Blow-up and global existence profile of a class of fully nonlinear degenerate parabolic equations

Jing Li¹, Jingxue Yin³, Chunhua Jin²,³*

1 College of Science, Minzu University of China, Beijing, 100081, China
2 School of Mathematical Sciences, Dalian University of Technology, Dalian 116024, China
3 School of Mathematical Sciences, South China Normal University, Guangzhou, 510631, China

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Abstract: This paper is mainly concerned with the blow-up and global existence profile for the Cauchy problem of a class of fully nonlinear degenerate parabolic equations with reaction sources.

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1. Introduction

This paper is concerned with the blow-up and global existence profile for the following Cauchy problem

\[ \frac{\partial u}{\partial t} = u^q |\Delta u|^{m-1} \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0, \quad (1) \]

\[ u(x, 0) = u_0(x), \quad x \in \mathbb{R}^N, \quad (2) \]

where \( q \geq 0, \ m \geq 1, \ p > 0, \) and \( u_0 \) is nonnegative and appropriately smooth.

The equation (1) is a typical fully nonlinear parabolic equation, which appears in modeling many phenomena in physics, chemistry and other natural sciences. If \( q = 0, \) it is just the dual porous medium equation with source which appears

* E-mail: jinchhua@126.com
in some problems in elasticity with damping [4] as well as in problems of Bellman–Dirichlet type [13]. However, if \( m = 1 \) then it reduces to a class of typical parabolic equations in non-divergence form which have been proposed as mathematical models of physical problems in many fields such as the resistive diffusion of a force-free magnetic field [14, 15], dynamics of biological groups [1], curve shortening flow [2, 5], spread of infectious disease [17] and so on.

The study of critical exponents began in 1966 in the famous work [6] of Fujita, where he considered the Cauchy problem of the semilinear equation

\[
\frac{\partial u}{\partial t} = \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0.
\]

It was shown that this problem does not have any nontrivial, nonnegative global solution if \( 1 < p < p_c = 1 + 2/N \). Whereas if \( p > p_c \), there exist both global (with small initial data) and non-global (with large initial data) solutions. People call \( p_c \) the critical Fujita exponent and such results the blow-up theorems of Fujita type.

Among the numerous answers to challenging questions on critical exponents in different situations studied since these pioneering works (see the survey papers [3, 9, 11]), the following degenerate parabolic equation

\[
\frac{\partial u}{\partial t} = u^q \Delta u + u^p, \quad x \in \mathbb{R}^N, \quad t > 0,
\]

is considered. For the case \( q \in (0, 1) \), the equation (3) is called the forced porous medium equation and one of the results derived in [7, 8] reads as follows: If \( 1 < p < p_c = q + 1 + 2(1 - q)/N \), there are no global (positive) solutions, while if \( p > p_c \), there are both global solutions and solutions blowing up in finite time. Furthermore, as the complementarity of the results in [7, 8], Winkler [19] explores the Cauchy problem of (3) with \( q \in [1, \infty) \) and the main results are formulated as follows: If \( 1 \leq p < q + 1 \) (resp. \( 1 \leq p < 3/2 \) for \( q = 1 \)), all positive solutions are global but unbounded, provided that \( u_0 \) decreases sufficiently fast in space. If \( p = q + 1 \), all positive solutions blow up in finite time, while if \( p > q + 1 \), there are both global and non-global positive solutions, depending on the size of \( u_0 \).

For the fully nonlinear parabolic equation, the only result we have known is obtained in [10], where Galaktionov investigated the blow-up and critical exponents for the dual porous medium equation

\[
\frac{\partial u}{\partial t} = |\Delta u|^{m-1} \Delta u + u^p, \quad (x, t) \in \mathbb{R}^N \times \mathbb{R}_+, \quad m > 1, \quad p > m,
\]

with given nonnegative bounded initial data \( u_0(x) \). The non-divergence form of the diffusion operator makes problematic applying many effective methods from the classical theory of weak solutions. Moreover, the pure dPME does not admit explicit similarity solutions of the first kind (i.e., obtained by the dimensional analysis). It was shown that for the fully nonlinear equation (4), a simple explicit expression for the critical exponent \( p_c \) does not exist. It turns out to be of a “transcendental” algebraic nature, i.e., cannot be obtained from simple algebraic dimensional equations of parameters. It was Galaktionov who proposed and developed a new approach, which makes it possible to calculate critical Fujita exponents for fully nonlinear equation. In fact, he gave an approximation of the Fujita exponent:

\[
p^* = m \left( 1 + \frac{2(\lambda + 1 + \mu)}{N - 2\mu} \right),
\]

which essentially varies according to the functional setting associated with nonlinear integral multipliers involved.

