Robust semi-coarsening multilevel preconditioning of biquadratic FEM systems

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Abstract: While a large amount of papers are dealing with robust multilevel methods and algorithms for linear FEM elliptic systems, the related higher order FEM problems are much less studied. Moreover, we know that the standard hierarchical basis two-level splittings deteriorate for strongly anisotropic problems. A first robust multilevel preconditioner for higher order FEM systems obtained after discretizations of elliptic problems with an anisotropic diffusion tensor is presented in this paper. We study the behavior of the constant in the strengthened CBS inequality for semi-coarsening mesh refinement which is a quality measure for hierarchical two-level splittings of the considered biquadratic FEM stiffness matrices. The presented new theoretical estimates are confirmed by numerically computed CBS constants for a rich set of parameters (coarsening factor and anisotropy ratio). In the paper we consider also the problem of solving efficiently systems with the pivot block matrices arising in the hierarchical basis two-level splittings. Combining the proven uniform estimates with the theory of the Algebraic MultiLevel Iteration (AMLI) methods we obtain an optimal order multilevel algorithm whose total computational cost is proportional to the size of the discrete problem with a proportionality constant independent of the anisotropy ratio.

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1. Introduction

Let us consider the elliptic boundary value problem

\begin{align}
\begin{aligned}
-\nabla \cdot (a(x) \nabla u(x)) &= f(x) & \text{in } & \Omega, \\
 u &= 0 & \text{on } & \Gamma_D, \\
(a(x) \nabla u(x)) \cdot n &= 0 & \text{on } & \Gamma_N.
\end{aligned}
\end{align}

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where \( \Omega \) is a bounded rectangle polygon in \( \mathbb{R}^2 \), \( f(x) \) is a given function in \( L^2(\Omega) \), \( \mathbf{n} \) denotes the outward unit vector normal to the boundary \( \Gamma = \partial \Omega \), \( \Gamma = \Gamma_D \cup \Gamma_N \), and the matrix of the coefficients

\[
\mathbf{a}(x) = \begin{bmatrix}
a_{11} \\
a_{22}
\end{bmatrix}
\]

is assumed to be symmetric positive definite and uniformly bounded in \( \Omega \). We split the domain \( \Omega \) into finitely many rectangle subdomains and denote the introduced partition with \( \mathcal{T}_h \).

After finite element method (FEM) discretization using conforming biquadratic elements defined on \( \mathcal{T}_h \), the elliptic problem (1) can be reduced to finding a solution to the linear system of algebraic equations

\[
\mathbf{A}_h \mathbf{u}_h = \mathbf{F}_h,
\]

where \( \mathbf{A}_h \) stands for the global stiffness matrix, \( \mathbf{F}_h \) represents the global right hand side vector and \( h \) is the mesh parameter for the underlying partition \( \mathcal{T}_h \).

Following the standard FEM assembling procedure we present the global stiffness matrix in the form

\[
\mathbf{A}_h = \sum_{e \in \mathcal{T}_h} R_e^T \mathbf{A}_e R_e,
\]

where \( \mathbf{A}_e \) is the element stiffness matrix and \( R_e \) is the restriction mapping of a global vector to the corresponding local vector defined on the element \( e \in \mathcal{T}_h \).

In order to obtain a numerical solution of (1) with prescribed accuracy we perform a recursive semi-coarsening refinement procedure on the initially introduced mesh \( \mathcal{T}_0 \), and construct the nested meshes \( \mathcal{T}_0 \subset \mathcal{T}_1 \subset \ldots \subset \mathcal{T}_\ell = \mathcal{T}_h \) as shown in Figure 1.

