Tricyclic graphs with exactly two main eigenvalues

Xiaoxia Fan\(^1\)\(\ast\), Yanfeng Luo\(^1\)\(\dagger\), Xing Gao\(^1\)\(\ddagger\)

1 Department of Mathematics, Lanzhou University, Tianshui Road South 222, Lanzhou, Gansu 730000, China

Received 25 May 2012; accepted 2 November 2012

Abstract: An eigenvalue of a graph \(G\) is called a main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. Let \(G_0\) be the graph obtained from \(G\) by deleting all pendant vertices and \(\delta(G)\) the minimum degree of vertices of \(G\). In this paper, all connected tricyclic graphs \(G\) with \(\delta(G_0) \geq 2\) and exactly two main eigenvalues are determined.

MSC: 05C50

Keywords: Main eigenvalues • Tricyclic graphs • 2-walk \((a, b)\)-linear graphs

1. Introduction

All graphs considered in this paper are finite, undirected and simple. Let \(G = (V, E)\) be a graph with vertex set \(V(G)\) and edge set \(E(G)\). Denote by \(A(G)\) the adjacency matrix of \(G\). The eigenvalues of \(G\) are those of \(A(G)\). An eigenvalue of a graph \(G\) is called a main eigenvalue if it has an eigenvector the sum of whose entries is not equal to zero. It is well known that a graph is regular if and only if it has exactly one main eigenvalue.


A vertex of a graph \(G\) is said to be pendant if it has degree one. Denote by \(C_n\) and \(P_n\) the cycle and path of order \(n\), respectively. A connected graph is said to be tricyclic (resp., unicyclic and bicyclic), if \(|E(G)| = |V(G)| + 2\) (resp., \(|E(G)| = |V(G)|\) and \(|E(G)| = |V(G)| + 1\). Hou and Tian [5] showed that the graphs \(C^*_{k, r}\) for some positive integers \(k, r\)

\(\ast\) E-mail: x25fan@gmail.com
\(\dagger\) E-mail: luoyf@lzu.edu.cn
\(\ddagger\) E-mail: gaoxing@lzu.edu.cn
with \( r \geq 3 \), where \( C^k_r \) is the graph obtained from \( C_r \) by attaching \( k > 0 \) pendant vertices to every vertex of \( C_r \), are the only connected unicyclic graphs with exactly two main eigenvalues. Hu et al. [7] and Shi [8] characterized independently all connected bicyclic graphs with exactly two main eigenvalues. This paper will continue the line of this research and determine a class of connected tricyclic graphs with exactly two main eigenvalues.

For any tricyclic graph \( G \), the base of \( G \), denoted by \( B_G \), is the minimal tricyclic subgraph of \( G \). Clearly, \( B_G \) is the unique tricyclic subgraph of \( G \) containing no pendant vertex, and \( G \) can be obtained from \( B_G \) by attaching trees to some vertices of \( B_G \). It follows from [3] that there are eight types of bases for tricyclic graphs, say, \( T_i \), \( i = 1, \ldots, 8 \), which are depicted in Figure 1.

**Figure 1.** The eight types of bases for tricyclic graphs.

\[
\begin{align*}
T_1 & \quad T_2 \\
T_3 & \quad T_4 \\
T_5 & \\
T_6 & \quad T_7 \\
T_8 & 
\end{align*}
\]

### 2. Preliminaries

In this section, we will present some notation and known results which will be used in the next section. The reader is referred to [1] for any undefined notation and terminology on graphs in this paper.

Let \( G \) be a graph. As usual, we denote by \( d(v) = d_G(v) \) and \( N(v) = N_G(v) \) the degree of vertex \( v \) and the set of all neighbors of \( v \) in \( G \). Let

\[
S(v) = \sum_{u \in N(v)} d(u),
\]

A graph \( G \) is called 2-walk \((a, b)\)-linear if there exist unique rational numbers \( a, b \) such that

\[
S(v) = ad(v) + b
\]

holds for every vertex \( v \in V(G) \).

An internal path of \( G \) is a walk \( v_0v_1\ldots v_s \) such that the vertices \( v_0, v_1, \ldots, v_s \) are distinct, \( d(v_0) > 2 \), \( d(v_s) > 2 \), and \( d(v_i) = 2 \) for \( 0 < i < s \). An internal cycle of \( G \) is a closed walk \( v_0v_1\ldots v_s \) such that \( v_0 = v_s \), \( d(v_0) = d(v_s) > 2 \), and \( d(v_i) = 2 \) for \( 0 < i < s \). If \( R \) is an internal path or an internal cycle of \( G \), the length of \( R \), denoted by \( |R| \), is defined as the number of vertices of \( R \) minus 1.

**Lemma 2.1 ([4]).**

A graph \( G \) has exactly two main eigenvalues if and only if \( G \) is 2-walk \((a, b)\)-linear.

**Lemma 2.2 ([5]).**

Let \( G \) be a 2-walk \((a, b)\)-linear graph. Then both \( a \) and \( b \) are integers.

**Lemma 2.3 ([7]).**

Let \( G \) be a 2-walk \((a, b)\)-linear graph and \( v, u \) be two vertices of \( G \) with unequal degrees \( d(v), d(u) \), respectively. Then

\[
a = \frac{S(v) - S(u)}{d(v) - d(u)}, \quad b = \frac{d(u)S(v) - d(v)S(u)}{d(v) - d(u)}.
\]
In the following, for convenience, we always assume that \( \mathcal{G} \) is the set of connected tricyclic graphs with exactly two main eigenvalues, \( x \) is a pendant vertex of \( G \) (if existing). For each \( G \in \mathcal{G} \), let \( G_0 \) be the graph obtained from \( G \) by deleting all pendant vertices. Let \( \delta(G) \) be the minimum degree of vertices of \( G \). [7, Lemma 3.1] holds for any graph \( G \). This is because the authors did not use any information on the graph to be bicyclic. Hence we have the following two lemmas.

**Lemma 2.4.**
Let \( G \in \mathcal{G} \) and \( R = x_1x_2\ldots x_t \) be an internal path or internal cycle of length at least 2 in \( G \). Then \( l(R) \leq 3 \). In particular, if \( l(R) = 3 \), then there exists no path \( Q = y_1y_2y_3 \) in \( G \) such that \( d(y_1) = d(y_3) = d(x_1) \) and \( d(y_2) = 2 \).

[7, Lemmas 3.3, 3.4 (ii), 3.5–3.7] give a characterization of bicyclic graphs with two main eigenvalues. In fact, their results and their proof hold for an arbitrary graph with at least one cycle and one pendant vertex and \( \delta(G_0) \geq 2 \). We have the following lemma.

**Lemma 2.5.**
Let \( G \in \mathcal{G} \) and \( v \in V(G_0) \). If \( \delta(G) = 1 \) and \( \delta(G_0) \geq 2 \), then

(i) \( G_0 \in \mathcal{I}_v \), \( i = 1, \ldots, 8 \), see Figure 1;
(ii) \( d(v) = d_{c_0}(v) \) or \( a + b \);
(iii) if \( G \) has at least one pendant vertex \( x \), then \( S(x) = a + b \geq 3 \) and \( a \geq 2 \);
(iv) for an internal cycle \( C = x_1x_2\ldots x_t \) of \( G_0 \) with \( d_{c_0}(x_1) \geq 3 \), \( d_{c_0}(x_2) = 2 \), if \( G \) has at least one pendant vertex, then there is an integer \( i \in \{1, 2, \ldots, t\} \) such that \( d(x_i) \neq a + b \).

### 3. Tricyclic graphs with exactly two main eigenvalues

In this section, we will determine all connected tricyclic graphs with exactly two main eigenvalues and \( \delta(G_0) \geq 2 \). By Lemma 2.1, it is sufficient to determine all 2-walk \((a, b)\)-linear tricyclic graphs.

**Lemma 3.1.**
Let \( G \in \mathcal{G} \) have at least one pendant vertex and \( \delta(G_0) \geq 2 \), \( R = x_1x_2\ldots x_t \) be an internal path or an internal cycle of length at least 3 in \( G_0 \) with \( d_{c_0}(x_1) = d_{c_0}(x_t) \in \{3, 4\} \) or \( d_{c_0}(x_1) = 3, d_{c_0}(x_t) = 5 \). And let all vertices in \( N(x_1) \) and \( N(x_t) \) lie in some internal paths or internal cycles of \( G \) when \( d_{c_0}(x_1) = d_{c_0}(x_t) = 4 \). Then:

(i) \( d(x_2) = d(x_3) = \ldots = d(x_{t-1}) \in \{2, a + b\} \) and \( d(x_1) = d(x_t) \);
(ii) if \( d(x_2) = 2 \), then \( l(R) = 3 \);
(iii) if \( R \) is an internal cycle of \( G_0 \) with \( d_{c_0}(x_1) \in \{3, 4\} \), then \( l(R) = 3 \) and \( d(x_2) = d(x_3) = 2 \); in particular, if \( d_{c_0}(x_1) = 3 \), then \( a = 2 \);
(iv) if \( R \) is an internal cycle of \( G_0 \) with \( d_{c_0}(x_1) = 5 \), then \( l(R) = 3 \), \( d(x_2) = d(x_3) \in \{2, 3\} \); in particular, if \( d(x_2) = d(x_3) = 3 \), then \( d(x_1) = 5 \) and \( a = 3, b = 0 \).

**Proof.** (i) By way of contradiction, assume that there is an integer \( i \in \{2, 3, \ldots, t-2\} \) such that \( d(x_i) \neq d(x_{i+1}) \). By Lemma 2.5 (ii), we may assume that \( d(x_i) = 2 \) and \( d(x_{i+1}) = a + b \). Applying (3) with \((v, u) = (x_{i+1}, x)\), we have

\[
a = \frac{S(x_{i+1}) - S(x)}{d(x_{i+1}) - d(x)} = \frac{a + b - 2 + d(x_{i+2}) - (a + b)}{a + b - 1} = \frac{d(x_{i+2})}{a + b - 1}.
\]

This together with Lemma 2.5 (iii) implies that

\[
d(x_{i+2}) = a(a + b - 1) \geq 2(a + b - 1) \geq a + b + 1 > \max \{a + b, 3\}.
\] (4)
Hence \( d(x_{i+2}) = d_{G_0}(x_{i+2}) > 3 \) by Lemma 2.5(ii). Note that we have \( d(x_j) \in \{a+b,2\} \) for \( 2 \leq j \leq t-1 \). Thus \( x_{i+2} = x_i \) or \( x_{i+2} = x_{i+1} \). If \( d_{G_0}(x_1) = d_{G_0}(x_3) = 3 \), then \( (d(x_1),d(x_3)) \subseteq \{3,a+b\} \), contrary to (4).

If \( d_{G_0}(x_1) = d_{G_0}(x_3) = 4 \), then \( (d(x_1),d(x_3)) \subseteq \{4,a+b\} \). It follows from (4) that \( d(x_{i+2}) = a(a+b-1) = 4 \). This together with Lemma 2.5(iii) implies that \( a = 2, b = 1 \). If \( d(x_2) = 2 \), then \( S(x_2) = 5 \) by (2). On the other hand, \( d_{G_0}(x_1) = 4 > a+b \), so \( d(x_1) = 4 \) by Lemma 2.5(ii). Thus by (1), \( S(x_1) = d(x_1) + d(x_3) = 4 + d(x_3) > 5 \), a contradiction. If \( d(x_2) = a+b = 3 \), then \( S(x_1) = ad(x_1) + b = 9 = 3 + d(w_1) + d(w_3) + d(w_3) \), where \( w_1,w_2,w_3 \in N(x_1) \). Note that all vertices in \( N(x_1) \) lie in cycles. So \( d(w_1) = 2, i = 1,2,3 \). Thus \( S(w_1) = 2d(w_1) + 1 = 5 = d(u_1) + d(u) \), where \( u \in N(w_1) \). This implies that \( d(u) = 1 \), which is impossible since \( w_1 \) lies in an internal path or internal cycle.