In this paper, we consider the Cauchy problem (1)–(2) for all \( q \geq 0, m \geq 1 \) and \( p > 0 \). It should be noticed that, due to the possible vanishing of \( u^q \) for the case \( q > 0 \), completely different from the case \( q = 0 \) (the dual porous medium equation), the equation (1) could not have classical solutions in general. So, we consider solutions in some weak sense. In fact, for the case \( p \geq 1 \), the local existence of a class of generalized solutions has been obtained in our previous work [12]. However, for the case \( 0 < p < 1 \), the corresponding result cannot be obtained due to the different nonlinear degree of source term. Instead, we consider the local existence of solutions for \( u_0 \) with positive lower bound. Moreover, due to the non-divergence form of the diffusion operator for the case \( q > 0 \), the solutions of the problem (1)–(2) might not be unique even for the special case \( m = 1 \). In this paper, we just consider limit solutions, that is the generalized solutions obtained as the unique limit of a sequence of classical solutions to the related regularized problem, for which
the comparison principle holds (see [20]). Based on the local existence and comparison principle of limit solutions, we investigate their long-term behavior. It was shown that there exist two critical exponents of $p$: the global existence exponent $p_0$ and the critical blow-up exponent $p_c$. Firstly, we introduce nonlinear local capacity, which makes the method proposed in [10] available for our generalized solutions and then investigate the blow-up profile for $p > q + m$. We calculate the critical exponent

$$p_c = \begin{cases} 
q + m \left(1 + \frac{4}{N-2}\right) & \text{if } N > 2, \\
+\infty & \text{if } N \leq 2.
\end{cases}$$

Furthermore, by introducing upper and lower solutions, we investigate the blow-up and global existence profile for the case $0 < p \leq q + m$ and the global existence profile for $p > p_c$. What is more interesting, we find that the exponent $q$ of the diffusion coefficient exerts great influence on the global existence exponent of $p$. In fact, different from the former research on the critical exponents of $p$, we find there exists a critical exponent for $q$:

$$q_0 = m,$$

such that

$$p_0 = \begin{cases} 
1 & \text{if } 0 \leq q \leq q_0, \\
q + m & \text{if } q > q_0.
\end{cases}$$

Roughly speaking, what we will obtain can be seen from the following picture:

This paper is arranged as follows. In Section 2, as a preliminary, we study the local existence and comparison principle of solutions. Moreover, for the case $p > q + m$, we give some important estimates for the fully nonlinear operators by introducing nonlinear local capacity. In Section 3, the blow-up theorems and the global existence theorems are established, where we consider the cases $p > q + m$ and $0 < p \leq q + m$ by optimal upper asymptotic estimates of capacity and the upper and lower solutions method respectively.

2. Preliminary

As mentioned in the introduction, the local existence of generalized solutions to (1)–(2) has been investigated in our previous work [12]. In this section, for convenience, we first present the local existence theorems without proofs and establish the related uniqueness and comparison principle. We also give some estimates for the fully nonlinear operators for $p > q + m$, which are important for establishing blow-up theorems.
Assume that \( \frac{\partial u}{\partial t} \in L^{1+\alpha}_{\text{loc}}(\mathbb{R}^N \times (0, T)), \Delta u \in L^{m+1}_{\text{loc}}(C_0) \) and 
\[
\int_{C_0} u^q |\Delta u|^{m+1} \, dx \, dt < +\infty,
\]
where \( C_0 = \{(x, t) \in \mathbb{R}^N \times (0, T) : u(x, t) > 0\} \).

For any \( 0 \leq \varphi \in C_0^\infty(\mathbb{R}^N \times [0, T]) \), the following integral equality holds:
\[
\int_{\mathbb{R}^N \times (0, T)} \frac{\partial u}{\partial t} \varphi \, dx \, dt \geq \int_{C_0} u^q |\Delta u|^{m-1} \Delta u \varphi \, dx \, dt + \int_{\mathbb{R}^N \times (0, T)} u^q \varphi \, dx \, dt,
\]
and respectively
\[
\int_{\mathbb{R}^N \times (0, T)} \frac{\partial u}{\partial t} \varphi \, dx \, dt \leq \int_{C_0} u^q |\Delta u|^{m-1} \Delta u \varphi \, dx \, dt + \int_{\mathbb{R}^N \times (0, T)} u^q \varphi \, dx \, dt.
\]

Furthermore, if \( u \) is a generalized upper solution as well as a generalized lower solution, we call it a generalized solution.

In our previous work [12] we have obtained the following theorem.

**Theorem 2.2.**
Assume that \( 0 \leq u_0(x) \in C_0(\mathbb{R}^N) \). If \( p \geq 1 \), then there exists \( T > 0 \) such that the Cauchy problem (1)--(2) admits a generalized solution \( u \in C([0, T); C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \). In particular, the generalized solution is global for \( p = 1 \).

For the case \( 0 < p < 1 \), using the same method we can obtain the following local existence theorem for \( \varepsilon \leq u_0 \in C_0(\mathbb{R}^N) \), where by \( f \in C_0(\mathbb{R}^N) \) we mean that \( f - \varepsilon \in C_0(\mathbb{R}^N) \) with \( \varepsilon > 0 \).

**Theorem 2.3.**
Assume that \( \varepsilon \leq u_0(x) \in C_0(\mathbb{R}^N) \). If \( 0 < p < 1 \), then there exists \( T > 0 \) such that the Cauchy problem (1)--(2) admits a generalized solution \( \varepsilon \leq u \in C([0, T); C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \).