We assume that over each element \( e \in \mathcal{T}_0 \) the functions \( a_{ii}, i = 1, 2 \), are constants.

\[\text{Figure 1. Mesh semi-coarsening discretization of } \Omega\]

We consider the biquadratic finite element spaces \( V_0 \subset V_1 \subset \ldots \subset V_\ell = V_h \) and the finite element stiffness matrices \( \mathbf{A}^{(0)}, \mathbf{A}^{(1)}, \ldots, \mathbf{A}^{(\ell)} = \mathbf{A}_h \) associated with these meshes. We aim now to find a numerical solution to (1) over the finest mesh \( \mathcal{T}_\ell = \mathcal{T}_h \).
Preconditioners based on various multilevel extensions of two-level FEMs lead to iterative methods which have an optimal order computational complexity with respect to the size of the system. The methods can be based on block matrix factorization, recursively extended via certain matrix polynomial approximations of the arising Schur complement matrices. The resulting spectral equivalence holds uniformly with respect to jumps in the coefficients of the differential operator if the jumps are resolved by the coarsest mesh partition. Such methods, known as multiplicative Algebraic MultiLevel Iteration (AMLI) methods, were first presented in [4]. The monographs [10, 20] provide a systematic presentation of AMLI methods and a discussion of related multilevel/multigrid preconditioners. We could refer, e.g., to [1, 2, 5, 6, 9, 11, 12, 18, 19] for some more recent results related to robust AMLI methods and algorithms.

The following two conditions are fundamental for the robustness of AMLI algorithms based on recursive hierarchical basis splittings, see e.g. [10]: (i) proper uniform estimate of the related CBS constant with respect to mesh and coefficient anisotropy, and/or other problem parameters, and (ii) optimal order solver/preconditioners for the systems with the diagonal block matrix, which arises on each finer level in a recursive refinement (coarsening) step and corresponds to the added degrees of freedom on that level.

The focus of the study is on the robust optimality of the AMLI algorithm in the case of semi-coarsening mesh refinement for biquadratic elements. It is in the spirit of [14, 15] whereas a novel approach is used in the analysis of the CBS constant overcoming the difficulties in the more complicated parameter dependent macroelement problems. This work is additionally motivated by the potential of the presented approach for solving non-symmetric and nonlinear problems, see e.g., [7, 16, 17].

The remainder of the paper is organized as follows. In Section 2 the applied semi-coarsening procedure is described. We prove new uniform estimates of the constant in the strengthened CBS inequality in Section 3 from where the main multilevel result given in Section 4 follows. Solving systems with the pivot blocks of the hierarchical basis stiffness matrices is discussed in Section 5. Section 6 presents a numerical study of the behavior of the CBS constant varying the semi-coarsening factor and the anisotropy ratio. Some concluding remarks are given at the end.

2. FEM matrices in semi-coarsening refinement

We consider a uniform semi-coarsening procedure where at the current refinement step all elements from $\mathcal{T}_k$ are split into $\rho$ congruent subelements each by adding lines parallel to the vertical edges as shown in Figure 2.

![Semi-coarsening macroelement for $\rho = 2$](image)

Let $e$ be an arbitrary biquadratic element from $\mathcal{T}_k$ whose length and height we denote by $h^{(k)}_x$ and $h_y$ respectively. Then if we set

$$
\varepsilon = \frac{a_{22}}{a_{11}} \left( \frac{h^{(k)}_x}{h_y} \right)^2
$$

and

$$
\delta^{(k)}_e = \frac{a_{11}}{45} \frac{h_y}{h^{(k)}_x},
$$

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Let us remind that the diffusion coefficients

Then the macroelement stiffness matrix corresponding to

the element stiffness matrices

The rectangular subelements obtained after \( \ell \) refinement steps of the element \( e \) have equal height \( h_y = h_y^{(0)} \) and lengths 

\[ h_x^{(0)} = \rho^{-\ell} h_x^{(0)} \]. Here \( h_x^{(0)} \) is the coarsest mesh size along the \( y \)-axis which remains unchanged during the refinement and \( h_x^{(0)} \) is the coarsest mesh size along the \( x \)-axis.