If \( d_{G_0}(x_1) = 3, d_{G_0}(x_2) = 5 \), then by (4), \( d(x_{i+2}) = d_{G_0}(x_1) = 5 \) and \( (a+b-1) = 5 \). This is impossible since \( a \geq 2, a+b \geq 3 \). Hence \( d(x_3) = d(x_3) = \ldots = d(x_{t-1}) = 2, a+b \). Therefore \( S(x_2) = S(x_{t-1}) \) by (2). On the other hand, by (1), \( S(x_2) = d(x_2) + d(x_1) + d(x_3) \), \( S(x_{t-1}) = d(x_{t-1}) + d(x_{t-2}) + d(x_{t-3}) \). It follows that \( d(x_1) = d(x_3) \).

(ii) Suppose that \( l(R) \geq 4 \). Then \( d(x_i) = d(x_4) = 2 \) by (i). Applying (3) with \( (v,u) = (x_3,x) \), we have \( a = 4 - (a+b) \), which is impossible by Lemma 2.5(iii).

(iii) By Lemma 2.5(ii), we have \( d(x_2) \in \{2,a+b\} \). If \( d(x_2) = a+b \), then \( d(x_3) = a+b \) for \( 2 \leq i \leq t-1 \) by (i). Thus \( d(x_i) = d_{G_0}(x_1) \neq a+b \) by Lemma 2.5(iv). Applying (3) with \( (v,u) = (x_2,x) \), we have

\[
a = \frac{a+b-2 + d(x_1) + a+b - (a+b)}{a+b-1} = 1 + \frac{d(x_1) - 1}{a+b-1} = 1 + \frac{d_{G_0}(x_1) - 1}{a+b-1}.
\]

Note that \( d_{G_0}(x_1) = 3,4 \) and \( d_{G_0}(x_1) \neq a+b \geq 3 \). It follows from (5) that \( a \) can not be an integer. This contradicts Lemma 2.2. Hence \( d(x_2) = 2 \). Therefore \( l(R) = 3 \) and \( d(x_3) = d(x_2) = 2 \).

In particular, if \( d_{G_0}(x_1) = 3 \), then \( a = 2 + d(x_1) - (a+b) \) by applying (3) with \( (v,u) = (x_2,x) \). If \( d(x_1) = a+b \), then \( a = 2 \). If \( d(x_1) = d_{G_0}(x_1) = 3 \), then \( a = 5 - (a+b) \). It follows from Lemmas 2.2 and 2.5(iii) that \( a = 2 \).

(iv) By Lemma 2.5(ii), \( d(x_2) \in \{2,a+b\} \). If \( d(x_2) = 2 \), then \( l(R) = 3 \) and \( d(x_3) = 2 \) by (i) and (ii). If \( d(x_2) = a+b \), then \( d(x_3) = a+b \) for \( 2 \leq i \leq t-1 \) by (i). So \( d(x_1) = d_{G_0}(x_1) = 5 \neq a+b \) by Lemma 2.5(iv). Applying (3) with \( (v,u) = (x_2,x) \), we have \( a = 1 + 4l(a+b-1) \). Thus \( a+b = 3 \) and \( a = 3 \) by Lemmas 2.2 and 2.5(iii). Suppose that \( l(R) \geq 4 \). Then \( d(x_2) = d(x_3) = d(x_4) = a+b = 3 \) by (ii). Applying (3) with \( (v,u) = (x_2,x) \), we have \( a = 2 \). It is a contradiction. Therefore \( l(R) = 3 \).

Lemma 3.2.

Let \( G \in \mathcal{F} \) have at least one pendant vertex and \( \delta(G_0) \geq 2 \), \( R = x_1x_2 \ldots x_{t} \) be an internal cycle of length at least 3 in \( G_0 \) with \( d_{G_0}(x_1) = 6 \). Then exactly one of the following holds:

(i) \( l(R) = 3 \) and \( d(x_2) = d(x_3) = 3 \);
(ii) \( l(R) = 5 \), \( d(x_1) = 6 \), \( d(x_2) = d(x_3) = 4 \) and \( d(x_4) = d(x_5) = 2 \);
(iii) \( l(R) = 4 \), \( d(x_1) = 6 \), \( d(x_2) = d(x_3) = 3 \) and \( d(x_4) = 2 \).

Proof. Let \( x_{i+1} = x_i \). Clearly, either \( d(x_2) = d(x_3) = \ldots = d(x_4) \), or there exist \( i \in \{2,3,\ldots,t\} \) such that \( d(x_i) \neq d(x_{i+1}) \). First assume that \( d(x_2) = d(x_3) = \ldots = d(x_4) \). Then by Lemma 2.5(ii), we have \( d(x_2) = d(x_3) = \ldots = d(x_4) \in \{2,a+b\} \). Furthermore, note that we have \( d_{G_0}(x_1) = 6, d_{G_0}(x_2) = 2 \) and \( G_0 \) has at least one pendant vertex. By Lemma 2.5(iv), there exists \( k \in \{1,\ldots,t\} \) such that \( d(x_k) \neq a+b \). We claim that \( k \in \{2,3,\ldots,t\} \). Otherwise, let \( k = 1 \), then \( d(x_1) = d_{G_0}(x_1) = 6, d(x_2) = d(x_3) = \ldots = d(x_4) = a+b \). Applying (2) with \( (v,u) = (x_2,x) \), we have

\[
a = \frac{a+b-2 + d(x_1) + a+b - (a+b)}{a+b-1} = 1 + \frac{5}{a+b-1}.
\]

By Lemma 2.5(iii), \( a+b \geq 3, \) so \( a \) is not an integer, which is impossible by Lemma 2.2. Hence \( k \in \{2,\ldots,t\} \). Therefore \( d(x_2) = d(x_3) = \ldots = d(x_4) = 2 \). This together with Lemma 2.4 implies that \( t = 3 \) and \( l(R) = 3 \). (i) follows.
Now assume that there exists \( i \in \{2, 3, \ldots, t-1\} \) such that \( d(x_i) \neq d(x_{i+1}) \). By Lemma 2.5(ii), we may assume that \( d(x_i) = 2 \) and \( d(x_{i+1}) = a + b \). Applying (2) with \((v, u) = (x_{i+1}, x_i)\), with the same argument as in the proof of Lemma 3.1(i), we have

\[
d(x_{i+2}) = a(a + b - 1) = d_G(x_{i+2}) > 3.
\]

Thus \( x_{i+2} = x_1, d(x_1) = d_G(x_1) = 6 \) and \( a(a + b - 1) = 6 \). By Lemma 2.5(iii), we have \( a = b = 2 \) or \( a = 3, b = 0 \), we consider the following two cases:

**Case 1:** \( a = b = 2 \). By Lemma 2.5(ii), we have \( d(x_2) = 2a + b = 6 \). Note that \( d(x_1) = 6 \), we have \( d(x_1) = 0 \), which is impossible. If \( d(x_2) = a + b = 4 \), then \( S(x_2) = 2d(x_2) + b = 10 = d(x_1) + d(x_2) + 2 = 8 + d(x_1) \). So \( d(x_1) = 2 \). This implies that \( S(x_3) = 6 = d(x_3) + d(x_2) \), which implies that \( d(x_1) = 4 \). Therefore \( S(x_3) = 10 = d(x_3) + d(x_2) + 2 \) which implies that \( d(x_0) = 6 \notin \{2, a + b\} \). Thus \( x_0 = x_1 \) and \( t = 5 \), (ii) follows.

**Case 2:** \( a = 3, b = 0 \). Similarly, we have \( d(x_2) = 2a + b = 6 \). If \( d(x_2) = 2 \), then \( S(x_2) = ad(x_2) + b = 6 = d(x_1) + d(x_2) \). This implies that \( d(x_3) = 0 \), which is impossible. If \( d(x_2) = a + b = 3 \), then \( S(x_2) = ad(x_2) + b = 9 = d(x_1) + d(x_2) + 1 \) which implies that \( d(x_3) = 2 \). So \( S(x_3) = ad(x_3) + b = 6 = d(x_2) + d(x_3) \). Thus \( d(x_1) = 3 \). Hence \( S(x_3) = ad(x_3) + b = 9 = d(x_3) + d(x_3) + 1 \). This implies that \( d(x_i) = 6 \notin \{2, a + b\} \). Therefore \( x_0 = x_1 \) and \( t = 4 \), (iii) follows.

**Lemma 3.3.**

Let \( G \in \mathcal{G} \) with \( \delta(G_0) \geq 2 \) and \( G_0 \in T_1 \), see Figure 1. Then \( G = H_i \) for \( i = 1, 2, 3 \), see Figure 3.
Figure 3. Graphs $H_i$ for $i = 1, \ldots, 32$.

**Proof.** If $G_0 \in \mathcal{T}_1$, then $d_G(u_1), d_G(v_1), d_G(w_1) \in \{3, 4, 5, 6\}$. If $d_G(u_1), d_G(v_1), d_G(w_1) \in \{3, 4, 5\}$, then each internal cycle of $G_0$ has the length equal to 3 by Lemmas 2.4 and 3.1. Hence $G_0 = \mathcal{T}_1$, see Figure 2, where $n, m, k \geq 1$ and $\max\{n, m, k\} \geq 2$. For convenience, we set $w_k = v_m = u_n$. We consider the following two cases:

**Case 1:** $n = m = k = 1$. If $G$ has no pendant vertex, then by Lemma 2.4, $G = H_1$, see Figure 3. By (3), $H_1$ is 2-walk $(1,6)$-linear.

If $G$ has at least one pendant vertex, then for the sake of convenience, we denote the three internal cycles of $G_0$ by $x_1 x_2 \ldots x_i x_1, y_1 y_2 \ldots y_j y_1, z_1 z_2 \ldots z_l z_1$. Let $u = x_1 = y_1 = z_1$ be the vertex in $G_0$ with degree 6. Then by Lemma 3.2 and symmetry, we consider the following three cases:

**Subcase 1:** $i = 3$ and $d(x_2) = d(x_3) = 2$. We have $S(x_2) = d(x_1) + d(x_3) > 8$. We claim that for $y_1 y_2 \ldots y_j y_1, z_1 z_2 \ldots z_l z_1$, we also have $j = l = 3$ and $d(y_j) = d(y_3) = d(z_l) = d(z_i) = 2$. Otherwise, without loss of generality, assume that $j \neq 3$ and $d(y_2) = d(y_3) = 2$ do not hold. By Lemma 3.2, we have $j = 4$, $d(y_2) = d(y_4) = 3$, $d(y_3) = 2$ or $j = 5$, $d(y_2) = d(y_5) = 4$, $d(y_3) = d(y_4) = 2$. In both cases, we have $d(x_1) = d(x_3) = d(z_1) = d(u) = 6$ and $d(x_2) = d(x_3) = 2$. So by (2), $S(x_2) = S(y_2) = S(z_2)$. On the other hand, by (1), $S(x_2) \neq S(y_2)$, a contradiction. Hence all the pendant vertices are adjacent to vertex $u$ and $d(u) = a + b > 6$. Applying (2) with $(v, u) = (u_1, x)$, we have $a = (a + b - 6 + 12 - (a + b))/(a + b - 1) < 2$. This is impossible by Lemma 2.5(iii).