It should be noticed that, due to the non-divergence form of the diffusion operator for the case \( q > 0 \), the solutions of the problem (1)--(2) need not be unique even for the special case \( m = 1 \). Fortunately, in our previous work [20], we have proved that the existing generalized solutions (limit solutions) are obtained as limits of monotonous sequences of classical solutions to the related regularized problems, for which the comparison principle holds. Thus we can conclude that the limit solution of (1)--(2) is unique and the related comparison principle holds. That is,

**Theorem 2.4.**
Assume \( p \geq 1 \). For \( 0 \leq u_0(x) \in C_0(\mathbb{R}^N) \), the limit solution of (1)--(2) is unique. Furthermore, let \( 0 \leq u_1, u_2 \in C([0, T); C(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)) \) be two limit solutions of the Cauchy problem (1)--(2) with
\[
0 \leq u_1(x, 0) \leq u_2(x, 0).
\]

Then
\[
0 \leq u_1(x, t) \leq u_2(x, t), \quad (x, t) \in \mathbb{R}^N \times [0, T).
\]
**Theorem 2.5.**
Assume $0 < p < 1$. For $\varepsilon \leq u_0(x) \in C_c(\mathbb{R}^N)$, the limit solution of (1)–(2) is unique. Furthermore, let $\varepsilon \leq u_1, u_2 \in C([0, T), C(\mathbb{R}^N) \cap L^{2m}(\mathbb{R}^N))$ be two limit solutions of the Cauchy problem (1)–(2) with

$$\varepsilon \leq u_1(x, 0) \leq u_2(x, 0).$$

Then

$$\varepsilon \leq u_1(x, t) \leq u_2(x, t), \quad (x, t) \in \mathbb{R}^N \times [0, T).$$

In what follows, along the lines of [10], we consider the case $p > q + m$ and give some estimates for fully nonlinear operators using purely algebraic techniques. The main idea of our analysis is based on the concept of nonlinear local capacity. The development of this approach for essentially nonlinear partial differential equations uses special multipliers with parameters, which enable us to accomplish the necessary manipulations with the equation under consideration.

Now we choose

$$M_1(u) = |\Delta u| u^\varphi$$

as the first multiplier of (1), where $\lambda > 0$, $\varphi \in C^{2,1}_0(G_0)$. Integrating over $\mathbb{R}^N \times (0, T)$, we have

$$\int_{\mathbb{R}^N} \frac{\partial}{\partial t} |\Delta u|^m u \varphi \, dx = \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^m u \varphi \, dx + \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u| \varphi \, dx dt.$$

After integration by parts, we obtain

$$2 \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^m u \varphi \, dx dt + \left(1 + \frac{p}{1 + \lambda}\right) \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u| \varphi \, dx dt$$

$$+ \frac{p(p - 1)}{\lambda + 1} \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^{m-1} \Delta u \varphi \, dx dt + \lambda \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^{m-1} \Delta u \varphi \, dx dt$$

$$= -2 \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^m \nabla u \cdot \nabla \varphi \, dx$$

$$- \frac{1}{\lambda + 1} \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^{m-1} \Delta u \nabla \varphi \, dx - \frac{1}{\lambda + 1} \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u| \varphi \, dx |_{t=0}.$$

Now we take

$$M_2(u) = \Delta u u^\varphi$$

as the second multiplier of (1), where $\lambda > 0$, $\varphi \in C^{2,1}_0(G_0)$. In a similar way we obtain the following identity:

$$2 \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^{m+1} \varphi \, dx dt + \left(1 + \frac{p}{1 + \lambda}\right) \int_{\mathbb{R}^N} u^{\varphi + 1} \Delta u \varphi \, dx dt$$

$$+ \frac{p(p - 1)}{\lambda + 1} \int_{\mathbb{R}^N} u^{\varphi + 1} \Delta u |\nabla u|^2 \varphi \, dx dt + \lambda \int_{\mathbb{R}^N} u^{\varphi + 1} \Delta u |\nabla u|^2 \varphi \, dx dt$$

$$= -2 \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^{m+1} \nabla u \cdot \nabla \varphi \, dx$$

$$- \frac{1}{\lambda + 1} \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u|^{m+1} \Delta u \nabla \varphi \, dx - \frac{1}{\lambda + 1} \int_{\mathbb{R}^N} u^{\varphi + 1} |\Delta u| \varphi \, dx |_{t=0}.$$

Adding the above two equations we obtain the main identity

$$\int_{\mathbb{R}^N} A_1 u^{\varphi + 1} |\Delta u|^m \varphi \, dx dt + \int_{\mathbb{R}^N} A_2 u^{\varphi + 1} \Delta u \varphi \, dx dt$$

$$+ \int_{\mathbb{R}^N} A_3 u^{\varphi + 1} |\nabla u|^2 \varphi \, dx dt + \int_{\mathbb{R}^N} A_4 u^{\varphi + 1} \nabla u |\nabla u|^2 \varphi \, dx dt$$

$$= \int_{\mathbb{R}^N} B_1 u^{\varphi + 1} |\Delta u|^{m+1} \nabla u \cdot \nabla \varphi \, dx dt + \int_{\mathbb{R}^N} B_2 u^{\varphi + 1} |\Delta u|^{m+1} \Delta u \nabla \varphi \, dx dt$$