Then the macroelement stiffness matrix corresponding to \( E \in \mathcal{T}_k \) at the \( (k+1) \)-st refinement step can be assembled by the element stiffness matrices \( A_e^{(k+1)} \) and presented in the following two by two block form:

\[
A_E^{(k+1)} = \begin{bmatrix}
A_{e,11}^{(k+1)} & A_{e,12}^{(k+1)} \\
A_{e,21}^{(k+1)} & A_{e,22}^{(k+1)}
\end{bmatrix}
\]  

(5)

Here the first diagonal block corresponds to the unknowns from \( \mathcal{T}_{k+1} \mid \mathcal{T}_k \) while the second diagonal block is a \( 9 \times 9 \) matrix related to the nodes from \( \mathcal{T}_k \). We will use (5) to analyze the local constant in the strengthened Cauchy–Bunyakovski–Schwarz (CBS) inequality.

We introduce also the hierarchical macroelement stiffness matrix \( \tilde{A}_E^{(k+1)} \) corresponding to the two-level hierarchical nodal basis. The two macroelement matrices are related by

\[
\tilde{A}_E^{(k+1)} = J^T A_E^{(k+1)} J,
\]  

(6)

where \( J \) is the transformation matrix between the standard and the hierarchical nodal bases and has the form

\[
J = \begin{bmatrix}
I_k & Z \\
0 & I_6
\end{bmatrix},
\]  

(7)

where \( k = 6(\rho - 1) \).

For \( \rho = 2 \) and the numbering of nodes according to Figure 2, the first diagonal block of \( J \) is a \( 6 \times 6 \) identity matrix and the interpolation matrix \( Z \) looks like

\[
Z = \frac{1}{8} \begin{bmatrix}
3l_1 & -l_1 \\
6l_2 & 6l_2 \\
-l_3 & 3l_3
\end{bmatrix}^T.
\]
For $\rho = 5$ and the numbering of nodes according to Figure 3, the first diagonal block of $J$ is a $24 \times 24$ identity matrix and the interpolation matrix $Z$ can be written in the form

$$Z = \frac{1}{25} \begin{bmatrix}
18 I_3 & 12 I_3 & 7 I_3 & 3 I_3 & -2 I_3 & -3 I_3 & -3 I_3 & -2 I_3 \\
9 I_3 & 16 I_3 & 21 I_3 & 24 I_3 & 21 I_3 & 16 I_3 & 9 I_3 \\
-2 I_3 & -3 I_3 & -3 I_3 & -2 I_3 & 3 I_3 & 7 I_3 & 12 I_3 & 18 I_3
\end{bmatrix}^T.$$

Following the introduced splitting of the unknowns the hierarchical basis matrix $\tilde{A}^{(k+1)}_E$ can be presented also in a two-by-two block form

$$\tilde{A}^{(k+1)}_E = \begin{bmatrix}
\tilde{A}^{(k+1)}_{E:11} & \tilde{A}^{(k+1)}_{E:12} \\
\tilde{A}^{(k+1)}_{E:21} & \tilde{A}^{(k+1)}_{E:22}
\end{bmatrix}.$$

The global hierarchical basis matrix $\tilde{A}^{(k+1)}$ is assembled from the introduced macroelement hierarchical basis matrices and after an appropriate ordering of the degrees of freedom it can be similarly written in the form

$$\tilde{A}^{(k+1)} = \begin{bmatrix}
\tilde{A}^{(k+1)}_{11} & \tilde{A}^{(k+1)}_{12} \\
\tilde{A}^{(k+1)}_{21} & \tilde{A}^{(k+1)}_{22}
\end{bmatrix}$$

where the first diagonal block is associated with the unknowns from $T_{k+1} \setminus T_k$ while the second diagonal block is related to the nodes from $T_k$.

