} \alpha_{ij} t_i$ for $1 \le j \le w$. Hence $g_j = b_j - \sum_{i=1}^{p} \alpha_{ij} t_i$ for $1 \le j \le r$. Thus $g_j \in k[Y]$ for $1 \le j \le r$.

The set $\{g_1, \ldots, g_r\}$ is a border basis as a consequence of Proposition 3.1. Namely, $S(g_i, g_j) \in k[Y]$ for $1 \le i, j \le r$. Moreover, if g_i, g_j are neighbors, then $S(g_i, g_j) = c_1 g_1 + \cdots + c_w g_w$ for $c_1, \ldots, c_w \in k$. Suppose that there exists $s \ge r$ such that $c_s \ne 0$. Then at least one monomial $c_{s_0} b_{s_0}$ for $s_0 \ge r$ cannot be reduced, which is a contradiction. Hence $S(g_i, g_j) = c_1 g_1 + \cdots + c_r g_r$. □

Theorem 4.1 provides a simple means for computing the border basis of the restriction of an ideal to a polynomial ring in a smaller number of variables. This is also the starting point for subsequent constructions.

An ideal I in $k[X]$ is zero-dimensional if and only if $I \cap k[x_i] \ne (0)$ for every $1 \le i \le n$ (3.7.1 of [6]). Let A, B be zero-dimensional ideals. Then $(A \cap B) \cap k[x_i] = (A \cap k[x_i]) \cap (B \cap [x_i])$. Since $k[x_i]$ is a principal ideal domain, then $A \cap k[x_i] = (f)$ and $B \cap [x_i] = (g)$ for nonzero polynomials f, g. Then $fg \in (A \cap k[x_i]) \cap (B \cap [x_i])$. Hence, the intersection of zero-dimensional ideals is also zero-dimensional. The following elementary proposition is presented in [2].

Proposition 4.1.

Let A, B be ideals (not necessarily zero-dimensional) in $k[X]$. Let $I \lhd k[t, X] = k[t, x_1, \ldots, x_n]$ be an ideal, such that, $I = (tA, (t-1)B)$. Then $A \cap B = I \cap k[X]$.

Therefore, to determine the border basis of the intersection of two zero-dimensional ideals, it suffices to compute the border basis of I and then apply Theorem 4.1. We only need I to be zero-dimensional. The latter is equivalent to ideals A and B having no common zero because $V(I) = \{0\} \times V(B) \cup \{1\} \times V(A) \cup (k \setminus \{0, 1\}) \times (V(A) \cap V(B))$. This example is a generic case, hence, algorithms of [4] work here in a generic case.

Note also that in [1] a generalization of the notion of border basis is introduced for positive dimensional ideals together with a corresponding algorithm. However the latter algorithm does not compute every border basis but only a border basis with respect to the specified ordering. For some applications, this is sufficient. Further development of the border basis theory for positive dimensional ideals may make the condition on the dimension not substantial.

Proposition 4.2 ([2], 3.2).
Let $k[Y] = k[y_1, \dots, y_p]$, $k[X] = k[x_1, \dots, x_n]$. Let $\varphi : k[Y] \to k[X]$ be a k-algebra homomorphism. If $B = (y_1 - \varphi(y_1), \dots, y_p - \varphi(y_p)) \subseteq k[X, Y]$, then $\ker \varphi = B \cap k[Y]$. □

Proposition 4.2 and Theorem 4.1 provide an algorithm for determining a border basis of the kernel of a homomorphism of polynomial rings (under usual assumption of dimension zero or applying results for positive dimensions where possible). It gives a border basis of the ideal of relations $A_{(\alpha)}$ in the case where $\varphi_1, \dots, \varphi_p$ are polynomials, because $R \subseteq k[X]$. As a consequence, we obtain information on rational constants of the associated factorizable derivation (see Section 2). We can give an algorithm also in the case where $\varphi_1, \dots, \varphi_p$ are not polynomials, but it is quite laborious and it uses Gröbner bases in one of the steps. In this case we don't know an algorithm based solely on border bases. We remark that the proof of Theorem 4.1 employs Gröbner bases but the theorem itself does not. Hence, the algorithms derived from Propositions 4.1 and 4.2 are based purely on border bases.

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