$$+ \int_{\mathbb{R}^N} B_3 u^{\varphi + 1} \Delta u \varphi \, dx dt + \int_{\mathbb{R}^N} B_4 u^{\varphi + 1} \varphi \, dx |_{t=0}. \quad (5)$$
where

\[ A_k = A_k^0 (|\Delta u| + \Delta u), \quad B_k = B_k^0 (|\Delta u| + \Delta u), \]

and \( A_k^0 \) and \( B_k^0 \) denote the following constants depending on \( \lambda \):

\[
\begin{align*}
A_0^0 &= 2, \quad A_1^0 = 1 + \frac{p}{\lambda + 1}, \quad A_2^0 = \frac{p(p - 1)}{\lambda + 1}, \quad A_3^0 = \lambda, \\
B_0^0 &= 1 - \frac{1}{\lambda + 1}, \quad B_1^0 = -\frac{1}{\lambda + 1}, \quad B_2^0 = -\frac{1}{\lambda + 1}.
\end{align*}
\]

Noticing \( p > q + m, \lambda > 0 \), it is easy to see that all the leading coefficients \( A_k^0 \) are strictly positive.

The following two lemmas are algebraic estimates, which will be used in the functional estimates.

**Lemma 2.6.**
Let \( m \geq 1, p > q + m, \) and \( \lambda > 0 \). Then there exist

\[ s = 1 + \frac{p - q - m}{m(\lambda + p + 1)} \]

and a constant \( C_1 > 0 \) such that

\[
(u^{\lambda+p+1} |\Delta u|^{m-1} (|\Delta u| + \Delta u))^s \leq C_1 \left( u^{\lambda+q} |\Delta u|^{m} + u^{\lambda+p} \right) (|\Delta u| + \Delta u).
\]

**Proof.** This purely algebraic inequality is obviously true if \( \Delta u \leq 0 \). Assume now that \( \Delta u > 0 \) but for convenience we keep using \( |\Delta u| \). Denote \( t = |\Delta u| u^{\lambda+p+1} \) and hence \( |\Delta u| = t^{1/m} u^{-(\lambda+q+1)/m} \). Then we have

\[ t^s \leq 2^{1-s} C_1 \left( t^{(m+1)/m} u^{-\lambda-q-1} t^{m-1} + t^{1/m} u^{-(\lambda+q+1)/m+\lambda+p} \right). \]

It is easy to see that, under the above hypothesis, the right-hand side attains its minimum at \( u_* = C_2 t^{\theta} \), where

\[ \theta = \frac{1}{\lambda + p + 1}. \]

Substituting \( u_* \) into the previous inequality completes the proof.

**Lemma 2.7.**
Let \( m \geq 1, p > q + m, \) and \( \lambda > 0 \). Then there exist

\[ r = 1 + \frac{m(p - 1)}{p - q + m(\lambda + 1)} \]

and a constant \( C_2 > 0 \) such that

\[
(u^{\lambda+q} |\Delta u|^{m} + u^{\lambda+p} (|\Delta u| + \Delta u))^r \leq C_2 \left( u^{\lambda+q} |\Delta u|^{m} + u^{\lambda+p} \right) (|\Delta u| + \Delta u).
\]

**Proof.** Similarly, we deal with \( \Delta u > 0 \) and set \( t = |\Delta u| u^{\lambda+q+1} \), hence \( |\Delta u| = t u^{-\lambda-1} \). Then we obtain

\[ t^r \leq 2^{1-r} C_2 \left( t^{m+1} u^{-(\lambda+q+1)} + t u^{\lambda-1} \right). \]

The minimum of the right-hand side relative to \( u \geq 0 \) is attained at \( u_* = C_2 t^{\theta_1} \) with

\[ \theta_1 = \frac{m}{p - q + m(\lambda + 1)}, \]

and upon substitution we complete the proof.
Next we will estimate the terms on the right-hand side of (5) by means of those on the left-hand side on the basis of standard Young’s and Hölder’s inequalities. In what follows, $A_0^{\alpha}$’s are assumed to be positive. Using Young’s inequality, we derive

$$B_1 |u^{k+q}| |\Delta u|^{m-1} \nabla u \nabla \varphi| \leq \lambda u^{k+q-1} |\Delta u|^{m-1} (|\Delta u| + |\Delta u|) |\nabla u|^2 \varphi + \frac{1}{\lambda} u^{k+q+1} |\Delta u|^{m-1} (|\Delta u| + |\Delta u|) \frac{|\nabla \varphi|^2}{\varphi}.$$  