3. Uniform estimates of the constant in the strengthened CBS inequality

The quality of the hierarchical basis splitting is measured by the CBS constant which is the infimum of $\gamma \geq 0$ satisfying the inequality

$$|w_1^T \tilde{A}^{(k+1)}_{12} w_2| \leq \gamma \left| w_1^T \tilde{A}^{(k+1)}_{11} w_1 w_2^T \tilde{A}^{(k+1)}_{22} w_2 \right|^{1/2} \quad \text{for all} \quad w^T = (w_1^T, w_2^T).$$

It is interpreted as the cosine of the abstract angle between the two subspaces of the hierarchical splitting (8).

From a practical point of view it is important that the constant $\gamma$ can be estimated locally at the level of macroelements, see e.g. [4, 10], that is

$$\gamma \leq \max_{E \in T_{k+1}} \gamma_E.$$

Here $\gamma_E$ is the local CBS constant that can be determined by

$$\gamma_E^2 = 1 - \mu_1.$$
where $\mu_i$ is the minimal eigenvalue of the generalized eigenproblem
\[
S^{(k+1)}_E v_{E,2} = \mu A^{(k+1)}_{E,22} v_{E,2}, \quad v_{E,2} \neq \text{const},
\] (10)
and the Schur complement $\tilde{S}^{(k+1)}_E$ is defined by
\[
\tilde{S}^{(k+1)}_E = A^{(k+1)}_{E,22} - A^{(k+1)}_{E,21} A^{-1}_{E,11} A^{(k+1)}_{E,12}.
\]

From (7) we have directly that $S^{(k+1)}_E = A^0$ and it can easily be shown also that $\tilde{S}^{(k+1)}_E = S^{(k+1)}_E$, which allows us to rewrite (10) in the form
\[
S^{(k+1)}_E v_{E,2} = \mu A^0 v_{E,2}, \quad v_{E,2} \neq \text{const}.
\] (11)

When lower order (say linear) finite elements are considered, the local problem (11) is easily solvable. Here the sizes of the macroelement matrices (depending on the coarsening factor $\rho$) and the involved anisotropy ratio make the solution of the parameter dependent eigenvalue problem more complicated. In what follows we derive estimates of the local CBS constants further developing the approach used in [14]. It is known from the linear algebra that the Schur complement $S^{(k+1)}_E$ has the extremal property:
\[
v_{E,2}^T S^{(k+1)}_E v_{E,2} = \min \left\{ v_{E,1}^T A_{E,21} A_{E,22}^{-1} A_{E,12} v_{E,1} \middle| v_{E,2}^T A_{E,22} v_{E,2} = 1 \right\} \text{ for all } v_{E,2}. \] (12)

As our purpose is to find a lower bound for the eigenvalues of (11), we consider the inequality
\[
v_{E,2}^T S^{(k+1)}_E v_{E,2} \geq \mu v_{E,2}^T A^0 v_{E,2} \quad \text{for all } v_{E,2}.
\]

Using (12) we transform it further to
\[
\min \left\{ v_{E,1}^T A_{E,21} A_{E,22}^{-1} A_{E,12} v_{E,1} \middle| v_{E,2}^T A_{E,22} v_{E,2} = 1 \right\} \geq 0 \quad \text{for all } v_{E,2}. \] (13)

Then if we denote
\[
B = \begin{bmatrix}
A^{(k+1)}_{E,11} & A^{(k+1)}_{E,12} \\
A^{(k+1)}_{E,21} & A^{(k+1)}_{E,22} - \mu A^0
\end{bmatrix},
\] (14)

it is enough to find a constant $\mu > 0$ for which $B$ is symmetric positive semi-definite in order to provide a lower bound for $\mu$ in (11) and therefore an estimate for $v^2$.