**Subcase 2:** $i = 5$, $d(x_1) = 6$, $d(x_2) = d(x_3) = 4$ and $d(x_4) = d(x_5) = 2$. By symmetry and Subcase 1, we have $d(y_2), d(y_4), d(z_2), d(z_4) \in \{3, 4\}$. For the pendant vertex $x$ adjacent to $x_2$, by (1) and (2), we have $S(x) = a + b = 4$, $S(x_1) = 2a + b = 6$. So we have $a = 2, b = 2$. For the vertex $u$, $S(x_1) = ad(x_4) + b = 14 = 4 + 4 + d(y_2) + d(y_4) + d(z_2) + d(z_4)$. So $d(y_2) + d(y_4) + d(z_2) + d(z_4) = 6$, which is impossible since $d(y_2), d(y_4), d(z_2), d(z_4) \in \{3, 4\}$.
Subcase 3: $i = 4$, $d(x_1) = 6$, $d(x_2) = d(x_3) = 3$ and $d(x_4) = 2$. By symmetry and Subcases 1 and 2, we have $j = l = 4$, $d(y_1) = d(y_2) = d(z_1) = d(z_2) = 3$ and $d(y_3) = d(z_3) = 2$. $G \cong H_2$, see Figure 3. It is easy to see that $H_2$ is 2-walk $(3,0)$-linear.

Case 2: $n \geq 2$. Then $G_0 = T_1$, see Figure 2. We consider the following two cases:

Subcase 1: $G$ has no pendant vertex. Then $S(x_1) = 2 + d(w_1) = S(x_3) = 5$ by (1) and (2). Hence $d(w_1) = 3$. It implies that $k \geq 2$. Similarly, we have $d(v_1) = 3$ and $m \geq 2$. By Lemma 2.4, we have $n, m, k \in \{2, 4\}$. Without loss of generality, suppose that $n \geq m \geq k$. If $n = 2$, then $m = k = 2$. Hence $S(u_1) = 7$, $S(u_2) = 9$ by (1). On the other hand, $d(u_1) = d(u_2) = 3$, so $S(u_1) = S(u_2)$ by (2), a contradiction. Hence $n \neq 4$. Similarly, we have $m = k = 4$. Therefore $G = H_3$, see Figure 3. By (3), $H_3$ is 2-2 walk $(1,3)$-linear.

Subcase 2: $G$ has at least one pendant vertex. In this case, we show that there is no such graph with exactly two main eigenvalues. Since $d_{G_0}(u_1) = 3$, we have $a = 2$ by Lemma 3.1 (iii). So $d(x_i) = 2$ for $i = 1, \ldots, 6$ by Lemma 3.1, (iii) and (iv).

We claim that $m, k \geq 2$. Otherwise, let $k = 1$. Then $S(x_1) = 2 + d(w_1) = S(x_3) = 2 + d(u_1)$ by (1) and (2). It follows from Lemma 2.5 (ii) and the fact that $d_{G_0}(u_1) \neq d_{G_0}(w_1)$ that $d(v_1) = d(u_1) = a + b$. Hence $S(v_1) = S(u_1)$ by (2). By (1),

$$S(u_1) = a + b - 3 + 4 + d(u_2), \quad S(w_1) = \begin{cases} a + b - 5 + 8 + d(u_{n-1}) & \text{if } m = 1, \\ a + b - 4 + 4 + d(v_{m-1}) + d(u_{n-1}) & \text{if } m \geq 2. \end{cases}$$

If $m = 1$, then $d(u_2) = d(u_{n-1}) + 2$. This is impossible by Lemma 3.1 (i). If $m \geq 2$, then $d(u_2) + 1 = d(v_{m-1}) + d(u_{n-1})$.

Obviously, $u_2 = u_{n-1}$ when $n = 3$ and $d(u_2) = d(u_{n-1})$ when $n \geq 4$ by Lemma 3.1(i). Hence $d(v_{m-1}) = 1$, it contradicts the fact that $d(v_{m-1}) \geq d_{G_0}(v_{m-1}) = 2$. Therefore $k \geq 2$. Dually, we have $m \geq 2$.

Note that $d(x_1) = d(x_3) = d(x_5)$, we have $S(x_1) = 2 + d(w_1) = S(x_3) = 2 + d(v_1) = S(x_5) = 2 + d(u_1)$ by (1) and (2). It implies that $d(v_1) = d(u_1) = d(u_3) = 3$ or $a + b$.

Let $d(u_1) = 3$. Applying (3) with $(v, u) = (x_5, x_i)$, we have $a = 5 - (a + b)$. So $a = 2$, $b = 1$. Furthermore, $S(u_1) = 4 + d(u_2) = 7$ by (1) and (2). Thus $d(u_2) = 3$. It follows from Lemma 3.1 (i) that $d(u_i) = 3$ for $2 \leq i \leq n - 1$. Similarly, we have $d(v_1) = d(w_1) = 3$ for $2 \leq i \leq n - 1$ and $2 \leq j \leq k - 1$. Note that $d_{G_0}(u_n) = a + b = 3$. We have $d(u_n) = 3$ and $S(u_n) = 7$ by (2). On the other hand, $S(u_n) = 9$ by (1), a contradiction.

If $d(u_1) = a + b \neq 3$, then $a + b \geq 4$ by Lemma 2.5 (iii). Applying (3) with $(v, u) = (u_1, x_1)$, we have $a = (a + b - 3 + 4 + d(u_2) - (a + b)) / (a + b - 1)$. Note that $d_{G_0}(u_1) = 2$ or 3, we have $d(u_2) = 2(a + b) - 3 \geq \max\{a + b, d_{G_0}(u_2)\}$. It contradicts Lemma 2.5 (ii).

Lemma 3.4.

Let $G \in \mathcal{S}$ with $\delta(G_0) \geq 2$ and $G_0 \in \mathcal{T}_2$, see Figure 1. Then $G = H_i$ for $i = 4, 5, 6, 7$, see Figure 3, or $G \in \mathcal{N}_j$ for $j = 1, 2$, see Figure 4.

Proof. If $G_0 \in \mathcal{T}_2$, then $d_{G_0}(w_1), d_{G_0}(r_1) \in \{3, 4\}$ and so each internal cycle of $G_0$ has the length equal to 3 by Lemmas 2.4 and 3.1 (iii). Hence $G_0 = T_2$ and $d(x_i) = 2$ for $i = 1, \ldots, 4$, see Figure 2, where $l, l \geq 1, n, m \geq 2$. For convenience, we set $w_k = v_1 = u_1$ and $v_n = u_n = r_1$. By (1) and (2), $S(x_1) = 2 + d(w_1) = S(x_3) = 2 + d(r_1)$. Hence $d(w_1) = d(r_1)$.

If $G$ has no pendant vertex, then $k, l \in \{1, 2, 4\}$ and $m, n \in \{2, 4\}$ by Lemma 2.4. First, let $k = 1$. Then $d(w_1) = d(r_1) = 4$. So $l = 1$. Therefore $G = H_i$, for $i = 4, 5$, see Figure 3. By (3), $H_i$ and $H_5$ are 2-2 walk $(2,2)$-linear and 2-2 walk $(1,4)$-linear, respectively. Next, let $k = 2$. Then $d(w_1) = d(r_1) = 3$. By (1) and (2), $S(r_1) = 4 + d(v_2) = S(w_1) = 7$. So $d(v_2) = 3$ and $l = 2$. Similarly, $S(u_1) = 3 + d(u_2) + d(v_2) = S(w_1) = 7$. So $d(u_2) = 3$. Note that $n, m = 2$ or 4, we have $n = m = 4$ and $d(u_3) = d(v_3) = 2$. Therefore, $G = H_6$, see Figure 3. By (3), $H_6$ is 2-2 walk $(2,1)$-linear. Finally, let $k = 4$. With a similar argument of the case $k = 2$, we have $G = H_7$ is 2-2 walk $(1,3)$-linear, see Figure 3.

If $G$ has at least one pendant vertex, we consider the following two cases:
Case 1: \( k = l = 1 \). Then \( d(w_1) = d(r_1) = 4 \) or \( a + b \) by Lemma 2.5 (ii).

Let \( d(w_1) = 4 \). Applying (3) with \( (v, u) = (x_1, x) \), we have \( a = 6 - (a + b) \). It follows from Lemmas 2.2 and 2.5 (iii) that \( a + b = 3 \) or \( 4 \). If \( a + b = 3 \), then \( a = 3, b = 0 \). So \( S(w_1) = 4 + d(u_2) + d(v_2) = 12 \) by (1) and (2). By Lemma 2.5 (ii), \( d(u_2), d(v_2) \in \{ 2, 4, a + b \} \). Thus \( d(u_2) = d(v_2) = 4 \). It implies that \( n = m = 2 \), which is impossible since \( G \) is simple. If \( a + b = 4 \), then \( a = 2, b = 2 \). So \( S(w_1) = 4 + d(u_2) + d(v_2) = 10 \) by (1) and (2). By Lemma 2.5 (ii), \( d(u_2), d(v_2) \in \{ 2, 4 \} \). Without loss of generality, suppose that \( d(u_2) = 2, d(v_2) = 4 \). Then \( d(v_i) = 4 \) for \( 2 \leq i \leq m - 1 \) by Lemma 3.1 (i). For the vertex \( u_2, S(u_2) = 4 + d(u_2) = 6 \) by (1) and (2). So \( d(u_2) = 2 \). Thus \( n \geq 4 \). Hence \( n = 4 \) by Lemma 3.1 (ii). Therefore \( G \in \mathcal{G}_1 \), see Figure 4, where \( l_1 \geq 1 \). It is easy to see that any graph \( G \in \mathcal{G}_1 \) is 2-walk \((2,2)\)-linear.

Let \( d(w_1) = a + b > d_{\mathcal{G}_0}(u_1) = 4 \). In this case, we show that there is no such graph with exactly two main eigenvalues. Applying (3) with \( (v, u) = (x_1, x) \) and \( (v, u) = (w_1, x) \), respectively, we have \( a = 2 \) and \( a = (a + b - 3 + 4 + d(w_2) - (a + b))/2 \). It implies that \( a + b \geq 4 \) and \( d_{\mathcal{G}_0}(w_2) = 4 \). We have \( d(w_2) = 2(a + b) - 3 > \max \{ a + b, d_{\mathcal{G}_0}(w_2) \} \). It contradicts Lemma 2.5 (ii). Hence \( d(w_1) = d(r_1) = 3 \). It implies that \( l \geq 2 \).

Next, applying (3) with \( (v, u) = (x_1, x) \), we have \( a = 5 - (a + b) \). So \( a + b = 3 \) and \( a = 2, b = 1 \). For the vertex \( w_1, S(w_1) = 4 + d(w_2) = 7 \) by (1) and (2). So \( d(w_2) = 3 \). Thus \( S(w_i) = 3 \) for \( 2 \leq i \leq k - 1 \) by Lemma 3.1 (i). Note that \( d_{\mathcal{G}_1}(w_i) = 3 \). We have \( d(w_k) = 3 \) by Lemma 2.5 (ii). Hence \( d(w_i) = 3 \) for \( 1 \leq i \leq k \). Similarly, we have \( d(r_i) = 3 \) for \( 1 \leq j \leq l \).