Substituting this estimate into (5), using again Hölder’s and Young’s inequalities with positive parameters $\delta$, $\rho$, by Lemma 2.6, Lemma 2.7, we can conclude that

$$\iint_{R^N \times (0,T)} (A_1 u^{k+q} |\Delta u|^m + A_2 u^{k+q}) \varphi \, dx \, dt \leq \iint_{R^N \times (0,T)} u^{k+q+1} |\Delta u|^{m-1} (|\Delta u| + |\Delta u|) \left( \frac{|\nabla \varphi|^2}{\lambda \varphi} + \frac{|\Delta \varphi|}{\lambda + 1} \right) \, dx \, dt$$

$$+ \frac{1}{\lambda + 1} \iint_{R^N \times (0,T)} \frac{\partial \varphi}{\partial t} \, dx \, dt + \frac{1}{\lambda + 1} \iint_{R^N \times (0,T)} \frac{\partial u^{k+1}}{\partial t} \varphi \, dx \, dt$$

$$\leq \frac{\delta}{\delta} \iint_{R^N \times (0,T)} (u^{k+q+1} |\Delta u|^{m-1} (|\Delta u| + |\Delta u|) |\nabla \varphi|^2 \varphi \, dx \, dt + \frac{1}{s} \delta \iint_{R^N \times (0,T)} \lambda \frac{|\nabla \varphi|^2}{\varphi} + \frac{|\Delta \varphi|}{\lambda + 1} \varphi^{1-s} \, dx \, dt$$

$$+ \frac{\rho}{r} \iint_{R^N \times (0,T)} (u^{k+1} (|\Delta u| + |\Delta u|) |\nabla \varphi|^2 \varphi \, dx \, dt + \frac{\rho}{s} \iint_{R^N \times (0,T)} \frac{\partial \varphi}{\partial t} |\nabla \varphi|^{1-r} \varphi^{1-s} \, dx \, dt$$

$$+ \frac{1}{\lambda + 1} \iint_{R^N \times (0,T)} \frac{\partial \varphi}{\partial t} |\nabla \varphi|^{1-r} \varphi^{1-s} \, dx \, dt + \frac{1}{\lambda + 1} \iint_{R^N \times (0,T)} \frac{\partial u^{k+1}}{\partial t} \varphi \, dx \, dt \bigg|_{t=0},$$

where $s' = s/(s-1)$, $r' = r/(r-1)$. Finally, choosing $\delta$ and $\rho$ sufficiently small, we obtain the main inequality characterizing evolution properties of the solution of our problem:

$$\iint_{R^N \times (0,T)} (u^{k+q} |\Delta u|^m + u^{k+q}) (|\Delta u| + |\Delta u|) \varphi \, dx \, dt$$

$$\leq C_1 \iint_{R^N \times (0,T)} \frac{|\nabla \varphi|^2}{\lambda \varphi} + \frac{|\Delta \varphi|}{\lambda + 1} \varphi^{1-s} \, dx \, dt + C_2 \iint_{R^N \times (0,T)} \frac{\partial \varphi}{\partial t} |\nabla \varphi|^{1-r} \varphi^{1-s} \, dx \, dt$$

$$- C_1 \iint_{R^N} u^{k+1} (|\Delta u| + |\Delta u|) \varphi \, dx \bigg|_{t=0},$$

where $C_i^+, i = 1, 2, 3$, are positive constants. Consider the functional associated with the main term on the right-hand side of (6).

$$\Phi(u) = \iint_{R^N \times (0,T)} \frac{|\nabla \varphi|^2}{\lambda \varphi} + \frac{|\Delta \varphi|}{\lambda + 1} \varphi^{1-s} \, dx \, dt + \iint_{R^N \times (0,T)} \frac{\partial \varphi}{\partial t} |\nabla \varphi|^{1-r} \varphi^{1-s} \, dx \, dt$$

for nonnegative functions $\varphi \in C^0_{\text{loc}}(G_0)$ such that $\Phi(\varphi)$ is finite. We denote this functional class by $W_0(G_0)$. Evidently, it is not empty. Next, we consider the nonlinear capacity generated by the nonlinear operator of the equation (1). For any $\varepsilon > 0$, we consider the domain

$$\Gamma_{RT}^\varepsilon = \{(x, t) : |x| \leq R, 0 \leq t \leq T |u(x, t) \geq \varepsilon, \Delta u(x, t) > 0\},$$
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where \( R, T > 0 \), and the class of \( \varphi \in W_0(\mathbb{C}_0) \) satisfying \( 0 \leq \varphi \leq 1 \) and \( \varphi \equiv 1 \) in \( \Pi_{R,T}^\varphi \) to be denoted by \( W_0(\Pi_{R,T}^\varphi) \). The local capacity is defined as follows:

\[
\text{Cap}(\Phi, \Pi_{R,T}^\varphi) = \inf \{ \Phi(\varphi) : \varphi \in W_0(\Pi_{R,T}^\varphi) \}. 
\]

(7)

It follows from (6) that

\[
\int\int_{\Pi_{R,T}^\varphi} (u^{s+q}|\Delta u|^m + u^{s+p}|\Delta u|) \, dx \, dt \leq C_5 \text{Cap}(\Phi, \Pi_{R,T}^\varphi) - C_5 \int_{\Pi_{R,T}^\varphi} u^{s+1}|\Delta u| \, dx \bigg|_{t=0},
\]

(8)

where \( \Pi_{R_T}^\varphi = \{ |x| \leq R : u_0(x) > \varepsilon, \Delta u_0(x) > 0 \} \) and \( C_5 \) is a positive constant.