The result of the analysis in this section is summarized in

**Lemma 3.1.**

The constant $\gamma$ in the strengthened CBS inequality corresponding to the semi-coarsening refinement for biquadratic elements is uniformly bounded with respect to the anisotropy ratio for $\rho = 2, 3, 4, 5$ and the following estimates hold true:

\[
\gamma^2 \leq \begin{cases}
\frac{5}{32}, & \rho = 2; \\
\frac{5}{21}, & \rho = 3; \\
\frac{30}{191 - \sqrt{13441}}, & \rho = 4; \\
\frac{196}{205}, & \rho = 5.
\end{cases}
\] (15)
Proof. We shall provide a proof of the lemma for the case $\rho = 5$. The same approach has been used to derive the other three estimates. First we note that for the element stiffness matrices $A_{v}^{(k+1)}$ and $A_{v}^{(k)}$ when $\rho = 5$

$$\delta_{v}^{(k+1)} = 5\delta_{v}^{(k)}.$$ 

Then as $\delta_{v}^{(k)}$ appears on both sides of (11), without loss of generality we can omit it from further consideration. The matrix $B$ defined in (14) can be assembled from the matrices

$$\mu A_{v}^{(k)} = \mu \begin{bmatrix} A_{v,11}^{(k)} & A_{v,12}^{(k)} & A_{v,13}^{(k)} \\ A_{v,21}^{(k)} & A_{v,22}^{(k)} & A_{v,23}^{(k)} \\ A_{v,31}^{(k)} & A_{v,32}^{(k)} & A_{v,33}^{(k)} \end{bmatrix} \quad \text{and} \quad A_{v}^{(k+1)} = \begin{bmatrix} A_{v,11}^{(k+1)} & A_{v,12}^{(k+1)} & A_{v,13}^{(k+1)} \\ A_{v,21}^{(k+1)} & A_{v,22}^{(k+1)} & A_{v,23}^{(k+1)} \\ A_{v,31}^{(k+1)} & A_{v,32}^{(k+1)} & A_{v,33}^{(k+1)} \end{bmatrix},$$

where the blocks of $A_{v}^{(k)}$ are of the form (4) and the blocks of $A_{v}^{(k+1)}$ look like

$$A_{v,11}^{(k+1)} = A_{v,23}^{(k+1)} = \begin{bmatrix} 14 + \frac{14}{25} \varepsilon & 7 - \frac{16}{25} \varepsilon & -7 + \frac{2}{25} \varepsilon \\ 7 - \frac{16}{25} \varepsilon & 50 + \frac{32}{25} \varepsilon & 7 - \frac{16}{25} \varepsilon \\ \frac{7}{2} & \frac{2}{25} \varepsilon & 14 + \frac{14}{25} \varepsilon \end{bmatrix} \quad \text{and} \quad A_{v,13}^{(k+1)} = \begin{bmatrix} 2 - \frac{7}{50} \varepsilon & 1 + \frac{4}{25} \varepsilon & -2 - \frac{\varepsilon}{50} \\ \frac{1}{2} + \frac{4}{25} \varepsilon & 8 - \frac{8}{25} \varepsilon & 1 + \frac{4}{25} \varepsilon \\ -\frac{1}{2} & \frac{4}{25} \varepsilon & 2 - \frac{\varepsilon}{7} \varepsilon \end{bmatrix},$$

$$A_{v,12}^{(k+1)} = A_{v,23}^{(k+1)} = \begin{bmatrix} -16 + \frac{7}{25} \varepsilon & -8 & 4 + \frac{\varepsilon}{25} \\ -8 - \frac{8}{25} \varepsilon & -64 + \frac{16}{25} \varepsilon & -8 + \frac{8}{25} \varepsilon \\ \frac{4}{25} \varepsilon & -8 & 16 + \frac{7}{25} \varepsilon \end{bmatrix} \quad \text{and} \quad A_{v,22}^{(k+1)} = \begin{bmatrix} 32 + \frac{56}{25} \varepsilon & 16 - \frac{64}{25} \varepsilon & -8 + \frac{8}{25} \varepsilon \\ 16 - \frac{64}{25} \varepsilon & 128 + \frac{128}{25} \varepsilon & 16 - \frac{64}{25} \varepsilon \\ -8 + \frac{8}{25} \varepsilon & 16 & 32 + \frac{56}{25} \varepsilon \end{bmatrix}.$$ 