For the vertex \( w_k, S(w_k) = 3 + d(u_2) + d(v_2) = 7 \). Note that \( d(u_2), d(v_2) \in \{ 2, 3 \} \) by Lemma 2.5 (ii). We have \( d(u_2) = d(v_2) = 2 \). It follows from Lemmas 2.4 and 3.1 that \( m = n = 4 \) and \( d(u_2) = d(v_2) = 2 \). Therefore \( G \in \mathcal{G}_2 \), see Figure 4, where \( \max \{ k_1, k_2 \} \geq 1 \). It is easy to see that any graph \( G \in \mathcal{G}_2 \) is 2-walk \((2,1)\)-linear.

Lemma 3.5.
Let \( G \in \mathcal{G} \) with \( \delta(G_0) \geq 2 \) and \( G_0 \in \mathcal{J}_0 \), see Figure 1. Then \( G = H_b \), see Figure 3.
Proof. If \( G_0 \in \mathcal{T}_3 \), then \( d_{G_0}(r_1) \in \{3, 5\} \) and so the internal cycle of \( G_0 \) has the length equal to 3 by Lemmas 2.4 and 3.1. Hence \( \mathcal{G}_0 = T_3 \), see Figure 2, where \( n, m, k \geq 2, l \geq 1 \). For convenience, we set \( u_1 = v_1 = w_1 \) and \( u_n = v_n = w_k = r_1 \).

We first observe that \( G \) contains at least one pendant vertex. On the contrary, suppose that \( G \) has no pendant vertex. Then \( n, m, k, l \leq 4 \) by Lemma 2.4. Without loss of generality, suppose that \( n \geq m \geq k \). If \( l = 1 \), then \( S(x_1) = 7 \) and \( S(u_2) = 5 \) or 8 by (1). So \( S(u_2) \neq S(x_1) \). On the other hand, \( S(u_2) = S(x_1) \) by (2), a contradiction. If \( l \geq 2 \), then \( d(u_2) = 4 \), \( d(r_1) = 3 \). So \( S(x_1) = 5 \) and \( S(u_{n-1}) = 6 \) or 7 by (1). On the other hand, \( S(x_1) = S(u_{n-1}) \) by (2), a contradiction. Therefore, \( G \) has at least one pendant vertex. We consider the following two cases:

Case 1: \( l = 1 \). By Lemma 3.1(iv), \( d(x_1) = d(x_2) = 3 \) or \( d(x_1) = d(x_2) = 2 \).

If \( d(x_1) = d(x_2) = 3 \), then \( a = 3, b = 0, d(u_2) = 5 \) by Lemma 3.1(iv). So \( d(u_1) = 3 \neq d(u_2) \) by Lemma 2.5(ii). It follows from Lemma 3.1(i) that \( n, m, k \leq 3 \). Without loss of generality, suppose that \( n = m = 3, k = 2 \) or 3. Then \( S(u_3) = 6 + d(u_2) + d(v_2) + d(w_{k-1}) = 15 \) by (1) and (2). Note that \( d(u_3), d(v_2), d(w_{k-1}) = 2 \) or 3 by Lemma 2.5(ii). We have \( d(u_3) = d(v_2) = d(w_{k-1}) = 3 \). So \( S(u_3) = 6 + d(v_2) = 9 \) by (1) and (2). Thus \( d(v_2) = 3 \). It implies that \( k = 3 \). Therefore \( G = H_k \), see Figure 3. By (3), \( H_k \) is \( (3, 0) \)-linear.

If \( d(x_1) = d(x_2) = 2 \), we show that in this case there is no such graph with exactly two main eigenvalues. We consider the following two cases:

Subcase 1: \( \max \{n, m, k\} \geq 4 \). Without loss of generality, suppose that \( n \geq 4 \). Then \( d(u_1) = d(u_n) \) and \( d(u_2) = d(u_n) \) for \( 2 \leq i \leq n - 1 \) by Lemma 3.1(i). Note that \( d_{G_0}(u_1) \neq d_{G_0}(u_n) \). We have \( d(u_3) = d(u_n) = a + b \geq d_{G_0}(u_n) = 5 \) by Lemma 2.5(ii). Hence

\[
S(u_1) = a + b - 3 + d(v_2) + d(v_3), \quad S(u_n) = a + b - 5 + 4 + d(u_{n-1}) + d(v_{n-1}).
\]

We next observe that \( d(v_2) = d(v_{n-1}) \). If \( m = 2 \), then \( v_2 = u_n, v_{n-1} = u_1 \). So \( d(v_2) = d(u_n) = d(u_1) = d(v_{n-1}) \). If \( m = 3 \), clearly, \( d(v_2) = d(v_{n-1}) \). If \( m \geq 4 \), then \( d(v_2) = d(v_{n-1}) \) by Lemma 3.1(i). Therefore \( d(v_2) = d(v_{n-1}) \) for all \( m \geq 2 \). Similarly, we have \( d(v_2) = d(v_{n-1}) \) for all \( k \geq 2 \). Hence \( S(u_1) \neq S(u_n) \). On the other hand, \( S(u_1) = S(u_n) \) by (2), a contradiction.

Subcase 2: \( \max \{n, m, k\} \leq 3 \). Without loss of generality, suppose that \( n = m = 3 \) and \( k = 2 \) or 3. We claim that \( d(u_2) = a + b \). Otherwise, let \( d(u_2) = d_{G_0}(u_2) = 2 \). Then \( d(u_1) + d(u_3) = S(u_3) = S(x_1) = 2 + d(u_3) \) by (1) and (2), which is impossible since \( d(u_1) \geq d_{G_0}(u_1) = 3 \). Hence \( d(u_2) = a + b \). Similarly, we have \( d(v_2) = a + b \).

Note that \( d(u_1) \in \{a + b, 5\} \) by Lemma 2.5. Let \( d(u_1) = a + b \geq d_{G_0}(u_3) = 5 \). Applying Lemma 2.5(iv) with \( C = u_1u_2v_2u_1v_1 \), we have \( d(u_1) \neq a + b \). So \( d(u_1) = 3 \). From (3) with \( (v, u) = (u_1, x) \) and \( (v, u) = (x_1, x) \), we get \( a = (a + b + d(u_3))/2 \) and \( a = 2 \), respectively. Hence \( d(v_2) = 4 - (a + b) < 0 \), a contradiction. If \( d(u_1) = d_{G_0}(u_1) = 5 \neq a + b \), then \( a = 7 - (a + b) \) by applying (3) with \( (v, u) = (x_1, x) \). It follows from Lemma 2.5(iii) that \( a + b = 3 \) or 4. If \( a + b = 3 \), then \( a = 4, b = 1 \). Hence \( S(u_2) = d(u_1) + 6 = 11 \) by (1) and (2). This is impossible since \( d(u_1) = 3 \) by Lemma 2.5(ii). If \( a + b = 4 \), then \( a = 3, b = 1 \). Thus \( S(u_2) = d(u_1) + 7 = 13 \) by (1) and (2). This is also impossible since \( d(u_1) = 3 \) or 4.

Case 2: \( l \geq 2 \). We show in this case there is no such graph with exactly two main eigenvalues. By Lemma 3.1(iii), we have \( a = 2 \) and \( d(x_1) = d(x_2) = 2 \). Applying (3) with \( (v, u) = (x_1, x) \), we have \( 2 = 2 + d(r_1) - (a + b) \). So \( d(r_1) = a + b \). From (3) with \( (v, u) = (r_1, x) \) we get \( 2 = (a + b + 3 + 4 + d(r_2) - (a + b))/2 \). Hence \( d(r_2) = 2(a + b) - 3 \). By Lemma 2.5(ii), \( d(r_2) \in \{2, 4, a + b\} \). If \( d(r_2) = 2 \) or 4, then \( a + b = b \) is not an integer, a contradiction. If \( d(r_2) = a + b \), then \( a + b = 3 \). So \( a = 2, b = 1 \). By Lemmas 2.5(ii) and 3.1(i), we have \( d(r_1) = 4 \) and \( d(r_{n-1}) = d(r_2) = 3 \). Hence \( S(r_1) = 3 + d(u_{n-1}) + d(v_{n-1}) + d(v_{n-1}) = 9 \) by (1) and (2). It follows that \( d(u_{n-1}) = d(v_{n-1}) = 2 \). Thus \( S(u_{n-1}) = 5 \) by (2). On the other hand, \( S(u_{n-1}) = 4 + d(u_{n-2}) \geq 6 \) by (1), a contradiction. \( \square \)

Lemma 3.6.

Let \( G \in \mathcal{G} \) with \( \delta(G_0) \geq 2 \) and \( G_0 \in \mathcal{T}_3 \), see Figure 1. Then \( G = H_i \) for \( i = 9, 10, 11 \), see Figure 3, or \( G \in \mathcal{G}_{j_3} \), see Figure 4.
**Proof.** If $G_0 \in \mathcal{T}_5$, then $d_{G_0}(v_1) \in \{3, 4\}$ and so the internal cycle of $G_0$ has the length equal to 3 by Lemmas 2.4 and 3.1 (iii). Hence $G_0 = T_3$, see Figure 2, where $k \geq 1$ and $p, q, n, m \geq 2$. For convenience, we set $u_i = v_i = s_i$, $u_n = v_n = t_1$ and $s_p = t_q = w_k$.

If $G$ has no pendant vertex. Without loss of generality, we may assume that $n \geq m$, $p \geq q$ and consider the following two cases:

**Case 1:** $k = 1$. Then $S(v_1) = 6$ by (1). We claim that $p = 2$. Otherwise, let $p \geq 3$. Then $d(s_2) = 2$ and $S(s_2) = 5$ or 7 by (1). So $S(v_1) \neq S(s_2)$. On the other hand, $S(v_1) = S(s_2)$ by (2), a contradiction. Hence $p = 2$. Similarly, we have $q = 2$, $n = 3$, $m = 2$ or 3. If $m = 2$, then applying (3) with $(v, u) = (w_1, u_2)$ and $(v, u) = (w_1, u_1)$, respectively, we have $a = 2$ and $a = 1$, respectively. A contradiction. If $m = 3$, then $G = H_0$, see Figure 3. By (3), $H_0$ is 2-walk $(2, 2)$-linear.

**Case 2:** $k > 1$. By Lemma 2.4, we have $k, p, q, m, n = 2$ or 4.

If $k = 2$, then $S(w_1) = 7 = S(v_2) = 3 + d(s_{p-1}) + d(t_{q-1})$ by (1) and (2). So $d(s_2) = d(t_2) = 2$. It implies that $p = q = 4$. Hence $n = 4$, $m = 2$. Otherwise, let $m = n = 4$. Then $S(u_1) = 6 \neq S(w_1) = 7$ by (1). On the other hand, $S(u_1) = S(w_1)$ by (2), a contradiction. Therefore $n = 4$, $m = 2$ and $G = H_0$, see Figure 3. By (3), $H_0$ is 2-walk $(2, 1)$-linear. If $k = 4$, with a similar argument, we have $G = H_1$ is 2-walk $(1, 3)$-linear, see Figure 3.