Now we are interested in an optimal upper estimate of the local capacity as \( R, T \to \infty \). To do this, we perform scaling

\[
t = T \tau, \quad x = R \xi
\]

and next use the test function of the related self-similar form

\[
\varphi(x,t) = \varphi_0(\xi, \tau),
\]

such that \( 0 \leq \varphi_0 \leq 1 \), \( \varphi_0 \equiv 1 \) in \( \{ (\xi, \tau) : |\xi| \leq 10 \leq \tau \leq 1 u(R\xi, T\tau) \geq \varepsilon, \Delta u(R\xi, T\tau) > 0 \} \) and \( \Phi(\varphi_0) \) is finite. Then the local capacity defined in (7) is estimated as follows:

\[
\text{Cap}(\Phi, \Pi_{R,T}^\varphi) \leq C_6 \left( TR^{N-2s'} + R^N T^{1-r'} \right).
\]

To obtain an optimal estimate, we introduce a relation between parameters \( R \) and \( T \) as follows:

\[
T = R^\theta \to \infty \quad \text{as} \quad R \to \infty,
\]

where \( \theta > 0 \) is a free parameter. It is easy to check that the optimal \( \theta \) is attained when the two terms are balanced in the sense that

\[
N - 2s' + \theta = N + (1 - r')\theta \quad \implies \quad \theta = \frac{2s'}{r'} = \frac{2m(p-1)}{p-q-m} > 0.
\]

(9)

3. Critical exponents

Now we are in a position to discuss the blow-up and global existence profile for solutions of the problem (1)–(2). In this section, we just consider the limit solutions for which a priori bounds in \( L^\infty \) are sufficient to guarantee the existence of solutions. For blow-up and global existence, we mean

**Definition 3.1.**

The limit solution \( u \) of the Cauchy problem (1)–(2) is said to **blow up in some finite time** \( T \) with \( 0 < T < \infty \) if

\[
\lim_{t \to T^-} \|u(\cdot, t)\|_{L^\infty(\mathbb{R}^N)} = +\infty.
\]

If \( u \) does not blow up at any finite time, we say \( u \) is **global**.
Remark 3.2.
We should remark that a priori bounds for solutions of (1)–(2) in $L^\infty(\mathbb{R}^N \times (0, \infty))$ are sufficient to guarantee the existence of solutions, which justifies Definition 3.1. In fact, under the assumption that $u$ is bounded in $L^\infty$, denoting $|\Delta u|^p = \min\{ |\Delta u|, \kappa \}$ with $\kappa$ a positive constant, the Dirichlet problem of

$$\frac{\partial u}{\partial t} = (u + \epsilon)^q |\Delta u|^p + \sigma u^p$$

is uniformly parabolic and the global existence of classical solutions follows [18]. Furthermore, from the proofs in [12], it is easy to verify that a priori bounds for solutions in $L^\infty$ are sufficient to guarantee all necessary estimates on generalized solutions that are uniformly with $k$, $\sigma$, and $\epsilon$. Then the global existence of generalized solutions for (1)–(2) follows immediately by letting $k \to \infty$, $\sigma \to 0$ and then $\epsilon \to 0$.

For the case $p > q + m$, we consider the blow-up and global existence profile of solutions by optimal upper asymptotic estimates of local capacity and upper and lower solution methods, respectively, and show that $p_c$ is the critical blow-up exponent.

Theorem 3.3.
Assume that $\Delta u_0 > 0$ in a subset of nonzero measure in $\mathbb{R}^N$. Then any nontrivial solution of the problem (1)–(2) blows up in finite time in the parameter range $q + m < p \leq p_c$, where

$$p_c = \begin{cases} q + m \left( 1 + \frac{4}{N - 2} \right) & \text{if } N > 2, \\ +\infty & \text{if } N \leq 2. \end{cases}$$

Proof. For the choice of (9), we have

$$\text{Cap}(\Phi, \Pi^\epsilon_{R}) \leq C_I R^{\gamma^*},$$

where $\gamma^* = N - 2(m + p - q + m\lambda)/(p - q - m)$. For the nonexistence conclusion, we need the capacity to vanish as $R \to \infty$. One can see this happens if $\gamma^* < 0$, which is equivalent to

$$N < \frac{m + p - q + m\lambda}{p - q - m},$$

and this yields

$$q + m < p < p^* = \begin{cases} q + m \left( 1 + \frac{2(\lambda + 2)}{N - 2} \right) & \text{if } N > 2, \\ +\infty & \text{if } N \leq 2. \end{cases}$$

Assume that $q + m < p < p^*$, so that $\gamma^* < 0$. Then passing to the limit as $R \to \infty$ in (8) and using (10), we obtain

$$\int_{\mathbb{R}^N} \left( u^{q+1}|\Delta u|^{p+1} + u^{q+p}|\Delta u|^{p} \right) \, dx \, dt \leq -C_3^{*} \int_{\mathbb{R}^N} u^{q+1}|\Delta u| \, dx \bigg|_{t=0},$$

where $\Pi^\epsilon = \{(x, t) \in \mathbb{R}^N \times \mathbb{R}_+: u(x, t) \geq \epsilon, \Delta u(x, t) > 0 \}$, $\Pi^\epsilon = \{x \in \mathbb{R}^N : u_0(x) > \epsilon, \Delta u_0(x) > 0 \}$. Furthermore, letting $\lambda \to 0$ yields $p^* \to p_c$ and

$$\int_{\mathbb{R}^N} \left( u^{q}|\Delta u|^{p+1} + u^{p}|\Delta u|^{p} \right) \, dx \, dt \leq -C_3 \int_{\mathbb{R}^N} u|\Delta u| \, dx \bigg|_{t=0},$$