This immediately enables us to write the matrix $B$ in the form

$$B = B_{0} + \mu B_{\varepsilon} + \varepsilon B_{\varepsilon} + \varepsilon^{2} B_{\varepsilon^{2}},$$

where $B_{0}$, $B_{\varepsilon}$, $B_{\varepsilon^{2}}$ and $B_{\varepsilon^{3}}$ do not depend on any parameters. Our aim now is to find the values of $\mu$ for which the matrix $B$ is positive semi-definite. As all of the matrices in (16) are symmetric, it is obvious that $B$ will be symmetric independently of $\mu$ and $\varepsilon$.

The eigenvalue analysis of each of the matrices shows that $B_{0}$ and $B_{\varepsilon}$ are positive semi-definite while $B_{\varepsilon^{2}}$ and $B_{\varepsilon^{3}}$ are negative semi-definite matrices. Now we rewrite (16) in the form:

$$B = (B_{0} + \mu B_{\varepsilon}) + \varepsilon (B_{\varepsilon} + \mu B_{\varepsilon^{2}}),$$

As we want $B$ to remain semi-positive definite for any parameter $\varepsilon > 0$, we need to find the values of $\mu$ which guarantee semi-positivity to both $B_{0} + \mu B_{\varepsilon}$ and $B_{\varepsilon} + \mu B_{\varepsilon^{2}}$. Solving the generalized eigenvalue problem\(^1\)

$$B_{0}v = \lambda (-B_{\varepsilon^{2}})v$$

we get the minimal eigenvalue

$$\lambda_{\min}^{*} = \frac{15}{19}.$$ 

\(^{1}\) Mathematica software tool is used in the presented computations.
Therefore, $B_0 + \mu B_{\varepsilon}$ is semi-positive definite for any nonnegative $\mu \leq 15/19$. In the same way we determine the minimal eigenvalue of the generalized eigenvalue problem

$$B_\varepsilon v = \lambda \left(-B_{\varepsilon}\right) v,$$

which is

$$\lambda''_{\text{min}} = \frac{29}{205}. \tag{19}$$

Then the necessary condition $B_\varepsilon + \mu B_{\varepsilon}$ to become semi-positive definite is

$$\mu \leq \frac{29}{205}.$$

Combining (18) and (19) we conclude that $29/205$ is the sharp uniform lower bound of the minimal eigenvalue $\mu_1$ of (11).

Putting together (9) and the last result we obtain

$$\gamma^2 \leq \max_{E \in \mathcal{T}_{k+1}} E \in \mathcal{T} \frac{E}{k+1} \gamma \leq 1 - \mu_1 \leq 1 - \frac{29}{205} = \frac{176}{205},$$

which completes the proof.

4. Semi-coarsening AMLI algorithm

In this section we utilize the estimates (15) in the analysis of the AMLI algorithm as introduced in [4], see also [10], to the case of semi-coarsening mesh refinement for biquadratic elements.

Let us consider the global stiffness matrix

$$A^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & A_{12}^{(k+1)} \\ A_{21}^{(k+1)} & A_{22}^{(k+1)} \end{bmatrix},$$

corresponding to the two-level splitting and the related global hierarchical basis stiffness matrix (see (8))

$$\tilde{A}^{(k+1)} = f^{(k+1)} A^{(k+1)} f^{(k+1)} = \begin{bmatrix} A_{11}^{(k+1)} & A_{12}^{(k+1)} \\ A_{21}^{(k+1)} & A_{22}^{(k+1)} \end{bmatrix},$$

corresponding to (6)–(7) where $f^{(k+1)}$ is the related global transformation matrix.