If $G$ has at least one pendant vertex, then $k > 1$. Otherwise, suppose that $k = 1$. Then $d(w_1) = 4$ or $a + b$ by Lemma 2.5 (ii). Let $d(w_1) = 4 \neq a + b$. Applying (3) with $(v, u) = (x_1, x)$ we get $a = 6 - (a + b)$. It follows from Lemma 2.5 (iii) that $a = 3$, $b = 0$. So $S(w_1) = 4 + d(s_{p-1}) + d(t_{q-1}) = 12$ by (1) and (2). This is impossible since $d(s_{p-1}), d(t_{q-1}) = 2$ or 3 by Lemma 2.5 (ii). Let now $d(w_1) = a + b \geq 4$. Apply (3) with $(v, u) = (x_1, x)$ and $(v, u) = (w_1, x)$.

We get $a = 2$ and $a = \{(a + b - 4 + d(t_{q-1}) + d(s_{p-1}) - (a + b))\}/(a + b - 1)$, respectively. Thus

$$d(t_{q-1}) + d(s_{p-1}) = (a + b) - 2.$$ (6)

Note that $d(t_{q-1}), d(s_{p-1}) \in \{2, 3, a + b\}$. We consider the following five cases by symmetry:

- If $d(t_{q-1}) = d(s_{p-1}) = 2$, then $a + b = 3$. This contradicts the fact that $a + b \geq 4$.
- If $d(t_{q-1}) = 2$, $d(s_{p-1}) = 3$, then $a + b = 3.5$. This contradicts Lemma 2.2.
- If $d(t_{q-1}) = 2$, $d(s_{p-1}) = a + b$, then $a + b + 4$ by (6). Note that $a = 2$. We have $b = 2$ and $d(s_{p-1}) = d(w_1) = 4$. By Lemma 3.1, $d(s_1) = 4$ for $2 \leq i \leq p - 1$. So $S(s_2) = 6 + d(u_1) = 10$ by (1) and (2). Thus $d(u_1) = 4$. Similarly, we have $d(u_n) = 4$. By (1) and (2), $S(u_1) = 5 + d(u_2) + d(v_2) = 10$. This is impossible since $d(u_2), d(v_2) = 2$ or 4.
- If $d(t_{q-1}) = 3$, $d(s_{p-1}) = a + b$, then $a + b = 5$ by (6). Recall that $a = 2$, we have $b = 3$ and $d(w_1) = 5$. Note that $d(t_{q-1}) = 3 \neq a + b$. We have $q = 2$. Thus $S(t_{q-1}) = 9 = 5 + d(u_{n-1}) + d(v_{n-1})$ by (1) and (2). By Lemma 2.5 (ii), $d(u_{n-1}), d(v_{n-1}) \in \{2, 3, 5\}$. Hence $d(u_{n-1}) = d(v_{n-1}) = 2$. Therefore $S(u_{n-1}) = 3 + d(u_{n-2}) = 7$, which is impossible since $d(u_{n-2}) \in \{2, 3, 5\}.

- If $d(t_{q-1}) = d(s_{p-1}) = a + b$, then $2(a + b) = 2(a + b) = 2$ by (6), a contradiction.

Hence $k > 1$. It follows from Lemma 3.1 (iii) that $a = 2$ and $d(x_1) = d(x_2) = 2$. From (3) with $(v, u) = (x_1, x)$ it follows

$$2 = 2 + d(w_1) - (a + b).$$

So $d(w_1) = a + b$. By (3) with $(v, u) = (w_1, x), 2 = (a + b - 3 + 4 + d(w_2) - (a + b))/|a + b - 1|$. Thus $d(w_2) = (a + b) - 3 \geq 3$. Note that $d(w_2) \in \{2, 3, a + b\}$ by Lemma 2.5 (ii). We have $d(w_2) = 3$ or $a + b$. It follows that $a + b = 3$ and $a = 2, b = 1$.

By Lemma 3.1 (i), $d(w_1) = d(w_2) = 3$ for $2 \leq i \leq k - 1$. By Lemma 2.5 (ii), $d(w_1) = d(w_2) = d(u_1) = d(u_n) = 3$. For the vertex $w_i$, we have $S(w_i) = 3 + d(s_{p-1}) + d(t_{q-1}) = 7$. It follows from Lemma 2.5 (ii) that $d(s_{p-1}) = d(t_{q-1}) = 2$ and $p, q \geq 3$. Hence $p = q = 4$ and $d(s_2) = d(t_2) = 2$ by Lemmas 2.4 and 3.1 (ii). For the vertex $u_1$, we have $S(u_1) = 2 + d(v_2) + d(u_2) = 7$. Note that $d(u_2), d(v_2) \in \{2, 3\}$. We may assume that $d(v_2) = 2, d(u_2) = 3$ by symmetry. It follows from Lemmas 2.4 and 3.1 (ii) that $m = 4, d(v_3) = 2$ and $d(u_3) = 3$ for $2 \leq i \leq n - 1$. Therefore $G \in \mathcal{G}_3$, see Figure 4, where $\max\{k_1, k_2\} \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_3$ is 2-walk $(2, 1)$-linear.

**Lemma 3.7.** There is no graph $G \in \mathcal{G}$ with $\delta(G_0) \geq 2$ and $G_0 \in \mathcal{T}_5$, see Figure 1.
**Proof.** By way of contradiction, suppose that $G \in \mathcal{S}$ with $G_0 \in \mathcal{T}_5$. Let $G_0 = T_5$, see Figure 2, where $n, m, k, p, q \geq 2$. For convenience, we set $u_1 = v_1 = s_1$, $u_n = w_k = t_q$, $v_m = s_p = w_1 = t_1$, and consider the following two cases:

**Case 1:** there is a vertex $v \in \{v_{m-1}, s_{p-1}, w_2, t_2\}$ such that $d_{G_0}(v) = 2$ and $d(v) = a + b$. Without loss of generality, let $v = v_{m-1}$. Then $d(v_i) = a + b$ for $2 \leq i \leq m - 1$ by Lemma 3.1 (i). In particular, $a \geq 2$, $a + b \geq 3$ by Lemma 2.5 (iii).

We first show that $d(u_1) = d(w_1) = a + b$.

If $m \geq 4$, then $d(u_1) = d(w_1)$ by Lemma 3.1 (i). Note that $d_{G_0}(u_1) \neq d_{G_0}(w_1)$. We have $d(u_1) = d(w_1) = a + b$ by Lemma 2.5 (ii). Let now $m = 3$. From (3) with $(v, u) = (v_2, x)$,

$$a = \frac{a + b - 2 + d(u_1) + d(w_1) - (a + b)}{a + b - 1} = \frac{d(u_1) + d(w_1) - 2}{a + b - 1}.$$  

By Lemma 2.5 (ii), $d(u_1) = 3$ or $a + b$ and $d(w_1) = 4$ or $a + b$.

If $d(u_1) = 3$, $d(w_1) = 4$, then $a = 5/(a + b - 1)$. Note that $a \geq 2$ is an integer. We have $a + b = 2$. It contradicts the fact that $a + b \geq 3$. If $d(u_1) = 3$, $d(w_1) = a + b \geq 4$, then $a = 1 + 2/(a + b - 1) < 2$, a contradiction. If $d(u_1) = a + b \geq 3$, $d(w_1) = 4 \neq a + b$, then $a = 1 + 3/(a + b - 1)$ is not an integer, a contradiction. Hence $d(u_1) = d(w_1) = a + b$.

By Lemma 2.5 (iv) applied to $C = u_1v_2 \ldots u_{m-1}w_1s_{p-1} \ldots s_2w_1$, we get $p \geq 3$ and $d(s_1) 
\neq a + b$ for some $2 \leq i \leq p - 1$. It follows from Lemmas 2.5 (ii) and 3.1 (i) that $d(s_i) < 2$ for $2 \leq i \leq p - 1$. By (3) with $(v, u) = (v_2, x)$ and $(v, u) = (u_1, x)$, we have $a = 2$ and $a = 1 + d(u_2)/(a + b - 1)$, respectively. This together with Lemma 2.5 (i) and the fact that $a + b \geq d_{G_0}(w_1) = 4$ implies that $d(u_2) = a + b - 1 = d_{G_0}(u_2) \geq 3$. Hence $n = 2$, $d(u_2) = d_{G_0}(u_2) = 3$. Therefore $a + b = 4$ and $a = b = 2$. By (1) and (2), $S(u_2) = 4 + d(w_{k-1}) + d(t_{q-1}) = 8$. By Lemma 2.5 (ii), $d(w_{k-1}) + d(t_{q-1}) = 2$ or 4. Thus $d(w_{k-1}) + d(t_{q-1}) = 2$. Hence $S(w_{k-1}) = 3 + d(w_{k-2}) = 6$ by (1) and (2), which is impossible since $d(w_{k-2}) = 2$ or 4.

**Case 2:** for any vertex $v \in \{v_{m-1}, s_{p-1}, w_2, t_2\}$, $d_{G_0}(v) \neq 3$ or $d(v) = 2$. It follows from Lemma 2.4 that $m, k, p, q \leq 4$. Without loss of generality, suppose that $m \geq p, k \geq q$. Hence $m, k = 3$ or 4.

**Subcase 1:** max $\{m, k\} = 4$. Without loss of generality, suppose that $m = 4$. Then $d(u_1) = d(w_1)$ by Lemma 3.1 (i). Note that $d_{G_0}(u_1) \neq d_{G_0}(w_1)$. We have $d(u_1) = d(w_1) = a + b \geq d_{G_0}(w_1) = 4$ by Lemma 2.5 (ii). Thus $G$ has at least one pendant vertex. Apply (3) with $(v, u) = (v_2, x)$ and $(v, u) = (u_1, x)$. We have $a = 2$ and $a = \{a + b - 3 + 2 + d(s_2) + d(u_2) - (a + b)/((a + b - 1)$, respectively. Hence $d(s_2) + d(u_2) = 2(a + b - 1) - 1$.

If $p > 2$, then $d(s_1) = 2$. It follows from the fact that $d_{G_0}(u_2) = 2$ or 3 and $a + b \geq 4$ that $d(u_2) = 2(a + b - 3 > max \{a + b, d_{G_0}(u_2)\}$, which is impossible by Lemma 2.5 (ii). If $p = 2$, then $d(s_2) = d(w_1) = a + b$. So $d(u_2) = a + b - 1 \geq 3$. It follows that $n = 2$, $d(u_2) = d_{G_0}(u_2) = 3$ and $a + b = 4$. Hence $a = b = 2$. By (2), $S(w_{k-1}) = 6$. On the other hand, $S(w_{k-1}) = d(w_{k-2}) + d(u_2) = 5$ or 7 by (1), a contradiction.

**Subcase 2:** $m = k = 3$. Then $d(u_1) + d(u_1) = S(v_2) = S(w_2) = d(w_1) + d(u_1)$ by (1) and (2). So $d(u_1) = d(u_2)$. Thus $S(u_1) = S(u_2)$ by (2).

We claim that $p = q = 2$ or 3. Otherwise, let $p \neq q$. Without loss of generality, suppose that $p = 3, q = 2$. Then $S(u_1) = d(u_1) - 3 + 4 + d(u_2)$ and $S(u_2) = d(u_1) - 3 + 2 + d(w_1) + d(w_{k-1})$. Note that $d(u_2) = d(w_{k-1}), d(w_1) > 2$. We have $S(u_1) \neq S(u_2)$, a contradiction. Hence $p = q = 3$. We consider the following two cases:

**Subcase 2.1:** G has no pendant vertex. Then $n = 2$. Otherwise, suppose that $n > 2$. Then $d(u_2) = d(v_2) = 2$. By (1) and (2), $3 + d(u_3) = S(u_2) = S(v_2) = 7$. This is impossible since $d(u_3) = 2$ or 3. Hence $n = 2$. Let $p = q = 2$. Apply (3) for $(v, u) = (u_1, v_2)$ and $(v, u) = (w_1, u_1)$, respectively, then $a = 2$ and $a = 1$, respectively, a contradiction. If $p = q = 3$, also applying (3) with $(v, u) = (u_1, v_2)$ and $(v, u) = (w_1, u_1)$, respectively, we have $a = 0$ and $a = 1$, respectively. Also a contradiction.