If $\Delta u_0 > 0$ in a subset of nonzero measure in $\mathbb{R}^N$, the right-hand side is strictly negative. By the arbitrariness of $\epsilon$, we conclude that if $q + m < p < p_c$, such a nontrivial global solution $u$ does not exist. In the critical case $\gamma^* = 0$, an extra effort and integration over expanding annuli is necessary to guarantee global nonexistence [16]. We have thus proved the theorem. \qed
Remark 3.4.
Notice that our results are mainly based on the analysis and manipulations with identities and estimates depending on $\Delta u$ in a nonlinear manner instead of the classical ones for $u$. This fact reflects that the critical exponent is precisely $p_c$.

Indeed, one can check that, applying a similar approach to the forced porous medium equation (3) with $q \in (0, 1)$, we obtain $p_c = q + 1 + 4/(N-2)$, which is greater than the correct Fujita one $p_f = q + 1 + 2(1-q)/N$. This means that all the solutions belonging to the necessary weighted spaces blow up in finite time for $1 < p \leq p_c$. Therefore, small global solutions, which are known to exist for $p$ above the Fujita exponent $p_f$, do not belong to these spaces.

Now we consider the case $p > p_c$.

Theorem 3.5.
Denote $\gamma^* = N-2(m+p-q)/(p-q-m)$. Assume that $C_3^* \int_{\mathbb{R}^N} |u_0|^\beta |\Delta u_0| \, dx \geq CR^\beta$ with constant $C > 0$ and a fixed exponent $\beta > \gamma^*$. Then any nontrivial solution of the problem (1)–(2) blows up in finite time in the parameter range $p > p_c$.

Proof. Notice that for $p > p_c$ we have $\gamma^* > 0$. Under the assumption that

$$C_3^* \int_{\mathbb{R}^N} |u_0|^\beta |\Delta u_0| \, dx \geq CR^\beta$$

with $\beta > \gamma^*$, we obtain

$$\iint_{\mathbb{R}^N} \left( |u|^\beta |\Delta u|^{n+1} + |u|^\beta |\Delta u| \right) \, dx \, dt \leq C_{R^\gamma} - CR^\beta.$$ 

Then passing to the limit as $R \to \infty$, we get

$$\iint_{\mathbb{R}^N} \left( |u|^\beta |\Delta u|^{n+1} + |u|^\beta |\Delta u| \right) \, dx \, dt < 0.$$ 

By the arbitrariness of $\epsilon$, we conclude that such a global nonnegative solution $u$ does not exist. The theorem is thus proved.

Theorem 3.6.
Let $p > p_c$. Then for $u_0$ sufficiently small, the limit solution of the Cauchy problem (1)–(2) exists globally.

Proof. Consider the classical semilinear elliptic problem

$$\Delta v + v^{(p-q)/m} = 0 \quad \text{in} \quad \mathbb{R}^N. \quad (13)$$

Under the assumption $p > p_c$, that is $(p-q)/m > (N+2)/(N-2)$, there exists at least one strictly positive classical solution $v(x)$ for (13). Then if $u_0 \leq v$, $v$ is an upper solution of the problem (1)–(2). Thus, by comparison, we conclude that the limit solution of (1)–(2) exists globally.

In what follows, we investigate the case $0 < p \leq q + m$ by introducing upper and lower solutions. The following two theorems show that $(1, q + m]$ belongs to the blow-up case if $q \in [0, m]$ and belongs to the global existence case if $q \in (m, \infty)$.

Theorem 3.7.
Let $0 \leq q \leq m$, $1 < p \leq q + m$. Then for any nontrivial $u_0$, the limit solution of the Cauchy problem (1)–(2) blows up in finite time.
Proof. Let

\[ u = B(T - t)^{\alpha} h(\eta), \quad h(\eta) = (1 - C\eta)^{1/\lambda}, \quad \eta = x^2(T - t)^{\beta}, \quad \alpha = \frac{1}{1 - p} < 0, \quad \beta = \frac{q + m - p}{m(p - 1)} \geq 0. \]

Choose

\[ \frac{2m}{q + m} < \lambda < \frac{2m - 1}{q + m - 1}. \]