The multiplicative AMLI preconditioner $M = M^{(k)}$ is defined recursively by

$$M^{(0)} = A^{(0)},$$

$$M^{(k+1)} = f^{(k+1)}^{-T} [A_{11}^{(k+1)} & 0 \\ A_{21}^{(k+1)} & P^{(k)}] \begin{bmatrix} I & A_{11}^{(k+1)} A_{12}^{(k+1)} \\ 0 & I \end{bmatrix} f^{(k+1)}^{-1}, \tag{20}$$

where $P^{(k)}$ is implicitly defined by the equation

$$P^{(k)} = \left[ I - \beta \left( M^{(k)} A^{(k)} \right) \right] A^{(k)}^{-1},$$

where $\beta$ stands for the degree of the stabilization Chebyshev polynomial $p_\beta$.

The preconditioner (20) is said to be of optimal order when

$$N(M^{-1}u_h) = O(N), \quad \kappa(M^{-1}A_h) = O(1),$$

where $N$ is the number of unknowns of (2) and $N(\cdot)$ stands for the computational complexity.

One of the fundamental results in the theory of the AMLI preconditioning methods is formulated in the next theorem (for more details see, e.g., [10]).
**Theorem 4.1.**
The multiplicative AMLI preconditioner (20) is of optimal order if the following condition holds true:

\[
\frac{1}{\sqrt{1 - \gamma}} < \beta < \rho.
\]  

(21)

5. **Solving systems with the pivot block** \(A_{11}^{(k+1)}\)

The idea of the AMLI algorithm is to reduce the linear system (2) to a sequence of smaller subproblems with the pivot block matrices \(A_{11}^{(k+1)}\). When the problem is isotropic the condition number of \(A_{11}^{(k+1)}\) is uniformly bounded with respect to the related number of degrees of freedom \(N^{(k+1)}\). However the situation significantly changes when the problem is anisotropic. In this case \(\kappa(A_{11}^{(k+1)})\) deteriorates with the anisotropy ratio.

Specially designed robust preconditioning techniques for the pivot blocks are needed when uniform refinement is used in the AMLI methods for parameter dependent ill-conditioned elliptic problems. Such robustness results for two dimensional anisotropic problems and linear (conforming and nonconforming) elements were successfully developed during the last decade, see e.g. [3, 11] and references therein.

An important advantage of the semi-coarsening AMLI algorithm is that the degrees of freedom can be ordered in such a way that the resulting blocks \(A_{11}^{(k+1)}\) become block diagonal matrices that have a uniformly bounded semi-bandwidth.

![Figure 4. Numeration of the pivot block unknowns for \(\rho = 3\)](image)

Let us come back to the case of semi-coarsening AMLI preconditioning of biquadratic FEM systems. And let us order the unknowns of the pivot block \(A_{11}^{(k+1)}\) following the pattern shown in Figure 4. Then the following relations hold true for the semi-bandwidth \(d(\rho)\) of \(A_{11}^{(k+1)}\):

\[
d(\rho) \leq 4(\rho - 1) + 1 \quad \text{for} \quad \rho = 2,
\]

\[
d(\rho) \leq 4(\rho - 1) + 2 \quad \text{for} \quad \rho = 3, 4,
\]

\[
d(\rho) \leq 4(\rho - 1) + 3 \quad \text{for} \quad \rho > 4,
\]

Moreover, for even \(\rho\), i.e. for \(\rho = 2k, k = 1, 2, \ldots\), the term \(4(\rho - 1)\) can be easily improved to \(2(\rho - 1)\) if the numbering is slightly modified as shown in Figure 5. Consequently, the computational complexity of any direct solver for banded matrices (using, say, LDL^T factorization) is of optimal order, that is

\[
N(A_{11}^{(k+1)-1}v) = O(N^{(k+1)}).
\]

Combining the last result with Theorem 4.1 and the uniform estimates (15) we get the main contribution of our study.
Theorem 5.1. The semi-coarsening AMLI preconditioner (20) with parameters $\rho = 5$ and $\beta = 3$ or $\beta = 4$ has an optimal order of computational complexity, uniformly with respect to mesh and coefficient anisotropy.