**Subcase 2.2:** G has at least one pendant vertex. Then $a \geq 2$, $a + b \geq 3$. First, let $p = q = 3$. Applying (3) with $(v, u) = (w_1, x)$, we get $a = (d(w_1) - 4 + 8 - (a + b))/(d(w_1) - 1)$. By Lemma 2.5 (ii), $d(w_1) = a + b$ or 4. It follows that $a < 2$, a contradiction.

Next, suppose that $p = q = 2$. By Lemma 2.5 (ii), $d(w_1) = a + b$ or 4. If $d(w_1) = a + b \geq 4$, applying (3) with $(v, u) = (w_1, x)$ and noting that $d(u_1) = d(u_2)$, we have $a = 2d(u_1) / (a + b - 1)$. If $d(u_1) = a + b$, then $a = 2 + 2/(a + b - 1)$ is not an
integer, which is impossible by Lemma 2.2. If \( d(u_1) = d_G(u_1) = 3 \neq a + b \), then \( a + b = 4 \) and \( a = b = 2 \). So \( S(v_2) = 6 \) by (2). On the other hand, \( S(v_2) = d(u_1) + d(w_1) = 7 \) by (1), a contradiction.

If \( d(w_1) = 4 \neq a + b \), then applying (3) with \((v, u) = (v_2, x)\), we have \( a = 4 + d(u_1) - (a + b) \). If \( d(u_1) = a + b \), then \( a = 4 \). For the vertex \( u_1 \), we have \( S(u_1) = 4(a + b) + b = a + b - 3 + 6 + d(u_2) \). It follows that \( d(u_2) = 4(a + b) - 7 > \max \{a + b, d_G(u_1)\} \), which is impossible by Lemma 2.5 (ii). If \( d(u_1) = 3 \neq a + b \), then \( a = 7 - (a + b) \). Note that \( a + b \neq 3, 4 \). We have \( a + b = 5 \) and \( a = 2, b = 3 \) by Lemma 2.5 (iii). By (1) and (2), \( S(w_1) = 11 = 4 + d(u_1) + d(u_4) \). This is impossible since \( d(u_1) = d(u_6) \).

\[ \square \]

**Lemma 3.8.**
Let \( G \in \mathcal{S} \) with \( \delta(G_0) \geq 2 \) and \( G_0 \in \mathcal{T}_6 \). Then \( G = H_i \) for \( i = 12, \ldots, 17 \), see Figure 3, or \( G \in \mathcal{S}_8 \), see Figure 4.

**Proof.** Let \( G_0 = T_6 \), see Figure 2, where \( n, m, p, k \geq 2 \). For convenience, we set \( u_1 = v_1 = w_1 = s_1 \) and \( u_n = v_n = w_k = s_k \).

**Case 1:** there is a vertex \( v \in \{u_2, v_2, w_2, s_2\} \) such that \( d_G(v) = 2 \) and \( d(v) = a + b \). Without loss of generality, suppose that \( v = u_2 \). From (3) with \((v, u) = (u_2, x)\), we get

\[
a = \frac{a + b - 2 + d(u_1) + d(u_3) - (a + b)}{a + b - 1} = \frac{d(u_1) + d(u_3) - 2}{a + b - 1}.
\]

By Lemma 2.5 (ii), \( d(u_1), d(u_3) = 4 \) or \( a + b \). We consider the following three cases:

**Subcase 1:** \( d(u_1) = d(u_3) = a + b \). Then \( a = 2 \) by (7).

We now show that \( d(u_i) = a + b \) for \( 1 \leq i \leq n \). If \( n = 3 \), then obviously, \( d(u_i) = a + b \) for \( 1 \leq i \leq n \). If \( n \geq 4 \), then \( d(u_i) = d(u_{i+1}) = a + b \) for \( 1 < i < n \) and \( d(u_i) = d(u_1) = a + b \) by Lemma 3.1 (i). Hence \( d(u_i) = a + b \) for \( 1 \leq i \leq n \).

By Lemma 2.5 (iv) applied to \( C = u_1u_2 \ldots u_nv_{n-1} \ldots u_2v_1 \), we have \( d(v_i) \neq a + b \) for some \( 2 \leq i \leq m - 1 \). It follows from Lemma 2.5 (ii) and 3.1 (i) that \( m > 3 \) and \( d(v_2) = 2 \) for all \( 2 \leq i \leq m - 1 \). Thus \( S(v_2) = 2a + b = a + b + d(v_3) \).

Note that \( a = 2 \) and \( d(v_3) = 2 \), so \( m \geq 4 \). It follows from Lemma 3.1 (ii) that \( m = 4 \). Similarly, we have \( k = p = 4 \) and \( d(v_2) = d(v_3) = d(s_3) = 2 \). For the vertex \( u_1 \), \( S(u_1) = 3(a + b) + b = a + b - 4 + 6 + a + b \) by (1) and (2), hence \( b = 2 \). It follows that \( d(u_1) = a + b = 4 \) for \( 1 \leq i \leq n \). Therefore \( G \in \mathcal{S}_8 \), see Figure 4, where \( l_1 \geq 1 \). It is easy to see that any graph \( G \in \mathcal{S}_8 \) is \( 2 \)-walk (2, 2)-linear.

**Subcase 2:** \( d(u_1) = a + b, d(u_4) = 4 \) or \( d(u_1) = 4, d(u_4) = a + b \). Then \( a = 1 + 3(a + b - 1) \) is not an integer by (7), a contradiction.

**Subcase 3:** \( d(u_1) = d(u_4) = 4 \neq a + b \). Then \( n = 3 \) and \( a = 6(a + b - 1) \). It follows from Lemma 2.5 (iii) that \( a + b = 3 \) and \( a = 3, b = 0 \). Thus \( d(u_i) = a + b = 3 \) for the vertex \( u_1 \), \( S(u_1) = 12 = 3d(v_2) + d(w_1) + d(s_2) \) by (1) and (2). Since \( d(v_2), d(w_1), d(s_2) \in \{2, 4, a, b\} \). By symmetry, we have either \( d(v_2) = d(w_1) = d(s_2) = a + b = 3 \) or \( d(v_2) = 2, d(w_1) = 3, d(s_2) = 4 \).

If \( d(v_2) = d(w_1) = d(s_2) = 3 \), then \( S(v_2) = 9 = 5 + d(v_4) \). Thus \( d(v_4) = 4 \neq a + b \). It implies that \( m = 3 \). Similarly, we have \( k = p = 3 \). Therefore \( G = H_{13} \), see Figure 3. By (3), \( H_{13} \) is \( 2 \)-walk (3, 0)-linear. If \( d(v_2) = 2, d(w_1) = 3, d(s_2) = 4 \), then \( p = 2 \) and \( S(w_1) = 9 = 4 + 1 + d(w_3), S(v_2) = 6 = 4 + d(v_3) \). So \( d(v_1) = 4 \) and \( d(v_3) = 2 \). This implies that \( k = 3 \) and \( S(v_3) = 6 = 2 + d(v_4) \). Thus \( d(v_4) = 4 \) and \( m = 4 \). Thus \( G \cong H_{13} \), see Figure 3. It is easy to see that \( H_{13} \) is \( 2 \)-walk (3, 0)-linear.

**Case 2:** for any vertex \( v \in \{u_2, v_2, w_2, s_2\} \), \( d_G(v) = 4 \) or \( d(v) = 2 \). By Lemma 2.4, \( n, m, p, k \leq 4 \). Without loss of generality, suppose that \( n \geq m \geq p \geq k \). Then \( n \geq m \geq p \geq 3 \) and \( d(u_2) = d(v_2) = d(s_2) = 2 \). By (1) and (2), \( S(u_2) = d(u_1) + d(u_3) = S(v_2) = d(u_1) + d(v_3) \), Thus \( d(v_2) = d(u_3) \). It implies that \( m = n \). Similarly, we have \( p = n \) and \( k = n \). Hence \( k = p = m = n \in \{3, 4\} \) or \( k = 2, p = m = n \in \{3, 4\} \).

If \( G \) has no pendant vertex, then \( G = H_i \) for \( i = 14, \ldots, 17 \), see Figure 3. By (3), \( H_{14} \) is \( 2 \)-walk (1, 6)-linear, \( H_{15} \) is \( 2 \)-walk (0, 8)-linear, \( H_{16} \) is \( 2 \)-walk (2, 2)-linear and \( H_{17} \) is \( 2 \)-walk (1, 4)-linear.

If \( G \) has at least one pendant vertex \( x \), then \( x \in N(u_1) \) or \( x \in N(u_n) \). Without loss of generality, suppose that \( x \in N(u_1) \), then \( d(u_1) = a + b \geq 5 \). Applying (3) with \((v, u) = (u_1, x)\), we have \( a = d(w_2) + 2(a + b - 1) \). By Lemma 2.5 (ii), \( d(w_2) = 2, 4 + a + b \). It implies that \( a < 2 \). This is impossible by Lemma 2.5 (iii).
Lemma 3.9. Let $G \in \mathcal{S}$ with $\delta(G_0) \geq 2$ and $G_0 \in \mathcal{T}_j$, see Figure 1. Then $G = H_j$ for $i = 18 \ldots, 25$, see Figure 3, or $G \in \mathcal{S}_j$, for $j = 5, 6, 7$, see Figure 4.

Proof. Let $G_0 = T_j$, see Figure 2, where $n, m, k, l, p, q \geq 2$. For convenience, we set $u_1 = w_1 = s_1, u_n = w_k = t_1, v_1 = r_1 = s_p$ and $v_n = r_1 = t_q$.

If $G$ has no pendant vertex, then $n, m, k, l, p, q \leq 4$ by Lemma 2.4. Without loss of generality, suppose that $n \geq k, m \geq l, p \geq q$, then $n = 3$ or $4$. By (1) and (2), $S(u_1) = d(u_1) + d(w_2) + d(s_2) = S(u_n) = d(u_n-1) + d(w_{k-1}) + d(t_2)$. Note that $d(u_2) = d(u_{n-1}), d(w_2) = d(w_{k-1})$. We have $S(s_2) = d(t_2)$. It implies that $p = q$. Similarly, we have $k = l$. If $n = 3$, then $m = 3, k, p = 2$ or $3$ by Lemma 2.4. Hence $G = H_j$ for $i = 18, 19, 20, 21$, see Figure 3. If $n = 4$, then $m = 4, k, p = 2$ or $4$ by Lemma 2.4. Hence $G = H_j$ for $j = 22, 23, 24, 25$, see Figure 3. By (3), $H_{18}$ is $2$-walk $(2, 2)$-linear, $H_{20}$ and $H_{22}$ are $2$-walk $(1, 4)$-linear, $H_{21}$ is $2$-walk $(0, 6)$-linear, $H_{23}$ and $H_{24}$ are $2$-walk $(2, 1)$-linear, and $H_{25}$ is $2$-walk $(1, 3)$-linear.