Then

\[ \frac{\partial u}{\partial t} \leq u\eta|\Delta u|^{m-1}\Delta u + u^p \]

in \( C_0 \) is equivalent to

\[ -B\alpha h(\eta) - B\beta h'(\eta) \leq B^{\alpha+q}(h(\eta))^{\alpha}[4\eta h''(\eta) + 2Nh'(\eta)|^{n-1}(4\eta h''(\eta) + 2h'(\eta)) + B^p(h(\eta))^p \]

for \( 0 \leq C\eta < 1 \), which is ensured by

\[ -\alpha(1 - C\eta)^{1/\lambda} + \beta\lambda C\eta(1 - C\eta)^{1 - 1/\lambda} \leq B^{\alpha+q-1}(1 - C\eta)^{q+1/(\lambda - 2) - n}(2C\lambda)^n \times \]

\[ \times |2C\eta(\lambda - 1) - N(1 - C\eta)_+|^{n-1}|2C\eta(\lambda - 1) - N(1 - C\eta)_+| + B^{p-1}(1 - C\eta)^{p\eta}. \] \( (14) \)

Set \( \tau = (\lambda + N - 1)/(2\lambda + N - 2) \). Then for \( \tau \leq C\eta < 1 \) we have

\[ 2C\eta(\lambda - 1) - N(1 - C\eta)_+ = C\eta(2\lambda + N - 2) - N \geq \lambda - 1 > 0. \]

Then (14) is ensured by

\[ (-\alpha(1 - \tau) + \beta\lambda)(1 - \tau)^{2m-1-\lambda(q+m-1)} \leq B^{\alpha+q-1}(2C\lambda)^n(\lambda - 1)^{\eta}. \] \( (15) \)

where \( 2m - 1 - \lambda(q + m - 1) > 0 \). For \( 0 \leq C\eta < \tau \), (14) is ensured by

\[ -\alpha + \beta\lambda\tau + B^{\alpha+q-1}(2NC\lambda)^n \leq B^{p-1}(1 - \tau)^{p\eta}. \] \( (16) \)

Let \( C = AB^{1-q-m}|^n \). So,

\[ B^{q+m-1}(2C\lambda)^n = (2A\lambda)^n. \]

Then (15) and (16) hold simultaneously if we first choose \( A \) properly large and then choose \( B \) sufficiently large.

Furthermore, for any \( u_0 \), \( u(x, 0) \leq u_0(x) \) is ensured by \( BT^p(1 - Cx^2 T^\beta)^{1/\lambda} \leq u_0 \), which holds when \( T \) is sufficiently large. Thus \( u \) is a generalized lower solution of the problem (1)–(2). Thus, by comparison, we conclude that the limit solution of (1)–(2) blows up in finite time. \( \square \)

**Theorem 3.8.**

Let \( q > m, 1 < p \leq q + m \). Then for any \( u_0 \) with compact support, the limit solution of the Cauchy problem (1)–(2) exists globally.
\textbf{Proof.} Let 
\[ \pi = C(A - x^2)^{\lambda}, \quad \frac{2m}{m + q} < \lambda < \min \left\{ 1, \frac{2m}{m + q - p} \right\}. \]

Then by a simple calculation,
\[ \frac{\partial \pi}{\partial t} \geq \pi^q |\Delta \pi|^{m-1} \Delta \pi + \pi^p \]

in \( C_0 \) is equivalent to
\[ C^{q+m-p}(2\lambda)^n(A - x^2)^{(1-2m+q)\lambda}|NA - (N + 2\lambda - 2)x^2|^m \geq (A - x^2)^{\lambda p} \]

for \( x^2 < A \), which is ensured by
\[ C^{q+m-p}[4\lambda(1 - \lambda)A]^m \geq (A - x^2)^{(p - q - m + 2m)} \]

when \( N + 2\lambda - 2 > 0 \), or
\[ C^{q+m-p}(2N\lambda A)^m \geq (A - x^2)^{(p - q - m + 2m)} \]

when \( N + 2\lambda - 2 \leq 0 \). Noticing that \( \eta = \lambda(p - q - m) + 2m \geq 0 \), it is clear there exists a sufficiently large constant \( C > 0 \) such that the above inequality holds.

Furthermore, for any \( u_0 \) with compact support, we also have \( \pi(x, 0) \geq u_0(x) \) when \( A \) is sufficiently large. Thus \( \pi \) is a generalized upper solution of the problem (1)–(2). Thus, by comparison, we conclude that the limit solution of (1)–(2) exists globally. The theorem is thus proved.

For \( p = 1 \) the global existence of solutions have been obtained in [12]. We now state the global existence of solutions to (1)–(2) for \( 0 < p < 1 \), where we restrict the initial data to be with positive lower bound.

\textbf{Theorem 3.9.} Let \( 0 < p < 1 \). Then for any \( \varepsilon \leq u_0 \in L^1(\mathbb{R}^N) \), the limit solution of the Cauchy problem (1)–(2) exists globally.

\textbf{Proof.} Let \( \pi = Me^t \). If we choose \( M > \|u_0\|_\infty + 1 \), then a simple calculation yields
\[ \frac{\partial \pi}{\partial t} \geq \pi^p |\Delta \pi|^{m-1} \Delta \pi + \pi^p \]

in \( \mathbb{R}^N \times (0, T) \) and \( \pi(x, 0) \geq u_0(x) \). Thus \( \pi \) is a generalized upper solution of the problem (1)–(2) with \( 0 < p < 1 \). Thus, by comparison, we obtain that the limit solution of (1)–(2) exists globally. The theorem is thus proved.

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\textbf{References}