The optimality referred in the theorem is understood in the sense that the number of iterations of the related Preconditioned Conjugate Gradient (PCG) method required to reach some in advance prescribed accuracy $\epsilon$ is independent of the size of the discrete problem and also that the total computational cost of the algorithm is proportional to the number of unknowns, with a proportionality constant independent of the coefficient jumps across element interfaces on the coarsest mesh partition and the anisotropy ratio.

6. Numerical tests

In this section we present a numerical study of the behavior of the CBS constant $\gamma^2$ and that of $\Pi(\gamma)$ as a function of $\epsilon$ (see (3)), where

$$\Pi(\gamma) = \frac{1}{\sqrt{1 - \gamma^2}}, \quad \epsilon = \frac{a_{22}}{a_{11}} \left( \frac{b_k}{h_y} \right)^2,$$

for different values of the refinement/coarsening parameter $\rho$. Lemma 3.1 implies that the refinement/coarsening parameter $\rho = 5$ is the smallest one that uniformly satisfies the AMLI optimality condition (21). The numerical results presented in Table 1 show that this condition is fulfilled for bigger values of $\rho$ as well.

The numerical tests confirm that our theoretical estimates are asymptotically sharp. According to the proof of Lemma 3.1, we observe that $\Pi(\gamma)$ grows (monotonically) with increasing anisotropy ratio $\epsilon$.

Let us also note that the uniform estimates of Lemma 3.1 and respectively the statement of Theorem 5.1 cover the most pessimistic possible scenarios with respect to coefficient and/or mesh anisotropy. In this respect, the numerically computed values of $\gamma$ could have a complementary practical value. Based on them, one could play with $\rho$ and $\beta$ (including varying values of $\beta_k$) to get a more efficient AMLI setting for a given particular problem.

7. Concluding remarks

In this paper we study theoretically a method that is based on hierarchical two-level splitting of the stiffness matrix and present estimates of the CBS constant which can be used to derive condition number estimates. Although initially it may seem that it is of limited practical application one can use it also to obtain corresponding estimates for other methods that do not rely on hierarchical basis transformations such as for the methods based on sparse Schur complement approximations, see [2, 10].
We develop semicoarsening AMLI methods for biquadratic conforming FEM systems and derive a set of parameters that allow for the design of optimal order AMLI methods. The presented estimates prove that the condition number of the multiplicative AMLI preconditioner is uniformly bounded with respect to the ratio of anisotropy in the elliptic model problem and the discretization parameter $h$.

Let us note that the efficient preconditioning of higher order FEM systems obtained after discretizations of elliptic problems with an anisotropic diffusion tensor is an open problem. It is a classical result [13] that the CBS constant $\gamma$ tends to one increasing the anisotropy ratio in the case of hierarchical splitting of $p$-FEM systems ($p = 2$). One can find some more related results in the recently published paper [8]. Here we presented a first robust preconditioner for biquadratic orthotropic systems.

The obtained results are relevant also in the case of a refinement procedure that (equivalently) defines an alternating semi-coarsening where at each level one is allowed to choose whether the semi-coarsening procedure is applied along the direction of the $x$-axis or along the direction of the $y$-axis. We should note again that in our analysis we consider
only two consecutive levels.

There are two more general remarks concerning the advantages of semi-coarsening methods: (i) the presented results can be naturally transferred to the case of 3D; (ii) the algorithms have a well expressed parallel structure based on a node numbering which is independent of the (possible) varying direction of dominating anisotropy.

At the end, let us briefly comment the case of full anisotropic coefficient matrix and nonuniform mesh. One possible approach to generalize the semicoarsening technique could be based on a proper abstract definition of the direction of dominating coefficient and/or mesh anisotropy, following e.g. the algebraic approach at a macroelement level used in [3].

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References