If $G$ has at least one pendant vertex, then $a \geq 2, a + b \geq 3$. We first show that $d(v_1) = d(v_n)$. If $m \{m, l\} \geq 4$, then $d(v_1) = d(v_n)$ by Lemma 3.1 (i). Let $m, l \leq 3$. By way of contradiction, suppose that $d(v_1) \neq d(v_n)$. By Lemma 2.5 (ii), we may assume that $d(v_1) = a + b > 3, d(v_n) = 3$ without loss of generality. By (3) with $(v, u) = (v_1, v_3)$,

$$a = \begin{cases} d(s_{p-1}) - d(t_{q-1}) \over a + b - 3 & \text{if } m = 3, l = 2 \text{ or } m = 2, l = 3, \\ 1 + d(s_{p-1}) - d(t_{q-1}) \over a + b - 3 & \text{if } m = 3, l = 3. \\ 
\end{cases}$$

By Lemma 2.5 (ii), $d(s_{p-1}), d(t_{q-1}) = 2, 3$ or $a + b$.

If $m = 3, l = 2$ or $m = 2, l = 3$, then $d(s_{p-1}) = a + b, d(t_{q-1}) = 2$ since $a \geq 2, a + b \geq 3$. So $a = 1 + 1/(a + b - 3)$. It implies that $a + b = 4$ and $a = b = 2$. By (1) and (2), $S(v_1) = 6 + d(v_2) = 8$. So $d(v_2) = 2$ and $S(v_2) = 6$ by (2). On the other hand, $S(v_2) = 7$ by (1), a contradiction.

If $m = l = 3$, then $d(s_{p-1}) = a + b \neq 3, d(t_{q-1}) = 3$ or $d(s_{p-1}) = a + b, d(t_{q-1}) = 2$ since $a \geq 2, a + b \geq 3$. If $d(s_{p-1}) = a + b \neq 3, d(t_{q-1}) = 3$, then $a + b \geq 4$ and $a = 2$. Thus $b \geq 2$. By (1) and (2), $S(v_1) = 6 + d(v_2) = 8$. Since $d(v_2) = 2$ or $d(v_2) = 3$ by Lemma 2.5 (ii), we have $d(v_2) = d(v_2) = 2$. Thus $S(v_2) = 13$ by (2). On the other hand, $S(v_2) = 9$ by (1), a contradiction. Hence $d(v_1) = d(v_n)$ by Lemma 2.5 (ii), we consider the following two cases.

Case 1: $d(v_1) = d(v_n) = 3 \neq a + b$. Then $a + b \geq 4$. Note that max $\{m, l\} \geq 3$. Without loss of generality, suppose that $m \geq 3$. By Lemma 2.5 (ii), we have $d(v_2) = a + b + 2$. Let $d(v_2) = a + b$. From (3) with $(v, u) = (v_2, x)$, it follows $a = (d(v_3) + 1)/(a + b - 1)$. By Lemma 2.5 (ii), $d(v_3) = 2, 3$ or $a + b$. This together with $a + b \geq 4$ implies that $a < 2$. It contradicts the fact that $a \geq 2$. Let now $d(v_2) = 2$. By (3) with $(v, u) = (v_2, x), a = 3 + d(v_3) - (a + b)$. So $d(v_3) = a + a + b - 3 \geq 3$. If $m \geq 3$, then by Lemma 3.1 (i), $d(v_1) = d(v_3) = 2$. Thus $m = 3$. By assumption, $d(v_3) = 3$ and $a = 6 - (a + b)$. Note that $a + b \geq 4, a \geq 2$, hence $a + b = 4$ and $a = b = 2$. By (1) and (2), $S(v_1) = 2 + d(v_2) + d(s_{p-1}) = 6$. So $d(v_2) = d(s_{p-1}) = 6$.

If $l = 2$, then $d(v_2) = d(v_3) = 3$. So $d(s_{p-1}) = 3 \in \{a + b, 2\}$. It implies that $p = 2$ and $d(u_1) = d(s_{p-1}) = 3$. Similarly, we get $q = 2$ and $d(u_n) = 3$. By (1) and (2), $S(u_1) = 3 + d(w_2) + d(u_2) = 8$. Note that $d(w_2), d(u_2) = 2, 3$ or $4$ by Lemma 2.5 (ii). It follows that $d(u_2) = 2, d(w_3) = 3$ or $d(u_2) = 3, d(w_3) = 2$. We may suppose that $d(u_2) = 2, d(w_3) = 3$ without loss of generality. Then $k = 3$ and $d(u) = 2$ for $2 \leq i \leq n - 1$ by Lemma 3.1 (i). Hence $G$ has no pendant vertex. This is a contradiction.

If $l \geq 3$, then $d(v_3) = 2$ or $4$. We claim that $d(v_3) = 2$. Otherwise, let $d(v_3) = 4$, then $S(v_3) = 10$ by (2). This is also impossible since $S(v_3) = 4 + 2 + 3 + d(r_i) \leq 9$ by (1). Hence $d(v_3) = 2$. By (1) and (2), we have $S(v_3) = 2d(v_3) + 2 = 6 = d(v_1) + d(r_1) = 3 + d(v_1)$. So $d(r_1) = 3 \neq a + b$ and hence $l = 3$. Furthermore, we have $S(v_1) = 2d(v_1) + 2 = 8 = d(v_2) + d(r_1) + d(s_{p-1}) = 4 + d(s_{p-1})$. Thus $d(s_{p-1}) = 4$. Thus $S(s_{p-1}) = 10$ by (2). If $p \geq 2$, then $S(s_{p-1}) = 4 + 2 + 3 + d(s_{p-2}) \leq 9$ by (1). A contradiction. So $p = 2$ and $s_{p-1} = u_1$. Thus $d(u_1) = d(s_{p-1}) = 4$. 
Furthermore, by (1) and (2), we have $S(u_1) = 2d(u_1) + 2 = 10 = 1 + 3 + d(u_2) + d(w_2)$. Hence $d(u_2) + d(w_2) = 6$. Note that $d(u_3) = d(u_1) = 4$ and $a + b = 4$, then by Lemma 25 (ii), we have $d(u_2) = 2$ or 4. Similarly, we have $d(w_2) = 2$ or 4. By symmetry, we may assume that $d(u_2) = 2$, $d(w_2) = 4$. By Lemma 3.1 (i), $d(w_i) = 4$ for all $i = 1, \ldots, k$. Furthermore, $S(u_2) = 6 = 4 + d(u_3)$. Thus $d(u_3) = 2$ and $S(u_3) = 6 = 2 + d(u_4)$. So $d(u_4) = 4 \neq d(u_2)$ and hence $n = 4$. Therefore $G \in \mathcal{F}_5$, see Figure 4. It is easy to see that every graph in $\mathcal{F}_5$ is 2-walk $(2, 2)$-linear.

Case 2: $d(v_1) = d(v_2) = a + b$. Dually, we have $d(u_1) = d(u_3) = a + b$. Apply Lemma 25 (iv) to $C = v_1v_2 \ldots v_mv_{r_1} \ldots v_{r_l}$. We get, without loss of generality, $m \geq 3$ and $d(v_1) \neq a + b$ for some $2 \leq i \leq m - 1$. So $d(v_i) = 2$ for all $2 \leq i \leq m - 1$ by Lemmas 25 (ii) and 3.1.

From (3) with $(v, u) = (v_1, x)$ and $(v, u) = (v_2, x)$, we have
\begin{equation}
\begin{aligned}
a &= \frac{d(r_2) + d(s_{p-1}) - 1}{a + b - 1}, \quad a = d(v_i),
\end{aligned}
\end{equation}

respectively. We claim that $m \geq 4$. Otherwise, suppose that $m = 3$. Then $a = d(v_2) = 2$ and $d(r_2) + d(s_{p-1}) = a(a + b - 1) + 1 \geq 3(a + b) - 2 > 2(a + b)$, which is impossible since $d(r_2), d(s_{p-1}) = 2$ or $a + b$ by Lemma 25 (ii). Hence $m \geq 4$. Therefore $m = 4$ by Lemma 3.1 (ii). Thus $a = d(v_2) = 2$. It follows from (8) that $d(r_2) + d(s_{p-1}) = 2(a + b) - 1$.

By Lemma 25 (iii), $d(v_2), d(s_{p-1}) = 2$ or $a + b$.

- If $d(r_2) = d(s_{p-1}) = 2$, then $a + b = 2.5$, a contradiction.
- If $d(r_2) = d(s_{p-1}) = a + b$, then $2(a + b) = 2(a + b) - 1$, a contradiction.
- If $d(r_2) = 2$, $d(s_{p-1}) = a + b$, then $a + b = 3$. So $a = 2, b = 1$. By Lemma 3.1 (i), $d(s_i) = d(s_{p-1}) = 3$ for $2 \leq i \leq p - 1$. For the vertex $u_2$, $S(u_2) = 3 + d(u_2) + d(w_2) = 7$ by (1) and (2). By Lemma 25 (ii), $d(u_3), d(w_2) = 2$ or 3. It follows that $d(u_3) = d(w_2) = 2$. So $S(u_3) = 3 + d(u_3) = 5$. Thus $d(u_3) = 2$ and $a + b = 4$. Hence $n = 4$ by Lemma 3.1 (ii). Similarly, we have $d(w_1) = d(r_2) = 2$ and $k = l = 4$. By (1) and (2), $S(u_4) = 4 + d(t_2) = 7$. So $d(t_2) = 3$. Hence $d(t_i) = 3$ for $1 \leq i \leq q - 1$ by Lemma 3.1 (i). Therefore $G \in \mathcal{F}_6$, see Figure 4, where max${\{k_1, k_2\}} \geq 1$. It is easy to see that any graph in $\mathcal{F}_6$ is 2-walk $(2, 1)$-linear.

- If $d(r_2) = a + b, d(s_{p-1}) = 2$, then with a similar argument of the case $d(r_2) = 2, d(s_{p-1}) = a + b$, we have $G \in \mathcal{F}_7$ and $G$ is 2-walk $(2, 1)$-linear, see Figure 4, where max${\{k_1, k_2\}} \geq 1$. \hfill \Box

Lemma 3.10.
Let $G \in \mathcal{F}$ with $\delta(G_0) \geq 2$ and $G_0 \in \mathcal{T}_b$. Then $G \cong H_i$ for $i = 26, \ldots, 32$, see Figure 3, or $G \in \mathcal{F}_j$ for $j = 8, 9$, see Figure 4.

Proof. Let $G_0 = \mathcal{T}_b$, see Figure 2, where $n, m, k, l, p, q \geq 2$. For convenience, we set $u_1 = v_1 = w_1, u_3 = t_1 = s_1$.

Case 1: there is a vertex $v \in \{w_2, v_2, u_2\}$ with $d_{C_0}(v) = 2$ and $d(v) = a + b$. Without loss of generality, suppose that $v = w_2$. Then $k \geq 3$ and $d(w_i) = a + b$ for $2 \leq i \leq k - 1$ by Lemma 3.1 (i). In particular, $a \geq 2, a + b \geq 3$ by Lemma 25 (iii).

Lemma 23 for $(v, u) = (w_2, x)$ implies $a = (d(w_i) + d(w_2) - 2)/(a + b - 1)$. By Lemma 25 (ii), we have $d(w_1), d(w_3) = 3$ or $a + b$. If $d(w_1) = d(w_3) = d(w_3) = 3 \neq a + b$, then $a + b \geq 4$ and $a = 4/(a + b - 1) < 2$, a contradiction. If $d(w_1) = a + b, d(w_3) = 3$ or $d(w_3) = a + b, d(w_1) = 3$ and $a + b \neq 3$, then $a = (a + b + 1)/(a + b - 1) = 1 + 2/(a + b - 1) < 2$, a contradiction. If $d(w_1) = d(w_3) = a + b$, then $a = 2$. We claim that $a + b = 3$. Otherwise, let $a + b > 3$. For the vertex $w_1$, $S(w_1) = 2(a + b) + b = a + b + 3 + a + b + d(w_2) + d(v_2)$. Thus $d(u_1) + d(v_2) = a + b + 1$. By Lemma 25 (ii), $d(u_3), d(v_2) = 2, 3$ or $a + b$. Note that $a + b > 3$. We have $d(u_3) = 2, d(v_2) = 3$ or $d(v_2) = 3, d(v_2) = 2$ or $d(u_3) = d(v_2) = 3$. Without loss of generality, we consider the following two cases:

- If $d(u_3) = 2, d(v_2) = 3$, then $a + b = 4$ and $m = 2$. Note that $a = 2$. We have $b = 2$. If $k \geq 4$, then $d(w_4) = d(w_1) = 4$ by Lemma 3.1 (i). If $k = 3$, then $d(w_4) = d(w_3) = 4$. So $d(r_3) = 2$ or 4 by Lemma 25 (ii). On the other hand, $S(v_2) = 4 + d(v_2) + d(t_{q-1}) = 8$ by (1) and (2). So $d(t_2) = d(t_{q-1}) = 2$. By (1) and (2), $S(v_2) = 6 = 3 + d(r_3)$. Thus $d(r_3) = 3, a$ contradiction.
If $d(u_2) = d(v_2) = 3$, then $a + b = 5$ and $m = n = 2$. Note that $a = 2$. We have $b = 3$. By (1) and (2), $S(v_2) = 5 + d(r_2) + d(t_{q-1}) = 9$. Thus $d(r_2) = d(t_{q-1}) = 2$. Hence $S(r_2) = 7 = 3 + d(r_2)$. This is impossible since $d(r_2) \in \{2, 3, 5\}$.

Therefore $a + b = 3$ and $a = 2, b = 1$. By (1) and (2), $S(w_1) = 7 = 3 + d(v_2) + d(u_2)$. So $d(v_2) = d(u_2) = 2$. Hence $S(v_2) = 5 + d(v_2)$. It follows that $d(v_1) = 2$ and $m \geq 4$. Therefore $m = 4$ by Lemma 3.1 (ii). Similarly, we have $n = l = p = 4, d(u_2) = d(v_2) = d(r_2) = d(s_2) = d(s_3) = 2$ and $d(t_i) = 3$ for $2 \leq i \leq q - 1$. Therefore $G \in \mathcal{G}_0$, see Figure 4, where $\max\{k_1, k_2\} \geq 1$. It is easy to see that any graph $G \in \mathcal{G}_0$ is 2-walk $(2,1)$-linear.

Case 2: For any vertex $v \in \{w_2, v_2, u_2\}$, we have $d_G(v) = 3$ or $d(v) = 2$. If $G$ has no pendant vertex, then $n, m, k, l, p, q \leq 4$ by Lemma 2.4. If $n = m = k = l = p = q = 2$, then $G$ is regular. It is well known that a graph is regular if and only if it has exactly one main eigenvalue. Thus $\max\{n, m, k, l, p, q\} \geq 3$. Without loss of generality, suppose that $n \geq 3$. We consider the following two cases:

Subcase 1: $n = 3$. Then $S(u_2) = 6$ and $m, k, l, p, q = 2$ or 3 by Lemma 2.4. If $m = k = 2$, then $S(u_1) = 8 = S(u_2) = 2 + d(t_2) + d(s_{p-1})$. So $d(t_2) = d(s_{p-1}) = 3$ and hence $p = q = 2$. Similarly, $S(u_1) = 8 = S(v_2) = 6 + d(r_2)$. Thus $d(r_2) = 2$ and $l = 3$. Therefore $G = H_{90}$, see Figure 3. By (3), $H_{90}$ is 2-walk $(2,2)$-linear. With a similar argument, we have: If $m = 2, k = 3$ or $m = 3, k = 2$, then $G = H_{72}$ is 2-walk $(1,4)$-linear, see Figure 3. If $m = k = 3$, then $G = H_{28}$ is 2-walk $(0,6)$-linear, see Figure 3.

Subcase 2: $n = 4$. Then $m, k, l, p, q = 2$ or 4 by Lemma 2.4. With a similar argument of Subcase 1, we have the following cases: If $m = k = 2$, then $G = H_{92}$ is 2-walk $(3,1)$-linear, see Figure 3. If $m = 2, k = 4$ or $m = 4, k = 2$, then $G = H_{90}$ is 2-walk $(2,1)$-linear, see Figure 3. If $m = k = 4$, then $G = H_{31}$ is 2-walk $(1,3)$-linear, see Figure 3.

If $G$ has at least one pendant vertex $x$, then $x \in N(v)$ for $v \in \{w_1, v_2, u_2, w_3, t_3, s_3\}$, where $2 \leq i_1 \leq l - 1, 2 \leq i_2 \leq q - 1, 2 \leq i_1 \leq p - 1$. If $v \in \{r_1, t_2, s_2\}$, then with a similar argument of Case 1, we have $G \in \mathcal{G}_0$. Hence we may suppose that $v \in \{w_1, v_2, u_2\}$. In particular, assume that $x \in N(w_1)$ without loss of generality. Then $d(w_1) = a + b \geq 4$ and $d(u) = d_G(u) \leq 3$ for $u \in \{r_1, t_2, s_2\}$, where $2 \leq i_1 \leq l - 1, 2 \leq i_2 \leq q - 1, 2 \leq i_1 \leq p - 1$. Apply Lemma 2.3 with $(v, u) = (w_1, x)$. We get

$$\begin{align*}
a &= \frac{a + b - 2 + d(u_2) + d(v_2) + d(w_2) - (a + b)}{a + b - 1}.
\end{align*}$$

Note that $d(u_2), d(v_2), d(w_2) = 2, a + b$ and $a + b \geq 4$. We consider the following cases without loss of generality:

- If $d(u_2) = d(v_2) = d(w_2) = 2$, then $a = 3/(a + b - 1) < 2$, a contradiction.
- If $d(v_2) = d(w_2) = 2, d(u_2) = 3$, then $a = 4/(a + b - 1) < 2$, a contradiction.
- If $d(u_2) = d(v_2) = 2, d(w_2) = a + b$, then $a + b \geq 2$, a contradiction.
- If $d(u_2) = d(v_2) = d(w_2) = 3$, then $a = 5/(a + b - 1) < 2$, a contradiction.
- If $d(u_2) = 2, d(v_2) = 3, d(w_2) = a + b$, then $m = 2$ and $a = 1 + 3/(a + b - 1)$. By Lemmas 2.2 and 2.5 (ii), we have $a + b = 4$ and $a = 2$. Hence $d(w_1) = d(w_2) = 4$ for $2 \leq i \leq k - 1$. By (1) and (2), $S(w_{i-1}) = 4 + 2 + d(w_1) = 10$. Thus $d(w_1) = 4$. Similarly, $S(v_2) = 4 + d(r_2) + d(t_{q-1}) = 8$. It implies that $d(r_2) = d(t_{q-1}) = 2$. Thus $S(r_2) = 6 + 3 + d(r_3)$. Hence $d(r_3) = 3$. On the other hand, $d(r_3) \in \{2, 4\}$ by Lemma 2.5 (ii) and the fact that $d(w_1) = 4$. This is a contradiction.
- If $d(u_2) = 2, d(v_2) = d(w_2) = a + b$, then $a = 2 + 1/(a + b - 1)$ is not an integer, a contradiction.
- If $d(u_2) = d(v_2) = d(w_2) = 3$, then $n = m = k = 2$ and $a = 6/(a + b - 1)$. It follows from Lemma 2.5 (iii) that $a + b = 4$ and $a = 2$. By (1) and (2), $S(v_2) = 4 + d(r_2) + d(t_{q-1}) = 8$. It implies that $d(r_2) = d(t_{q-1}) = 2$. Hence $S(r_2) = 3 + d(r_3) = 6$. It follows that $d(r_3) = 3$ and $l = 3$. Similarly, we have $p = q = 3$. Therefore $G = H_{28},$ see Figure 3. By (3), $H_{28}$ is 2-walk $(2,2)$-linear.
- If $d(u_2) = d(v_2) = 3, d(w_2) = a + b$, then $n = m = 2$ and $a = 4/(a + b - 1)$. Thus $a + b = 5$ and $a = 2$ by Lemma 2.5 (iii). Hence $d(w_1) = d(w_2) = a + b = 5$. For the vertex $v_2, S(v_2) = 5 + d(r_2) + d(t_{q-1}) = 9$. It implies that $d(r_2) = d(t_{q-1}) = 2$. Thus $S(r_2) = 3 + d(r_3) = 7$, which is impossible since $d(r_3) \in \{2, 3, 5\}$ by Lemma 2.5 (ii).
• If \( d(u_2) = 3, d(v_2) = d(w_2) = a + b \), then \( a = 2 + 2/(a + b - 1) \) is not an integer, a contradiction.

• If \( d(u_2) = d(v_2) = d(w_2) = a + b \), then \( n = m = k = 2 \) and \( a = 3 \). We claim that \( p = l = 2 \). Otherwise, let \( p, l > 2 \). By (1), \( S(w_1) = a + b - 3 + 3(a + b) \), \( S(w_2) = a + b - 3 + a + b + d(s_2) + d(r_{l-1}) \). Note that \( d(s_2), d(r_{l-1}) = 2 \) by assumption. We have \( S(w_1) > S(w_2) \). On the other hand, \( S(w_1) = S(w_2) \) by (2), a contradiction. Hence \( p = l = 2 \). Similarly, we have \( q = 2 \). Thus \( G \in S_b \), see Figure 4, where \( b \geq 1 \). It is easy to see that any graph \( G \in S_b \) is 2-walk \((3, b)\)-linear.

The following theorem follows directly from Lemmas 3.3–3.10.

**Theorem 3.11.**
Let \( G \) be a graph and \( G_0 \) be the graph obtained by deleting all pendant vertices in \( G \). The graphs \( H_i \) for \( i = 1, \ldots, 32 \) and those in \( S_j \) for \( j = 1, \ldots, 9 \) are all connected tricyclic graphs \( G \) with \( \delta(G_0) \geq 2 \) and exactly two main eigenvalues.

4. Conclusion

In Section 3, we determined all connected tricyclic graphs with \( \delta(G_0) \geq 2 \) and exactly two main eigenvalues. In fact, there exist tricyclic graphs with exactly two main eigenvalues and \( \delta(G_0) = 1 \). Figure 5 gives two examples. To determine all tricyclic graph with \( \delta(G_0) = 1 \) and exactly two main eigenvalues is still an open problem. Further studies could concentrate on investigating these graphs.

**Figure 5.** Tricyclic graphs with exactly two main eigenvalues and \( \delta(G_0) = 1 \).

Acknowledgements

We thank the referees whose useful suggestions and two examples of tricyclic graphs with exactly two main eigenvalues resulted in improvement to this article.

This research was partially supported by the National Natural Science Foundation of China (No. 10971086 and No. 11201201).

First author is currently a visiting Ph.D. student at the Department of Combinatorics and Optimization in University of Waterloo from September 2010 to September 2012.

References


